On the Shaft End Torque and the Forced Vibrations of an Asymmetrical Shaft Carrying an Asymmetrical Rotor*

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In a rotating asymmetrical shaft with an asymmetrical rotor, the increase rate of total energy and the torque applied to the shaft end change with the angular positions of the static and dynamic unbalances. The response curve also changes with the angular positions of the rotor unbalances. With a rotor mounted on the middle of an asymmetrical shaft, the parallel motion of the rotor is not connected with its conical motion, and the shaft end torque can be directly obtained from the equilibrium of forces and moments. The shaft end torque in an asymmetrical shaft and an asymmetrical rotor changes with the angular positions of rotor unbalances in a similar way to the response curve.

Key Words: Vibration of Rotating Body, Asymmetrical Shaft, Asymmetrical Rotor, Forced Vibration, Torque Applied to Shaft End

1. Introduction

It has been reported^{(1)~(5)} that the response curve is considerably influenced by the angles between the principal axis of inertia of an asymmetrical shaft or an asymmetrical rotor, and the direction of rotor unbalances. H. D. Taylor⁽¹⁾ reports analytical results that the response curves of an asymmetrical shaft are different due to the angular position of static unbalance of a rotor. K. Okijima and Y. Kondo⁽⁴⁾ discuss the effect of the angular position of rotor unbalance on the response curve of a rotating asymmetrical shaft, both ends of which are supported by flexible pedestals with directional inequality in stiffness. T. A. Henry and B. E. Okah-Avae⁽⁵⁾ indicate that the response curve of the shaft system with a crack depends on the angle between a crack and rotor unbalance.

The present study clarifies the effect of two angular positions ξ , η of the static unbalance e_0 and the dynamic unbalance τ on the increase rate of total energy and the torque^{(6)~(9)} applied to the shaft end in an asymmetrical shaft carrying an asymmetrical rotor. When a rotor is mounted on the middle of an asymmetrical shaft, the parallel motion of a rotor is not connected with its conical motion, and so a torque applied to the shaft end is directly obtained from the equilibrium of forces and moments acting upon the rotor. Moreover, it is seen in an asymmetrical shaft and/or in an asymmetri-

cal rotor that the shaft end torque changes with the orientation angles ξ , η of rotor unbalances in a manner similar to the response curve.

2. Equations of Motion

The angular positions of the rectangular coordinate system $S-X_2Y_2Z_1$ fixed to an asymmetrical rotor are denoted by Eulerian angles θ , ϕ and ψ as shown in Fig. 1, and an orientation angle⁽⁹⁾ between the inequality of shaft stiffness and that of rotor inertia is defined by $\zeta = \angle X_2SX_3 = \angle Y_2SY_3$. Let displacements of the center S of an asymmetrical rotor be x and y , and let projected angles of the inclinational angle θ of SZ_1 -axis to xOz and yOz planes be θ_x and θ_y , respectively. When the same symbols as in the previous study⁽⁹⁾ are used and terms higher than the 3rd order terms of small quantities θ_x and θ_y are neglected, the kinetic energy of an asymmetrical rotor T and the potential energy of an asymmetrical shaft V are given as follows⁽³⁾⁽⁹⁾:

Fig. 1 Eulerian angles θ , ϕ and ψ , and the orientation angle ζ

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$$\left. \begin{aligned}
2T &= m_0(\dot{x}^2 + \dot{y}^2) + I_p(\dot{\theta}^2 + \dot{\phi}(\dot{\theta}_x\theta_y - \theta_x\dot{\theta}_y)) \\
&\quad + I(\dot{\theta}_x^2 + \dot{\theta}_y^2) - \Delta I\{(\dot{\theta}_x^2 - \dot{\theta}_y^2)\cos 2\Theta + 2\dot{\theta}_x\dot{\theta}_y \sin 2\Theta\} \\
2V &= \alpha(x^2 + y^2) + 2\gamma(x\theta_x + y\theta_y) + \delta(\theta_x^2 + \theta_y^2) \\
&\quad - \Delta\alpha\{(x^2 - y^2)\cos 2(\Theta + \xi) + 2xy \sin 2(\Theta + \xi)\} \\
&\quad - 2\Delta\gamma\{(x\theta_x - y\theta_y)\cos 2(\Theta + \xi) \\
&\quad + (x\theta_y + y\theta_x)\sin 2(\Theta + \xi)\} \\
&\quad - \Delta\delta\{(\theta_x^2 - \theta_y^2)\cos 2(\Theta + \xi) + 2\theta_x\theta_y \sin 2(\Theta + \xi)\}
\end{aligned} \right\} \dots\dots\dots (1)$$

where $\Theta = \varphi + \psi$ is a rotational angle of the shaft end. Let c_1 and c_2 be viscous damping coefficients in relation to displacement and inclination, respectively. Dissipation function F is given as

$$2F = c_1(\dot{x}^2 + \dot{y}^2) + c_2(\dot{\theta}_x^2 + \dot{\theta}_y^2) \dots\dots\dots (2)$$

Substituting equations (1) and (2) into Lagrange's equation of motion (3), taking generalized coordinate q_s as Θ, x, y, θ_x and θ_y ;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = 0 \dots\dots\dots (3)$$

and neglecting the terms higher than the 2nd order term of small quantities, the equation of motion $\dot{\Theta} = 0$ is derived, and then we have

$$\dot{\Theta} = \omega = \text{constant}, \quad \Theta = \omega t \dots\dots\dots (4)$$

The following complex variables

$$\left. \begin{aligned}
z &= x + iy, \quad \bar{z} = x - iy \\
\theta_z &= \theta_x + i\theta_y, \quad \bar{\theta}_z = \theta_x - i\theta_y
\end{aligned} \right\} \dots\dots\dots (5)$$

are introduced. The equations of motion⁽⁹⁾ for z and θ_z are obtained by using equations (4) and (5):

$$\left. \begin{aligned}
m_0\ddot{z} + c_1\dot{z} + \alpha z + \gamma\theta_z \\
\quad - (\Delta\alpha\bar{z} + \Delta\gamma\bar{\theta}_z)e^{2i(\omega t + \xi)} = 0 \\
I\ddot{\theta}_z - iI_p\omega\dot{\theta}_z + c_2\dot{\theta}_z + \gamma z + \delta\theta_z - \Delta I \frac{d}{dt}(\bar{\theta}_ze^{2i\omega t}) \\
\quad - (\Delta\gamma\bar{z} + \Delta\delta\bar{\theta}_z)e^{2i(\omega t + \xi)} = 0
\end{aligned} \right\} \dots\dots\dots (6)$$

In general, there are both the static unbalance $e_0 = \overline{SG}$ and the dynamic one $\tau = \angle Z_1SZ_0$ as shown in Fig. 2. Displacements x_0 and y_0 of the gravity center G of a rotor are represented by complex variables $z_0 = x_0 + iy_0$, and the projected angles θ_{1x} and θ_{1y} of the principal axis SZ_1 by $\theta_{1z} = \theta_{1x} + i\theta_{1y}$. When the four terms underlined in equation (6), that is, $m_0\ddot{z}$, $I\ddot{\theta}_z$, $-iI_p\omega\dot{\theta}_z$ and $-\Delta I \cdot d/dt(\bar{\theta}_ze^{2i\omega t})$, are replaced by $m_0\ddot{z}_0$, $I\ddot{\theta}_{1z}$, $-iI_p\omega\dot{\theta}_{1z}$ and $-\Delta I \cdot d/dt(\bar{\theta}_{1z}e^{2i\omega t})$, respectively, the equations of motion with rotor unbalances are rewritten as follows:

$$\left. \begin{aligned}
m_0\ddot{z}_0 + c_1\dot{z}_0 + \alpha z + \gamma\theta_z \\
\quad - (\Delta\alpha\bar{z} + \Delta\gamma\bar{\theta}_z)e^{2i(\omega t + \xi)} = 0 \\
I\ddot{\theta}_{1z} - iI_p\omega\dot{\theta}_{1z} + c_2\dot{\theta}_z + \gamma z + \delta\theta_z - \Delta I \frac{d}{dt}(\bar{\theta}_{1z}e^{2i\omega t}) \\
\quad - (\Delta\gamma\bar{z} + \Delta\delta\bar{\theta}_z)e^{2i(\omega t + \xi)} = 0
\end{aligned} \right\} \dots\dots\dots (7)$$

In the case that both unbalances e_0 and τ exist, when the first equation of equation (1) is transformed by complex variables (5) and the variables $\dot{z}$, $\ddot{z}$, θ_z , $\dot{\theta}_z$ and $\ddot{\theta}_z$ are

Fig. 2 Angular positions ξ and η of static unbalance e_0 and dynamic unbalance τ

replaced by $\dot{z}_0$, $\ddot{z}_0$, θ_{1z} , $\dot{\theta}_{1z}$ and $\ddot{\theta}_{1z}$, respectively, and equation (4) is substituted into this obtained equation, the kinetic energy T is expressed as follows⁽⁹⁾:

$$\begin{aligned}
2T &= m_0\dot{z}_0\bar{\dot{z}}_0 + I_p\omega^2 + I_p\omega \text{Im}[\theta_{1z}\bar{\dot{\theta}}_{1z}] \\
&\quad + I\dot{\theta}_{1z}\bar{\dot{\theta}}_{1z} - \Delta I \text{Re}[\dot{\theta}_{1z}\bar{\theta}_{1z}e^{-2i\omega t}] \dots\dots\dots (8 \cdot a)
\end{aligned}$$

where the symbols $\text{Re}[\dots]$ and $\text{Im}[\dots]$ mean the real part and the imaginary one of a complex number $[\dots]$. By substituting the complex variables (5) into the second equation of equation (1), the potential energy V is rewritten as follows⁽⁹⁾:

$$\begin{aligned}
2V &= \alpha z\bar{z} + 2\gamma \text{Re}[z\bar{\theta}_z] + \delta\theta_z\bar{\theta}_z - \Delta\alpha \text{Re}[z^2e^{-2i(\omega t + \xi)}] \\
&\quad - 2\Delta\gamma \text{Re}[z\bar{\theta}_ze^{-2i(\omega t + \xi)}] - \Delta\delta \text{Re}[\theta_z^2e^{-2i(\omega t + \xi)}] \dots\dots\dots (8 \cdot b)
\end{aligned}$$

As shown in Fig. 2, the gravity center G locates in the SG direction given by a rotation of the SX_2 axis about the SZ_1 axis by angle $\xi = \angle X_2SG$, and the dynamic unbalance τ occurs in the $X_1Z_1(X_0Z_0)$ plane which is perpendicular to the SY_1 axis of the rotated SY_2 axis about the SZ_1 axis by angle $\eta = \angle Y_2SY_1 = \angle X_2SX_1$. The following relations⁽¹⁰⁾ are obtained by neglecting the terms higher than the 3rd order term of small quantities:

$$z_0 = z + e_0e^{i(\omega t + \xi)}, \quad \theta_{1z} = \theta_z + \tau e^{i(\omega t + \eta)} \dots\dots\dots (9)$$

Substituting equation (9) into equation (7), the equations of motion are rewritten as follows^{(3) (10)}:

$$\left. \begin{aligned}
m_0\ddot{z} + c_1\dot{z} + \alpha z + \gamma\theta_z - (\Delta\alpha\bar{z} + \Delta\gamma\bar{\theta}_z)e^{2i(\omega t + \xi)} \\
\quad = m_0e_0\omega^2e^{i(\omega t + \xi)} \\
I\ddot{\theta}_z - iI_p\omega\dot{\theta}_z + c_2\dot{\theta}_z + \gamma z + \delta\theta_z \\
\quad - \Delta I \frac{d}{dt}(\bar{\theta}_ze^{2i\omega t}) - (\Delta\gamma\bar{z} + \Delta\delta\bar{\theta}_z)e^{2i(\omega t + \xi)} \\
\quad = -\tau\omega^2\{(I_p - I)e^{i(\omega t + \eta)} - I\bar{e}^{i(\omega t - \eta)}\}
\end{aligned} \right\} \dots\dots\dots (10)$$

3. Relation between Shaft End Torque and Increase Rate of Total Energy

In the case that rotor unbalances e_0 and τ both exist, the increase rate of kinetic energy⁽⁸⁾ and that of potential energy⁽⁸⁾ are obtained by differentiating equations (8.a) and (8.b) with respect to time t :

$$\left. \begin{aligned} \dot{T} &= m_0 \operatorname{Re}[\ddot{z}_0 \dot{z}_0] + (I_p/2) \omega \operatorname{Im}[\dot{\theta}_{1z} \ddot{\theta}_{1z}] + I \operatorname{Re}[\ddot{\theta}_{1z} \dot{\theta}_{1z}] - (\Delta I/2) \frac{d}{dt} \operatorname{Re}[\dot{\theta}_{1z}^2 e^{-2i\omega t}] \\ \dot{V} &= \alpha \operatorname{Re}[z \ddot{z}] + \gamma \operatorname{Re}[\ddot{z} \theta_z + z \ddot{\theta}_z] + \delta \operatorname{Re}[\theta_z \ddot{\theta}_z] - \Delta \alpha \operatorname{Re}[z \ddot{z} e^{-2i(\omega t + \zeta)}] - \Delta \alpha \omega \operatorname{Im}[z^2 e^{-2i(\omega t + \zeta)}] \\ &\quad - \Delta \gamma \operatorname{Re}[(z \dot{\theta}_z + \dot{z} \theta_z) e^{-2i(\omega t + \zeta)}] - \Delta \gamma \omega \operatorname{Im}[2z \theta_z e^{-2i(\omega t + \zeta)}] \\ &\quad - \Delta \delta \operatorname{Re}[\theta_z \ddot{\theta}_z e^{-2i(\omega t + \zeta)}] - \Delta \delta \omega \operatorname{Im}[\dot{\theta}_z^2 e^{-2i(\omega t + \zeta)}] \end{aligned} \right\} \dots\dots\dots (11)$$

The first equation in equation (7) is multiplied by $\dot{z}$, the second one by $\dot{\theta}_z$, and these two equations are added together, the real part of which gives the following equation:

$$\begin{aligned} m_0 \operatorname{Re}[\ddot{z} \dot{z}] + c_1 |\dot{z}|^2 + \alpha \operatorname{Re}[z \ddot{z}] + \gamma \operatorname{Re}[\ddot{z} \theta_z + z \ddot{\theta}_z] + I \operatorname{Re}[\ddot{\theta}_z \dot{\theta}_z] + c_2 |\dot{\theta}_z|^2 + \delta \operatorname{Re}[\theta_z \ddot{\theta}_z] \\ - \operatorname{Re} \left[\Delta I \dot{\theta}_z \frac{d}{dt} (\dot{\theta}_{1z} e^{2i\omega t}) + \{ \Delta \alpha \ddot{z} + \Delta \gamma (\ddot{z} \theta_z + \dot{z} \ddot{\theta}_z) + \Delta \delta \ddot{\theta}_z \theta_z \} e^{2i(\omega t + \zeta)} \right] = 0 \end{aligned} \dots\dots\dots (12)$$

Displacement variable z' and inclinational angle variable θ'_z viewed from the rotating coordinate system $O-x'y'z$ turning about the bearing center line Oz at an angular velocity ω are related to z and θ_z viewed from the stationary coordinate system $O-xyz$ as follows:

$$z' = x' + iy' = ze^{-i\omega t}, \quad \theta'_z = \theta'_x + i\theta'_y = \theta_z e^{-i\omega t} \quad \dots (13)$$

Using equations (9), (11), (12) and (13), the increase rate of total energy $\dot{T} + \dot{V}$ is given as:

$$\begin{aligned} \dot{T} + \dot{V} &= -\omega \operatorname{Im}[\Delta I (-\omega \theta'_z + i \dot{\theta}'_z)^2 + (\Delta \alpha z'^2 + 2\Delta \gamma z' \theta'_z \\ &\quad + \Delta \delta \theta'^2_z) e^{-2i\zeta} + (I_p/2) \dot{\theta}'_z \ddot{\theta}'_z - m_0 e_0 \ddot{z} e^{-i(\omega t + \zeta)} \\ &\quad + \tau \dot{\theta}'_z \{ (I_p - I) e^{-i(\omega t + \eta)} - \Delta I e^{-i(\omega t - \eta)} \}] \\ &\quad - c_1 |\dot{z}'|^2 - c_2 |\dot{\theta}'_z|^2 \end{aligned} \dots\dots\dots (14)$$

Torque T_r applied to the shaft end should be a generalized force⁽⁷⁾⁽⁹⁾ with respect to rotation of shaft end θ , so the torque is obtained by putting $q_s = \theta$ in Lagrange's equation of motion (3):

$$\begin{aligned} T_r &= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} + \frac{\partial F}{\partial \theta} \\ &= -\operatorname{Im}[\Delta I (-\omega \theta'_z + i \dot{\theta}'_z)^2 \\ &\quad + (\Delta \alpha z'^2 + 2\Delta \gamma z' \theta'_z + \Delta \delta \theta'^2_z) e^{-2i\zeta} \\ &\quad + (I_p/2) \dot{\theta}'_z \ddot{\theta}'_z - m_0 e_0 \ddot{z} e^{-i(\omega t + \zeta)} \\ &\quad + \tau \dot{\theta}'_z \{ (I_p - I) e^{-i(\omega t + \eta)} - \Delta I e^{-i(\omega t - \eta)} \}] \end{aligned} \dots\dots\dots (15)$$

From equations (2), (14) and (15), the following relation holds:

$$\omega T_r = \dot{T} + \dot{V} + 2F \quad \dots\dots\dots (16)$$

Equation (16) means that the time rate of work applied to the shaft end ωT_r agrees with the sum of the increase rate of total energy and the dissipated rate by damping forces. The amplitude of whirling motion changes with the orientational angle ζ between the inequality of shaft stiffness and that of rotor inertia, the angular position ξ of static unbalance e_0 and η of dynamic unbalance τ , because the right hand side of equation (15) contains the three angles ζ , ξ and η .

4. Effect of Angular Position of Rotor Unbalances on Shaft End Torque for Asymmetrical Shaft

4.1 Case of parallel motion of rotor

Let us consider only a parallel motion of a rotor mounted on the middle of an asymmetrical shaft ($\theta_z = 0$), and put $\zeta = 0$. Three force vectors F , D and P in Fig. 3 indicate restoring, damping and inertial forces, respectively; the restoring and damping forces act upon the rotor center S , and an inertial force acts upon the gravity center of the rotor G . The three forces F , D and P are given as

$$\left. \begin{aligned} F &= -az + \Delta a \ddot{z} e^{2i\omega t} = (-az' + \Delta a \ddot{z}') e^{i\omega t} \\ D &= -c_1 \dot{z}, \quad P = -m_0 \ddot{z}_g \end{aligned} \right\} \dots\dots (17)$$

A restoring force vector F does not point⁽⁶⁾⁽⁷⁾ to the direction of lateral displacement OS . Components of a restoring force F in x' - and y' -directions are expressed as follows:

$$F'_x = -(\alpha - \Delta \alpha) x', \quad F'_y = -(\alpha + \Delta \alpha) y' \quad \dots\dots\dots (18)$$

Let F_2 be a perpendicular component of F to OS -direction. In order to turn the shaft end at a constant velocity ω , a counter-torque

Fig. 3 Three forces F , P and D acting upon a rotor with a parallel motion ($\theta_z = 0$, $\zeta = 0$)

T_{r1} must be applied to the shaft end^{(6) (7)} against a moment produced by a reaction force $-F_2$ about the bearing center line Oz ;

$$\begin{aligned} T_{r1} &= F_2 |z'| = -F_2' y' + F_2' x' \\ &= -2\Delta \alpha x' y' = -\Delta \alpha \operatorname{Im}[z'^2] \end{aligned} \quad (19)$$

As shown in Fig. 3, an inertial force P acting upon G is in balance to a vector sum $-P$ shown by a dotted line consisting of two vectors F and D . By using equations (9) and (17), an inertial force P is expressed as

$$P = P_x + iP_y = -m_0 \{\ddot{z} - e_0 \omega^2 e^{i(\omega t + \xi)}\} \quad (20)$$

An inertial force P makes a moment about S , because vector P does not point to the direction SG . The following counter-torque against a moment produced by P

$$\begin{aligned} T_{r2} &= P_x(y_0 - y) - P_y(x_0 - x) \\ &= m_0 e_0 \operatorname{Im}[\ddot{z} e^{-i(\omega t + \xi)}] \end{aligned} \quad (21)$$

must be applied to the shaft end in order

to turn a shaft at a constant velocity ω .

Therefore, the applied torque T_r is obtained from equations (19) and (21) as follows:

$$T_r = T_{r1} + T_{r2} = -\operatorname{Im}[\Delta \alpha z'^2 - m_0 e_0 \ddot{z} e^{-i(\omega t + \xi)}] \quad (22)$$

Equation (22) agrees precisely with the underlined terms obtained by setting $\zeta = 0$ in equation (15) excepting θ_z terms.

Next, let us investigate how the shaft end torque and amplitude change with the angular position ξ of the static unbalance e_0 .

The solutions for forced vibration of equation (10) are represented by

$$z = A e^{i\omega t}, \quad z' = z e^{-i\omega t} = A \quad (23)$$

where A is a complex constant. Substituting the solution (23) into equation (22), the torque is given as follows:

$$T_r = -\operatorname{Im}[\Delta \alpha A^2 + m_0 e_0 \omega^2 A e^{-i\xi}] \quad (24)$$

Let $\arg$ be the argument of complex number. Equation (24) is rewritten in the following form:

$$T_r = -\Delta \alpha |A|^2 \sin 2 \arg A - m_0 e_0 \omega^2 |A| \sin(\arg A - \xi) \quad (25)$$

When the solutions of forced vibration (23) are substituted into the first equation of equation (10), the real and imaginary parts of this derived equation are as follows:

$$\begin{aligned} (\alpha - \Delta \alpha - m_0 \omega^2) |A| \cos \arg A - c_1 \omega |A| \sin \arg A &= m_0 e_0 \omega^2 \cos \xi \\ c_1 \omega |A| \cos \arg A + (\alpha + \Delta \alpha - m_0 \omega^2) |A| \sin \arg A &= m_0 e_0 \omega^2 \sin \xi \end{aligned} \quad (26)$$

$|A| \cos \arg A$ and $|A| \sin \arg A$ in equation (26) can be solved as follows⁽²⁾:

$$\left. \begin{aligned} |A| \cos \arg A &= \frac{m_0 e_0 \omega^2 \{(\alpha + \Delta \alpha - m_0 \omega^2) \cos \xi + c_1 \omega \sin \xi\}}{(\alpha - m_0 \omega^2)^2 - \Delta \alpha^2 + c_1^2 \omega^2} \\ |A| \sin \arg A &= \frac{m_0 e_0 \omega^2 \{-c_1 \omega \cos \xi + (\alpha - \Delta \alpha - m_0 \omega^2) \sin \xi\}}{(\alpha - m_0 \omega^2)^2 - \Delta \alpha^2 + c_1^2 \omega^2} \end{aligned} \right\} \quad (27)$$

Substitution of equation (27) into equation (25) gives a driving torque T_r . Thus,

$$T_r = \frac{m_0^2 e_0^2 c_1 \omega^5 \{2c_1 \omega \Delta \alpha \sin 2\xi + 2\Delta \alpha (\alpha - m_0 \omega^2) \cos 2\xi + (\alpha - m_0 \omega^2)^2 + \Delta \alpha^2 + c_1^2 \omega^2\}}{((\alpha - m_0 \omega^2)^2 - \Delta \alpha^2 + c_1^2 \omega^2)^2} \quad (28)$$

The dimensionless torque $T_r/(ae_0^2)$ and amplitude $|A|/e_0$ derived from equations (27) and (28) are indicated by solid lines in Figs. 4(a), (b) where ξ is $-\pi/4$, 0 , $\pi/4$ and $\pi/2$ for $\Delta \alpha/\alpha = 0.322$ and $c_1/\sqrt{m_0 \alpha} = 0.5$. For comparison, the calculated values for a symmetrical shaft are also indicated by a dotted line in Figs. 4(a), (b). The shaft end torque shows a qualitative tendency similar to the response curve. The shaft end torque and response curve have the maxima for the angular position $\xi = \pi/4$, and have the minima for $\xi = -\pi/4$.

4.2 Case of conical motion of rotor

Let us consider only a conical motion of a rotor mounted on the middle of an asymmetrical shaft ($z=0$) putting $\zeta=0$, and derive torque T_r applied to the shaft end from the equilibrium of moments shown in Figs. 5(a), (b). Take two parallel planes to the xy -plane having a short distance h as shown in Fig. 5(a). The tangent TT' at the origin O intersects those two planes at T and T' , and the principal axis of rotor inertia UU' intersects those at U and U' . The restoring moment M_i , the damping moment M_d , and the moment by inertia M_p are indicated by vectors in Fig. 5(a). Two vectors M_i and M_d drawn from O come on a plane perpendicular to the tangent OT , and the vector M_p drawn from O does on a plane perpendicular to the principal axis of inertia OU . When the 2nd order terms of small quantities θ and θ_1 are neglected, three vectors projected on the xy -plane [Fig. 5(b)] can be represented by the same symbols M_i , M_d and M_p as those in Fig. 5(a). Components of restoring moment M_i in x' - and y' -directions are expressed as

$$M'_{ix} = (\delta + \Delta\delta)\theta'_y, \quad M'_{iy} = -(\delta - \Delta\delta)\theta'_x \quad (29)$$

The component M_{iz} of M_i on TOz plane promotes the conical motion of the rotor. The restoring moment M_i can be equivalently replaced by two forces F and $-F$ which act on two points T and T' . In order to maintain the shaft end revolution at a constant velocity ω against the reaction of M_{iz} , the following torque T_{r1} must be applied⁽⁶⁾:

$$T_{r1} = M_{iz} = -M'_{ix}\theta'_x - M'_{iy}\theta'_y = -2\Delta\delta\theta'_x\theta'_y = -\Delta\delta \operatorname{Im}[\theta_z'^2] \quad (30)$$

Components of moment by inertia M_p in x - and y -directions are obtained from the underlined terms in equation (7) as follows:

$$\left. \begin{aligned} M_{px} &= I\ddot{\theta}_{1y} - I_p\omega\dot{\theta}_{1x} = I\ddot{\theta}_y - I_p\omega\dot{\theta}_x + (I_p - I)\tau\omega^2 \sin(\omega t + \eta) \\ M_{py} &= -(I\ddot{\theta}_{1x} + I_p\omega\dot{\theta}_{1y}) = -I\ddot{\theta}_x - I_p\omega\dot{\theta}_y - (I_p - I)\tau\omega^2 \cos(\omega t + \eta) \end{aligned} \right\} \quad (31)$$

The moment by inertia M_p can be equivalently replaced by a couple of inertia forces P and $-P$ which act on the two points U and U' . Equivalent inertia forces P and $-P$ produce a moment about TOz axis. The counter-torque T_{r2} to this moment is obtained as follows:

$$T_{r2} = M_{px}(\theta_{1x} - \theta_x) + M_{py}(\theta_{1y} - \theta_y) = I\tau \operatorname{Im}[\ddot{\theta}_z e^{-i(\omega t + \eta)}] - I_p\tau\omega \operatorname{Re}[\dot{\theta}_z e^{-i(\omega t + \eta)}] \quad (32)$$

Thus, equations (30) and (32) give the torque T_r required to turn the asymmetrical shaft at a constant velocity ω :

$$T_r = T_{r1} + T_{r2} = -\operatorname{Im}[\Delta\delta \cdot \theta_z'^2 - I\tau\ddot{\theta}_z e^{-i(\omega t + \eta)}] - I_p\tau\omega \operatorname{Re}[\dot{\theta}_z e^{-i(\omega t + \eta)}] \quad (33)$$

When a rotor makes a conical motion, the solutions of forced vibration of equation (10) are expressed by

$$\theta_x = B e^{i\omega t}, \quad \theta'_x = \dot{\theta}_x = i\omega B e^{i\omega t} = B \quad (34)$$

The shaft end torque T_r is obtained by substituting $\zeta = 0$ and the solution (34) into the double-underlined terms in equation (15);

(a) Shaft end torque

(b) Response curve

Fig. 4 Shaft end torque and response curve for parallel motion of a rotor mounted on an asymmetrical shaft

(a)

(b)

Fig. 5 Three moments M_t , M_d and M_p acting upon a rotor with a conical motion ($\zeta = 0$, $\zeta = 0$)

$$T_r = -\text{Im}[\Delta\delta B^2 - \tau\omega^2 B(I_p - I)e^{-i\eta}] \quad (35)$$

When the solution (34) is substituted into equation (33) which is derived directly from balance of moments, equation (33) agrees with equation (35). Using an argument of a complex number, equation (35) is rewritten as

$$T_r = -\Delta\delta|B|^2 \sin 2 \arg B + \tau\omega^2(I_p - I)|B| \sin(\arg B - \eta) \quad (36)$$

When the solution (34) is applied to a conical motion in which zero is put for z , $\bar{z}$, ΔI and τ in the second equation in equation (10), the real and imaginary parts are

$$\left. \begin{aligned} \{\delta - \Delta\delta + (I_p - I)\omega^2\}|B| \cos \arg B - c_2\omega|B| \sin \arg B &= -\tau\omega^2(I_p - I) \cos \eta \\ c_2\omega|B| \cos \arg B + \{\delta + \Delta\delta + (I_p - I)\omega^2\}|B| \sin \arg B &= -\tau\omega^2(I_p - I) \sin \eta \end{aligned} \right\} \quad (37)$$

Equation (37) is solved for $|B| \cos \arg B$ and $|B| \sin \arg B$. Thus:

$$\left. \begin{aligned} |B| \cos \arg B &= \frac{-\tau\omega^2(I_p - I)[\{\delta + \Delta\delta + (I_p - I)\omega^2\} \cos \eta + c_2\omega \sin \eta]}{\{\delta + (I_p - I)\omega^2\}^2 - \Delta\delta^2 + c_2^2\omega^2} \\ |B| \sin \arg B &= \frac{-\tau\omega^2(I_p - I)[-c_2\omega \cos \eta + \{\delta - \Delta\delta + (I_p - I)\omega^2\} \sin \eta]}{\{\delta + (I_p - I)\omega^2\}^2 - \Delta\delta^2 + c_2^2\omega^2} \end{aligned} \right\} \quad (38)$$

Substituting equation (38) into equation (36), the shaft end torque T_r is given as

$$T_r = \frac{(I_p - I)^2 \tau^2 c_2 \omega^5 [2\Delta\delta \{\delta + (I_p - I)\omega^2\} \cos 2\eta + 2c_2\omega \Delta\delta \sin 2\eta + \{\delta + (I_p - I)\omega^2\}^2 + \Delta\delta^2 + c_2^2\omega^2]}{[\{\delta + (I_p - I)\omega^2\}^2 - \Delta\delta^2 + c_2^2\omega^2]^2} \quad (39)$$

the value of which changes with the angle η .

On the conical motion of a rotor mounted on an asymmetrical shaft, dimensionless torque $m_0 T_r / (a I \tau^2)$ and amplitude $|B|/\tau$ are indicated by solid lines in Figs. 6(a), (b) where $\eta = -\pi/4, 0, \pi/4$ and $\pi/2$ for $I_p/I = 0.8$, $m_0 \delta / (a I) = 1.060$, $m_0 \Delta\delta / (a I) = 0.322$ and $c_2 \sqrt{m_0} / a I = 0.3$. For comparison, the calculated values for a symmetrical shaft ($\Delta\delta = 0$) are indicated by a dotted line in Fig. 6. The shaft end torque and the response curve in Fig. 6 show a similar tendency to Fig. 4 for the parallel motion.

5. Effect of Angular Position of Rotor Unbalances on Shaft End Torque for an Asymmetrical Rotor

In this chapter, let us consider a coupled motion of an asymmetrical rotor mounted on a symmetrical shaft. When the solutions of forced vibration (23) and (34) are substituted into the equations of motion (10) and shaft asymmetries $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ are set equal to zero, the real and imaginary parts are represented as follows:

$$\left. \begin{aligned} (\alpha - m_0\omega^2)|A| \cos \arg A - c_1\omega|A| \sin \arg A + \gamma|B| \cos \arg B &= m_0 e_0 \omega^2 \cos \xi \\ c_1\omega|A| \cos \arg A + (\alpha - m_0\omega^2)|A| \sin \arg A + \gamma|B| \sin \arg B &= m_0 e_0 \omega^2 \sin \xi \\ \gamma|A| \cos \arg A + \{\delta + (I_p - I)\omega^2\}|B| \cos \arg B - c_2\omega|B| \sin \arg B &= -(I_p - I - \Delta I)\tau\omega^2 \cos \eta \\ \gamma|A| \sin \arg A + c_2\omega|B| \cos \arg B + \{\delta + (I_p - I)\omega^2\}|B| \sin \arg B &= -(I_p - I + \Delta I)\tau\omega^2 \sin \eta \end{aligned} \right\} \quad (40)$$

Substituting solutions of forced vibration (23) and (34) into equation (15) and putting zero for $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$, a torque T_r applied to shaft end is derived as

(a) Shaft end torque

(b) Response curve

Fig. 6 Shaft end torque and response curve for conical motion of a rotor mounted on an asymmetrical shaft

$$T_r = -\text{Im}[\Delta I(-\omega B)^2 + (I_p/2)(-\omega^2 B \bar{B}) + m_0 e_0 \omega^2 A e^{-i\theta} - \tau \omega^2 B((I_p - I)e^{-i\eta} - \Delta I e^{i\eta})] \\ = -\Delta I \omega^2 |B|^2 \sin 2 \arg B - m_0 e_0 \omega^2 |A| \sin(\arg A - \xi) + \tau \omega^2 |B|((I_p - I) \sin(\arg B - \eta) - \Delta I \sin(\arg B + \eta)) \quad (41)$$

When $|A| \cos \arg A$, $|A| \sin \arg A$, $|B| \cos \arg B$ and $|B| \sin \arg B$ obtained by solving equation (40) are substituted into equation (41), the shaft end torque can be calculated. Only amplitude A regarding the parallel motion of the rotor is shown hereafter, because the rotor's parallel motion is much larger than its conical one at the first major critical speed.

5.1 Influence of angular position ξ upon torque

Let us examine the influence of the angular position ξ , when only the static unbalance e_0 exists and no dynamic one. Putting zero for τ in equation (40), the real and imaginary parts of amplitudes A and B are expressed as follows⁽²⁾:

$$\left. \begin{aligned} |A| \cos \arg A &= (m_0 e_0 \omega^2 / K_0) [(\alpha - m_0 \omega^2)(X_1 X_2 + c_2^2 \omega^2) - \gamma^2 X_1] \cos \xi + \{c_1(X_1 X_2 + c_2^2 \omega^2) + \gamma^2 c_2\} \omega \sin \xi \\ |A| \sin \arg A &= (m_0 e_0 \omega^2 / K_0) [-\{c_1(X_1 X_2 + c_2^2 \omega^2) + \gamma^2 c_2\} \omega \cos \xi + \{(\alpha - m_0 \omega^2)(X_1 X_2 + c_2^2 \omega^2) - \gamma^2 X_2\} \sin \xi] \\ |B| \cos \arg B &= (m_0 e_0 \gamma \omega^2 / K_0) [-(\alpha - m_0 \omega^2) X_2 - c_1 c_2 \omega^2 - \gamma^2] \cos \xi - \{c_2(\alpha - m_0 \omega^2) + c_1 X_2\} \omega \sin \xi \\ |B| \sin \arg B &= (m_0 e_0 \gamma \omega^2 / K_0) [\{c_2(\alpha - m_0 \omega^2) - c_1 X_1\} \omega \cos \xi - \{(\alpha - m_0 \omega^2) X_1 - c_1 c_2 \omega^2 - \gamma^2\} \sin \xi] \end{aligned} \right\} \quad (42)$$

where

$$\left. \begin{aligned} K_0 &= \{(\alpha - m_0 \omega^2) X_1 - \gamma^2\} \{(\alpha - m_0 \omega^2) X_2 - \gamma^2\} + c_1^2 \omega^2 X_1 X_2 + c_2^2 \omega^2 (\alpha - m_0 \omega^2)^2 + 2c_1 c_2 \gamma^2 \omega^2 + c_1^2 c_2^2 \omega^4 \\ X_1 &= \delta + (I_p - I - \Delta I) \omega^2, \quad X_2 = \delta + (I_p - I + \Delta I) \omega^2 \end{aligned} \right\} \quad (43)$$

The shaft end torque can be calculated numerically by substituting equation (42) into equation (41) and putting zero for τ .

For $I_p/I = 1.987$, $\Delta I/I = 0.322$, $\gamma \sqrt{m_0/I} / \alpha = -0.855$, $m_0 \delta / (\alpha I) = 1.060$, $c_1 / \sqrt{m_0 \alpha} = 0.1$ and $c_2 \sqrt{m_0} / \alpha / I = 0.1$, dimensionless torque $T_r / (ae_0^2)$ and amplitude $|A|/e_0$ are indicated by solid lines with the parameter ξ in Figs. 7(a), (b). The calculated results for a symmetrical rotor ($\Delta I = 0$) are indicated by dotted lines. The shaft end torque and the amplitude with respect to an asymmetrical rotor show a tendency similar to that of an asymmetrical shaft. When $\xi = \pi/4$, the torque and the amplitude have maxima. When $\xi = -\pi/4$, on the other hand, the torque and the amplitude have minima, and they are smaller than those of the dotted lines.

5.2 Influence of angular position η upon torque

The real and imaginary parts of amplitudes A and B are obtained⁽²⁾ from equation (40) when there is no static unbalance but only a dynamic unbalance τ :

$$\left. \begin{aligned} |A| \cos \arg A &= (\gamma \tau \omega^2 / K_0) [(I_p - I - \Delta I) \{(\alpha - m_0 \omega^2) X_2 - c_1 c_2 \omega^2 - \gamma^2\} \cos \eta \\ &\quad + (I_p - I + \Delta I) \{c_1 X_1 + c_2 (\alpha - m_0 \omega^2)\} \omega \sin \eta] \\ |A| \sin \arg A &= (\gamma \tau \omega^2 / K_0) [- (I_p - I - \Delta I) \{c_1 X_2 + c_2 (\alpha - m_0 \omega^2)\} \omega \cos \eta \\ &\quad + (I_p - I + \Delta I) \{(\alpha - m_0 \omega^2) X_1 - c_1 c_2 \omega^2 - \gamma^2\} \sin \eta] \\ |B| \cos \arg B &= (\tau \omega^2 / K_0) [- (I_p - I - \Delta I) \{(\alpha - m_0 \omega^2)^2 X_2 - \gamma^2 (\alpha - m_0 \omega^2) + c_1^2 \omega^2 X_2\} \cos \eta \\ &\quad - (I_p - I + \Delta I) \{c_2 (\alpha - m_0 \omega^2)^2 + c_1^2 c_2 \omega^2 + c_1 \gamma^2\} \omega \sin \eta] \\ |B| \sin \arg B &= (\tau \omega^2 / K_0) [(I_p - I - \Delta I) \{c_1 (\alpha - m_0 \omega^2)^2 + c_1^2 c_2 \omega^2 + c_1 \gamma^2\} \omega \cos \eta \\ &\quad - (I_p - I + \Delta I) \{(\alpha - m_0 \omega^2)^2 X_1 - \gamma^2 (\alpha - m_0 \omega^2) + c_1^2 \omega^2 X_1\} \sin \eta] \end{aligned} \right\} \quad (44)$$

Torque applied to the shaft end can be calculated numerically by substituting equation (44) into equation (41) and putting $e_0 = 0$.

Figures 8(a), (b) show the shaft end torque and the response curve which are obtained from numerical calculation for the same parameter as for Fig. 7 except $e_0 = 0$ and $\tau \neq 0$. Figures 8(a), (b) show a tendency similar to those of Figs. 7(a), (b).

6. Conclusions

The obtained conclusions may be summarized as follows:

(1) In an asymmetrical shaft carrying an asymmetrical rotor, the increase rate of total energy of the shaft system and the torque applied to the shaft end are given by equations (14) and (15). These equations

explicitly include the angular positions ξ , η in which the static unbalance e_0 and the dynamic one τ exist.

(2) In the case of parallel motion of a rotor mounted on the middle of an asymmetrical shaft, the shaft end torque can be obtained from the equilibrium of forces. Near the major critical speed, the shaft end torque and the response curve have maxima when $\xi = \pi/4$, and minima when $\xi = -\pi/4$.

(3) In the case of conical motion of a rotor mounted on the middle of an asymmetrical shaft, the shaft end torque can be obtained from the equilibrium of moments. The same as with parallel motion, the shaft end torque shows a tendency similar to the response curves near the major critical speed. The shaft end torque and the response curve have maxima when $\eta = \pi/4$, and minima

(a) Shaft end torque

(b) Response curve

Fig. 7 Shaft end torque and response curve for an asymmetrical rotor ($\tau=0$)

when $\eta = -\pi/4$.

(4) When parallel motion of an asymmetrical rotor couples with conical motion, the shaft end torque and the response curve are obtained. In the case $\tau=0$, the shaft end torque and the response curve with respect to the parallel motion have maxima near the major critical speed when $\xi = -\pi/4$, and minima when $\xi = \pi/4$.

In the case $e_0=0$, the shaft end torque and the response curve with respect to the parallel motion show a tendency similar to that for the angular position η .

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(a) Shaft end torque

(b) Response curve

Fig. 8 Shaft end torque and response curve for an asymmetrical rotor ($e_0=0$)

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