Flow of Liquid Metals through Curved Circular Channels

with a Transverse Magnetic Field*

(1st Report, Flow in Electrically Insulated Channels)

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A steady flow of liquid metals in curved circular channels under a transversely applied magnetic field is investigated theoretically and experimentally. Firstly, a laminar flow at low Dean number region is analyzed numerically. And the internal mechanisms of the flow, which are velocity profiles, distributions of current densities and others, are made clear, and the resistance coefficient is found. Secondly, on the basis of numerical results, a laminar flow at high Dean number region is analyzed by means of the boundary layer approximate method, and the resistance coefficient is calculated. Finally, an experimental study is carried out for mercury flow. The resistance coefficient obtained by the experiments is shown to be in good agreement with that of the theoretical analysis. Furthermore, the effects of magnetic field on the resistance character in the transition and turbulent flow regions are examined.

Key Words: Fluid Mechanics, Magnetohydrodynamics, Curved Channel, Dean number, Hartmann number, Liquid Metal

1. Introduction

With the development of the atomic and metallurgical industries, the importance of MHD devices such as large sized MHD pumps, electromagnetic flow-meters and others has been recognized. And much interest has been aroused in the phenomena of the channel flows of liquid metals under magnetic fields and much research has been done actively. For example, important and interesting results are produced from the theoretical investigations of Hartmann⁽¹⁾, Shercliff⁽²⁾, and Lykoudis-Brouillette⁽³⁾ and the experimental ones of Hartmann-Lazarus⁽⁴⁾, Murgatroyd⁽⁵⁾ and Branover-Lielausis⁽⁶⁾. But all of them concern flows through straight channels and the investigations on ones through curved channels are very rare.

The investigations on the MHD flows of curved channels can hardly be found except for those of Boedeker et al.⁽⁷⁾, Gel'fgat et al.⁽⁸⁾, Baylis et al.⁽⁹⁾, Sahai⁽¹⁰⁾, Singh⁽¹¹⁾, and the authors.⁽¹²⁾⁻⁽¹⁵⁾

Boedeker et al. measured the flow rate and the pressure gradient for the Hartmann numbers 0 and 3.4 by flowing an electrolytic solution (HCL) in a spiral channel of square cross section under a transverse magnetic field (a uniform field normal to the surface involving the curvature radius of the channel and the channel axis). Consequently they suggested a suppressing effect of the field on the secondary flow, though their experimental results were insufficient. Gel'fgat et al. conducted the numerical analysis of an annular channel with a rectangular cross section for the Hartmann number 6 and they showed that the secondary flow in the channel was reduced by applying the field in the radius direction. Baylis et al. carried out an approximate analysis of the flow in an annular channel for large Hartmann number, which is driven by the electromagnetic force, and compared the analytical and experimental results of the resistance coefficient. Sahai analyzed the flows in curved circular channels with a transversely applied magnetic field by means of Galerkin's Method and Singh did a similar problem using the perturbation method. However, these analyses concern the flows of small Hartmann numbers or large Hartmann numbers and can not explain the effect of the magnetic field sufficiently. The authors obtained the resistance coefficient, which is important for engineering, experimentally and theoretically by flowing mercury into the curved channels of rectangular cross section under a transverse magnetic field and by conducting an analysis backed by the boundary layer theory. From the conformity of the results, they suggested that the magnetic field exhibits an effect

* Received 27th January, 1982.

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of suppressing the secondary flow. After that, the authors analyzed numerically the flows through the curved channels of square cross section in three cases of the fields being applied in a transverse, radial, and axial directions and made clear the internal mechanisms of the flow, which are the profiles of the velocity and the distributions of the magnetic fields and electric current densities. And consequently, they confirmed the suggestion as mentioned in their previous reports that the magnetic field suppresses the secondary flow.

The purpose of the present study is to make clear theoretically and experimentally the flow characteristics of the MHD flow in curved circular channels which have not been elucidated. Generally, the flow in curved channels is constituted of an axial flow and a secondary flow induced by the centrifugal force. As the influence of the centrifugal force is small on the flow with low velocity, the secondary flow is weak and consequently the flow is regarded as a similar flow to the one in straight channels, having only the velocity component of axial direction. But the influence of the centrifugal force becomes greater with an increase of the velocity and it forms a flow pattern peculiar to curved channels, which has a spiral structure constructed of a secondary flows induced symmetrically in the upside and the downside cross section of a channel respectively and circulating in each cross section of the channel. Therefore, the loss becomes greater than that in straight channels. It is guessed that the application of a magnetic field upon such a flow will suppress the secondary flow due to the Lorentz force and the flow similar to the one in straight channels will be formed even at high Reynolds numbers (Dean numbers). And further it is guessed that in turbulent flows, the effect of suppressing the turbulence by the field will be added to the Hartmann effect and the suppression effect of the secondary flow, and a very complicated flow, different from the turbulent flow in straight channels, will be formed. So, the numerical and approximate analyses were adopted in the present study to elucidate the internal mechanisms of the flow and obtain the resistance coefficient. Further experiment was conducted and the results of the analyses and the experiment were compared. On these results the explanations were given for the influence of the magnetic field on the flow characteristics.

2. Nomenclature (see Fig.1)

- a : radius of a pipe
 B_0 : non-dimensional applied magnetic flux density $= b_0 a / \sqrt{\rho \mu}$
 D : diameter of a channel $= 2R$
 f, g, h : magnetic flux densities
 F, G, H : non-dimensional magnetic flux densities $= fa / \sqrt{\rho \mu}, = ga / \sqrt{\rho \mu}, = ha / \sqrt{\rho \mu}$
 f_c : resistance coefficient of a curved channel $= -\frac{1}{R} \frac{\partial p}{\partial \theta} \frac{2a}{(1/2)\rho w^2}$

- f_s : resistance coefficient of a straight channel $= -\frac{dp}{dZ} \frac{2a}{(1/2)\rho w^2}$
 J : non-dimensional electric current density $= ja^2 / \sqrt{\rho \mu}$
 K : Dean number $= Re \sqrt{2a/D}$
 M : Hartmann number $= 2ab_0 \sqrt{\sigma / \rho \nu}$
 P : non-dimensional pressure $= a^2 p / \rho \nu^2$
 P_m : magnetic Prandtl number $= \sigma \mu \nu$
 r, θ, ϕ : co-ordinates
 R : channel radius of curvature
 $R_c = D/2a = 1/\beta$
 Re : Reynolds number $= 2aw/\nu$
 T : non-dimensional time $= t\nu/a^2$
 u, v, w : velocity components
 U, V, W : non-dimensional velocity components $= au/\nu, = av/\nu, = aw/\nu$
 β : radius ratio $= a/R = 1/R_c$
 δ : non-dimensional thickness of boundary layer $= \Delta/a$
 Δ : thickness of boundary layer
 η, ξ : non-dimensional co-ordinate $= r/a, = 1 - r/a$
 $\lambda = -\beta \frac{\partial p}{\partial \theta}$
 $\Lambda = R_c + \eta \sin \phi$
 μ : permeability
 ν : kinematic viscosity
 ρ : density
 σ : electric conductivity
 χ : non-dimensional stream function of magnetic field
 Ψ : non-dimensional stream function of flow field
 Ω : non-dimensional vorticity $= (1/\eta)(\partial U/\partial \phi) - (\partial V/\partial \eta + V/\eta)$
 Subscripts
 m : mean value
 r, ϕ, θ : values in r -, ϕ -, θ -directions
 tr : value of the transition point
 o : value of the flow without a magnetic field

3. Analysis of Laminar Flow

In the first place, a numerical analysis is conducted of a fully developed, steady laminar flow for low Dean numbers and then an approximate analysis, based on the boundary layer theory, is carried out of a flow with higher Dean numbers to which numerical analyses are not appropriate.

3.1 Numerical analysis

3.1.1 Basic equations and boundary conditions The basic equations of flows of incompressible, conductive fluids under a magnetic field are written in the vector form as follows:

Fig.1 Coordinate system

Equation of continuity:

$$\nabla \cdot \mathbf{V} = 0 \quad (1)$$

Equation of magnetic field:

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

Momentum equation:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) \quad (3)$$

Equation of induction:

$$\Omega = \left[\frac{\sin \varphi}{\Lambda^2} - \frac{1}{\eta \Lambda} \right] \frac{\partial \Psi}{\partial \eta} - \frac{1}{\Lambda} \frac{\partial^2 \Psi}{\partial \eta^2} + \frac{\cos \varphi}{\eta \Lambda^2} \frac{\partial \Psi}{\partial \varphi} - \frac{1}{\eta^2 \Lambda} \frac{\partial^2 \Psi}{\partial \varphi^2} \quad (5)$$

$$J_z = \left[\frac{\sin \varphi}{\Lambda^2} - \frac{1}{\eta \Lambda} \right] \frac{\partial \chi}{\partial \eta} - \frac{1}{\Lambda} \frac{\partial^2 \chi}{\partial \eta^2} + \frac{\cos \varphi}{\eta \Lambda^2} \frac{\partial \chi}{\partial \varphi} - \frac{1}{\eta^2 \Lambda} \frac{\partial^2 \chi}{\partial \varphi^2} \quad (6)$$

$$\begin{aligned} \frac{\partial W}{\partial t} + \frac{1}{\Lambda^2} \left[1 + \cos \varphi \frac{\partial \Psi}{\partial \eta} - \frac{\sin \varphi}{\eta} \frac{\partial \Psi}{\partial \varphi} \right] W + \frac{1}{\eta \Lambda} \left[\frac{\partial \Psi}{\partial \eta} - \cos \varphi \right] \frac{\partial W}{\partial \varphi} \\ - \left[\frac{1}{\eta} + \frac{\sin \varphi}{\Lambda} + \frac{1}{\eta \Lambda} \frac{\partial \Psi}{\partial \varphi} \right] \frac{\partial W}{\partial \eta} - \frac{\partial^2 W}{\partial \eta^2} - \frac{1}{\eta^2} \frac{\partial^2 W}{\partial \varphi^2} \\ = -\frac{1}{\Lambda} \frac{\partial P}{\partial \theta} - \left(B_0 \sin \varphi - \frac{1}{\Lambda} \frac{\partial \chi}{\partial \eta} \right) \left(\frac{1}{\eta} \frac{\partial H}{\partial \varphi} + \frac{\cos \varphi}{\Lambda} H \right) + \left(B_0 \cos \varphi - \frac{1}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi} \right) \left(\frac{\partial H}{\partial \eta} + \frac{\sin \varphi}{\Lambda} H \right) \end{aligned} \quad (7)$$

$$\begin{aligned} P_m \frac{\partial H}{\partial t} + \frac{1}{\Lambda^2} \left[1 + P_m \left(\frac{\sin \varphi}{\eta} \frac{\partial \Psi}{\partial \varphi} - \cos \varphi \frac{\partial \Psi}{\partial \eta} \right) \right] H + \frac{1}{\eta \Lambda} \left[P_m \frac{\partial \Psi}{\partial \eta} - \cos \varphi \right] \frac{\partial H}{\partial \varphi} \\ - \left[\frac{1}{\eta} + \frac{1}{\Lambda} \sin \varphi + \frac{P_m}{\eta \Lambda} \frac{\partial \Psi}{\partial \varphi} \right] \frac{\partial H}{\partial \eta} - \frac{\partial^2 H}{\partial \eta^2} - \frac{1}{\eta^2} \frac{\partial^2 H}{\partial \varphi^2} \\ = -P_m \left[\left(-B_0 \cos \varphi + \frac{1}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi} \right) \frac{\partial W}{\partial \eta} + \left(B_0 \sin \varphi - \frac{1}{\Lambda} \frac{\partial \chi}{\partial \eta} \right) \frac{1}{\eta} \frac{\partial W}{\partial \varphi} + \frac{W}{\Lambda^2} \left(\cos \varphi \frac{\partial \chi}{\partial \eta} - \frac{\sin \varphi}{\eta} \frac{\partial \chi}{\partial \varphi} \right) \right] \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial Q}{\partial t} + \frac{1}{\Lambda^2} \left[1 - \cos \varphi \frac{\partial \Psi}{\partial \eta} + \frac{\sin \varphi}{\eta} \frac{\partial \Psi}{\partial \varphi} \right] Q + \frac{1}{\eta \Lambda} \left[\frac{\partial \Psi}{\partial \eta} - \cos \varphi \right] \frac{\partial Q}{\partial \varphi} \\ - \left[\frac{1}{\eta} + \frac{\sin \varphi}{\Lambda} + \frac{1}{\eta \Lambda} \frac{\partial \Psi}{\partial \varphi} \right] \frac{\partial Q}{\partial \eta} - \frac{\partial^2 Q}{\partial \eta^2} - \frac{1}{\eta^2} \frac{\partial^2 Q}{\partial \varphi^2} \\ = -\frac{2W}{\Lambda} \left[\cos \varphi \frac{\partial W}{\partial \eta} - \frac{\sin \varphi}{\eta} \frac{\partial W}{\partial \varphi} \right] + \frac{2H}{\Lambda} \left(\cos \varphi \frac{\partial H}{\partial \eta} - \frac{\sin \varphi}{\eta} \frac{\partial H}{\partial \varphi} \right) \\ - \left(B_0 \sin \varphi - \frac{1}{\Lambda} \frac{\partial \chi}{\partial \eta} \right) \frac{1}{\eta} \frac{\partial J_z}{\partial \varphi} + \left(B_0 \cos \varphi - \frac{1}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi} \right) \frac{\partial J_z}{\partial \eta} + \frac{J_z}{\Lambda^2} \left(\frac{\sin \varphi}{\eta} \frac{\partial \chi}{\partial \varphi} - \cos \varphi \frac{\partial \chi}{\partial \eta} \right) \end{aligned} \quad (9)$$

$$\begin{aligned} P_m \left[\frac{\partial J_z}{\partial t} + \frac{2}{\eta} \left(\frac{1}{\Lambda} \frac{\partial^2 \Psi}{\partial \eta^2} - \frac{\sin \varphi}{\Lambda^2} \frac{\partial \Psi}{\partial \eta} \right) C + \frac{2}{\eta} \left(\frac{1}{\Lambda} \frac{\partial^2 \chi}{\partial \eta^2} - \frac{\sin \varphi}{\Lambda^2} \frac{\partial \chi}{\partial \eta} \right) D \right. \\ \left. + \frac{\partial C}{\partial \eta} - \frac{\partial Q}{\partial \varphi} \left(B_0 \cos \varphi - \frac{1}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi} \right) + \frac{1}{\eta} \frac{\partial Q}{\partial \varphi} \left(B_0 \sin \varphi - \frac{1}{\Lambda} \frac{\partial \chi}{\partial \eta} \right) + \frac{J_z D}{\eta} + \frac{1}{\eta \Lambda} \left(\frac{\partial \Psi}{\partial \eta} \frac{\partial J_z}{\partial \varphi} - \frac{\partial \Psi}{\partial \varphi} \frac{\partial J_z}{\partial \eta} \right) \right] \\ = \frac{\partial^2 J_z}{\partial \eta^2} + \left(\frac{1}{\eta} + \frac{\sin \varphi}{\Lambda} \right) \frac{\partial J_z}{\partial \eta} - \frac{J_z}{\Lambda^2} + \frac{\cos \varphi}{\eta \Lambda} \frac{\partial J_z}{\partial \varphi} + \frac{1}{\eta^2} \frac{\partial^2 J_z}{\partial \varphi^2} \end{aligned} \quad (10)$$

where

$$\begin{aligned} C &= -\frac{2}{\Lambda} \frac{\partial^2 \chi}{\partial \eta \partial \varphi} + \frac{2}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi} + \frac{\eta \cos \varphi}{\Lambda^2} \frac{\partial \chi}{\partial \eta} + \frac{\sin \varphi}{\Lambda^2} \frac{\partial \chi}{\partial \varphi} \\ D &= \frac{2}{\Lambda} \frac{\partial^2 \Psi}{\partial \eta \partial \varphi} - \frac{2}{\eta \Lambda} \frac{\partial \Psi}{\partial \varphi} - \frac{\eta \cos \varphi}{\Lambda^2} \frac{\partial \Psi}{\partial \eta} - \frac{\sin \varphi}{\Lambda^2} \frac{\partial \Psi}{\partial \varphi} \\ \Lambda &= R_c + \eta \sin \varphi \end{aligned}$$

In the above equations, Ψ and χ are non-dimensional expressions of the stream function of the velocity and magnetic flux density and defined as

$$\begin{aligned} U &= -\frac{1}{\eta \Lambda} \frac{\partial \Psi}{\partial \varphi}, \quad V = \frac{1}{\Lambda} \frac{\partial \Psi}{\partial \eta} \\ F &= -\frac{1}{\eta \Lambda} \frac{\partial \chi}{\partial \varphi}, \quad G = \frac{1}{\Lambda} \frac{\partial \chi}{\partial \eta} \end{aligned}$$

These equations are converted into finite-difference expressions at the upper, half cross section of a channel and solved numerically using the boundary conditions shown in Tab.1. 231 lattice points (11×21) are taken as shown in Fig.2 and a run is terminated when the ratio of the iterative residual to the preceding value of the resistance coefficient becomes less than 10^{-5} .

3.1.2 Analytical results and considerations In Figs.3 and 4, the contour maps of velocity in axial direction and stream function of the secondary flow (stream lines) are shown for the flows without a magnetic field. In the case of very small Dean numbers, the distribution

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\sigma \mu} \nabla^2 \mathbf{B} \quad (4)$$

where $\mathbf{V}$, $\mathbf{B}$ and $\mathbf{J}$ are vectors denoting the velocity, magnetic flux density and electric current density, respectively. Choosing the co-ordinate system as shown in Fig.1, the equations concerning the non-dimensional vorticity Ω , the current density in axial direction J_z , the velocity in axial direction W and the magnetic flux density in axial direction H can be written from Eqs.(1) ~ (4) as follows:

of the axial velocities in curved channels is almost the same as that in straight channels and the secondary flow is weak. But the intensity of the secondary flow increases with the Dean number and its influence on the axial velocity distribution appears (see Fig.3). Namely, the fluid in the central portion of a channel is pushed toward the outside wall by the secondary flow and the greatest axial velocity and the center of the secondary flow are near the outside wall. When the Dean number increases further (see Fig.4), the greatest axial velocity comes closer to the outside wall while the center of the secondary flow moves toward the inside wall.

Figs.5 and 6 show the results of the flows on which a transverse magnetic field is applied ($M = 20$). At first, Fig.5 for $K = 53$ shows that the secondary flow is suppressed remarkably by the magnetic field, its center comes close to $\phi = 0$ (see Fig.5(b)), and the axial velocity distribution becomes similar to that of a

straight channel(see Fig.5(a) and Fig.7(a)). Fig.5(c) shows the distribution of the electric current induced in the cross section of the channel(an equi-magnetic field line denotes a current line from $\nabla \times B = \mu J$), indicating that two sets of circulating current are formed in the cross section of a curved channel, like those induced in the cross section of a straight channel(see Fig.7(b)). Fig.5(d) shows the distribution of the electric current induced by the secondary flow, of which direction corresponds to the velocity distribution of the secondary flow. To be more specific, the current flows in the positive direction of channel axis at the central region of a channel where the secondary flow flows toward the outside wall; and it flows in the reverse direction near the wall where the secondary flow moves toward the inside wall. For $K = 156$ (see Fig.6), the secondary flow near the central portion of a vortex becomes weaker than the flow without a magnetic field and the distribution of it varies. But a variation of the axial velocity distribution can hardly be recognized. This is because the effect of the centrifugal force increases and consequently that of the magnetic field becomes weak relatively. However, it is predictable that an increase of the magnetic field gives a result similar to that for $K = 53$. Paying attention to the distribution of the axial velocity, it is seen that the velocity gradient in the central region of the channel is gentle but it is steep near the wall. This suggests that the influence of the viscosity persists in the thin layer near the wall and the analysis based on the boundary layer theory is effective. The electric current induced in the cross section and in the axial direction show a flow pattern similar to those for $K = 53$. In any case, the application of a magnetic field in-

duces electric currents in the cross section and in the axial direction of channels and suppresses the secondary flow due to the centrifugal force, which makes the

$$K = 53 \quad M = 0 \quad R_c = 20$$

(a) Velocity in axial direction

$$K = 53 \quad M = 0 \quad R_c = 20$$

(b) Stream line of secondary flow

Fig.3 Velocity distribution without a magnetic field($K = 53$, $R_c = 20$)

Table 1 Boundary conditions

variable	wall	$\varphi = \pm \frac{\pi}{2}$	center
Ω	$-\frac{1}{A} \frac{\partial^2 \psi}{\partial \eta^2}$	0	0
J_z	$-\frac{1}{A} \frac{\partial^2 x}{\partial \eta^2}$	$\frac{\partial f_z}{\partial \varphi} = 0$	$\frac{\partial f_z}{\partial \eta} \Big _{\dots} = 0$
ψ	0	0	0
x	0	$\frac{\partial x}{\partial \varphi} = 0$	$\frac{\partial x}{\partial \eta} \Big _{\dots} = 0$
W	0	$\frac{\partial W}{\partial \varphi} = 0$	$\frac{\partial W}{\partial \eta} \Big _{\dots} = 0$
H	0	0	0

Fig.2 Lattice in numerical analysis

$$K = 156 \quad M = 0 \quad R_c = 20$$

(a) Velocity in axial direction

$$K = 156 \quad M = 0 \quad R_c = 20$$

(b) Stream line of secondary flow

Fig.4 Velocity distribution without a magnetic field($K = 156$, $R_c = 20$)

flow pattern in curved channels similar to the one in straight channels.

The results of the resistance coefficient are shown together with the experimental results in chapter 4.

3.2 Approximate analysis

The flow field is divided into two parts, i.e. the core region where the vis-

cous effect is weak; and the boundary layer where the viscous effect can not be disregarded, and the boundary layer theory is applied. The profiles of the velocity, magnetic field and electric current density, which are necessary for the analysis are assumed in the form of a polynomial expression by referring to the results of

(a) Velocity in axial direction

(b) Stream line of secondary flow

(c) Stream line of electric current induced in cross section

(d) Electric current induced in axial direction

Fig.5 Distributions of velocities and electric current densities($K=53, R_c=20, M=20$)

(a) Velocity in axial direction

(b) Stream line of secondary flow

(c) Stream line of electric current induced in cross section

(d) Electric current induced in axial direction

Fig.6 Distributions of velocities and electric current densities($K=156, R_c=20, M=20$)

(a) Velocity in axial direction

(b) Electric current induced in cross section

Fig.7 Distributions of velocities and electric current densities in a straight channel ($R_e = 698, M = 20$)

the preceding paragraph and further the following assumptions are made:

(1) Near the wall of a channel, the stream lines of the secondary flow run through the boundary layer.

(2) Similarly, near the wall of a channel, the stream lines of the electric current run through the boundary layer.

(3) $a \ll R$

3.2.1 Velocity, magnetic field and electric current in core region The following equations are obtained by neglecting the terms concerning the viscosity and rewriting the basic equations (1)~(4) in the non-dimensional form:

$$\frac{\partial U}{\partial \eta} + \frac{U}{\eta} + \frac{1}{\eta} \frac{\partial V}{\partial \varphi} = 0 \quad (11)$$

$$\frac{\partial F}{\partial \eta} + \frac{F}{\eta} + \frac{1}{\eta} \frac{\partial G}{\partial \varphi} = 0 \quad (12)$$

$$-\beta W^2 \sin \varphi = -\frac{\partial P}{\partial \eta} + \frac{B_0}{\eta} \sin \varphi \left(\frac{\partial}{\partial \eta} \eta G - \frac{\partial F}{\partial \varphi} \right) \quad (13)$$

$$-\beta W^2 \cos \varphi = -\frac{1}{\eta} \frac{\partial P}{\partial \varphi} + \frac{B_0}{\eta} \cos \varphi \left(\frac{\partial}{\partial \eta} \eta G - \frac{\partial F}{\partial \varphi} \right) \quad (14)$$

$$U \frac{\partial W}{\partial \eta} + \frac{V}{\eta} \frac{\partial W}{\partial \varphi} = \lambda + B_0 \cos \varphi \frac{\partial H}{\partial \eta} - \frac{B_0}{\eta} \sin \varphi \frac{\partial H}{\partial \varphi} \quad (15)$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial G}{\partial \eta} + \frac{G}{\eta} - \frac{1}{\eta} \frac{\partial F}{\partial \varphi} \right) = -P_\infty \frac{\partial}{\partial \eta} (UB_0 \sin \varphi + VB_0 \cos \varphi) \quad (16)$$

$$\left(\frac{\partial}{\partial \eta} + \frac{1}{\eta} \right) \frac{\partial H}{\partial \eta} + \frac{1}{\eta^2} \frac{\partial^2 H}{\partial \varphi^2} = -P_\infty B_0 \left(\cos \varphi \frac{\partial W}{\partial \eta} - \frac{\sin \varphi}{\eta} \frac{\partial W}{\partial \varphi} \right) \quad (17)$$

The electric current density is expressed using Maxwell's equation $\nabla \times \mathbf{B} = \mu \mathbf{J}$, as follows:

$$J_r = -\frac{1}{\eta} \frac{\partial H}{\partial \varphi}, J_\varphi = \frac{\partial H}{\partial \eta}, J_z = \frac{1}{\eta} \frac{\partial F}{\partial \varphi} - \frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta G) \quad (18)$$

Here the profiles of the velocity and electric current in the radius and circumferential directions are assumed as

$$U = C_1 \sin \varphi, V = C_1 \cos \varphi \quad (19)$$

$$J_r = -C_2 \sin \varphi, J_\varphi = -C_2 \cos \varphi, J_z = C_3 \quad (20)$$

$C_i (i = 1 \sim 3)$ in the above equations is a constant to be determined later. Substituting Eqs.(19) and (20) into Eqs.(11)~(18) yields

$$W = \frac{(\lambda - C_2 B_0)}{C_1} \eta \sin \varphi + C_4 \quad (21)$$

$$P = \frac{\beta C_1}{3(\lambda - C_2 B_0)} \left\{ \frac{(\lambda - C_2 B_0)}{C_1} \eta \sin \varphi + C_4 \right\}^3 - C_2 B_0 \eta \sin \varphi - \lambda \theta + \text{const.} \quad (22)$$

$$F = 0, G = -\frac{1}{2} C_3 \eta, H = -C_2 \eta \cos \varphi \quad (23)$$

$$C_4 = P_\infty C_1 B_0 \quad (24)$$

where C_4 is a constant to be determined later.

3.2.2 Velocity, magnetic field and electric current in boundary layer Evaluating the order of magnitude of physical variables in the boundary layer yields

$$U \sim O(\delta), V \sim O(1), W \sim O(1), F \sim O(\delta), G \sim O(1), H \sim O(1), \frac{\partial}{\partial \eta} \sim O(\delta^{-1}), \frac{\partial}{\partial \varphi} \sim O(1) \quad (25)$$

The basic equations become, by omitting the small terms in conformity with the above evaluation, as follows:

$$-\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \varphi} = 0 \quad (26)$$

$$-\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \varphi} = 0 \quad (27)$$

$$-V^2 - \beta W^2 \sin \varphi = \frac{\partial P}{\partial \xi} - B_0 \sin \varphi \frac{\partial G}{\partial \xi} + \frac{\partial^2 V}{\partial \xi \partial \varphi} \quad (28)$$

$$-U \frac{\partial V}{\partial \xi} + V \frac{\partial V}{\partial \varphi} - \beta W^2 \cos \varphi = -\frac{\partial P}{\partial \varphi} - B_0 \cos \varphi \frac{\partial G}{\partial \xi} - \frac{\partial^2 V}{\partial \xi^2} \quad (29)$$

$$-U \frac{\partial W}{\partial \xi} + V \frac{\partial W}{\partial \varphi} = \lambda - B_0 \cos \varphi \frac{\partial H}{\partial \xi} + \frac{\partial^2 W}{\partial \xi^2} \quad (30)$$

$$\frac{\partial^2 G}{\partial \xi^2} = P_m B_0 \cos \varphi \frac{\partial V}{\partial \xi} \quad (31)$$

$$\frac{\partial^2 H}{\partial \xi^2} = P_m B_0 \cos \varphi \frac{\partial W}{\partial \xi} \quad (32)$$

The electric current density is written as

$$J_r = -\frac{\partial H}{\partial \varphi}, J_\varphi = -\frac{\partial H}{\partial \xi}, J_z = \frac{\partial G}{\partial \xi} \quad (33)$$

The flow rate of the secondary flow in the boundary layer must be equal to that of the fluid flowing into the boundary layer from the core region (see Fig.1) and therefore the following relation holds:

$$\int_0^\delta V d\xi = -C_1(1-\delta) \cos \varphi \quad (34)$$

V and W must satisfy the following boundary conditions,

$$\xi=0: V=W=0$$

$$\xi=\delta: V=C_1 \cos \varphi$$

$$\frac{\partial V}{\partial \xi}=0$$

$$W = \frac{(\lambda - C_2 B_0)}{C_1} (1-\delta) \sin \varphi + C_4$$

$$\frac{\partial W}{\partial \xi} = -\frac{(\lambda - C_2 B_0)}{C_1} \sin \varphi \quad (35)$$

Here V and W are written in the following polynomial expressions which satisfy the above conditions:

$$V = C_1 \cos \varphi \left\{ \left(6 - \frac{12}{\delta} \right) \left(\frac{\xi}{\delta} \right) - \left(9 - \frac{24}{\delta} \right) \left(\frac{\xi}{\delta} \right)^2 + \left(4 - \frac{12}{\delta} \right) \left(\frac{\xi}{\delta} \right)^3 \right\} \quad (36)$$

$$W = \frac{(\lambda - C_2 B_0)}{C_1} \sin \varphi \left\{ \left(2 - \delta \right) \left(\frac{\xi}{\delta} \right) - \left(\frac{\xi}{\delta} \right)^2 \right\} + C_4 \left\{ 2 \left(\frac{\xi}{\delta} \right) - \left(\frac{\xi}{\delta} \right)^2 \right\} \quad (37)$$

Like Eq.(34), the electric current in the cross section must satisfy the following condition of continuity:

$$\int_0^\delta J_z d\xi = \int_0^\delta -\frac{\partial H}{\partial \xi} d\xi = C_2(1-\delta) \cos \varphi \quad (38)$$

On the other hand, H and G must satisfy the following boundary conditions, respectively:

$$\left. \begin{aligned} \xi=0: H=0 \\ \frac{\partial^2 H}{\partial \xi^2} = P_m B_0 \cos \varphi \frac{\partial W}{\partial \xi} \\ \xi=\delta: H = -C_2(1-\delta) \cos \varphi \\ \frac{\partial H}{\partial \xi} = C_2 \cos \varphi \end{aligned} \right\} \quad (39)$$

$$\left. \begin{aligned} \xi=0: G=0 \\ \xi=\delta: G = -\frac{1}{2} C_3(1-\delta) \\ \frac{\partial G}{\partial \xi} = \frac{1}{2} C_3 \end{aligned} \right\} \quad (40)$$

Accordingly, the polynomial expressions are used as the profiles of H and G like the velocity profiles:

$$H = C_2 \cos \varphi \left\{ \left(-\frac{3}{2} + \varphi \right) \left(\frac{\xi}{\delta} \right) + \frac{1}{2} \left(\frac{\xi}{\delta} \right)^3 \right\} - C_2 P_m B_0 \cos \varphi \left\{ \frac{1}{2} \left(\frac{\xi}{\delta} \right) - \left(\frac{\xi}{\delta} \right)^2 + \frac{1}{2} \left(\frac{\xi}{\delta} \right)^3 \right\} \quad (41)$$

$$G = -C_3 \left\{ \left(1 - \frac{\delta}{2} \right) \left(\frac{\xi}{\delta} \right) - \frac{1}{2} \left(\frac{\xi}{\delta} \right)^2 \right\} \quad (42)$$

3.2.3 Determination of unknown constants The following relation holds from the balance between the pressure gradient in the axial direction and the shearing stress at the wall:

$$\lambda = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\partial W}{\partial \xi} \right)_{\xi=0} d\varphi = 2 \left(\frac{\partial W}{\partial \xi} \right)_{\xi=0, \pi} \quad (43)$$

Differentiating Eq.(37) with respect to ξ and replacing δ with δ_m , the average value of δ about ϕ , for simplicity, the following equation is obtained as an approximate equation of the right hand side of Eq.(43):

$$\left(\frac{\partial W}{\partial \xi} \right)_{\xi=0, \pi} \approx \frac{(\lambda - C_2 B_0)}{C_1} \left(\frac{2}{\delta_m} - 1 \right) \sin \varphi + \frac{2C_4}{\delta_m} \quad (44)$$

Substituting Eq.(44) into Eq.(43) yields

$$\lambda = \frac{4C_4}{\delta_m} \quad (45)$$

Hereafter, δ is treated in the same manner as in Eq.(44).

The mean velocity can be written as

$$W_m = \frac{R_z}{2} = \frac{1}{\pi} \left(\int_0^{2\pi} \int_0^{(1-\delta)} W \eta d\eta d\varphi + \int_0^{2\pi} \int_0^{\delta} W(1-\xi) d\xi d\varphi \right) \approx C_1 \left(1 - \frac{2}{3}\delta_m + \frac{1}{6}\delta_m^2 \right) \quad (46)$$

Substituting Eqs.(37) and (41) into Eq.(32) and integrating the result over the boundary layer, the following relational expression is obtained:

$$\frac{3C_2}{\delta_m} = P_m B_0 C_1 \quad (47)$$

Integrating Eq.(30) over the boundary layer and comparing the result with Eq.(44), the following expression is derived:

$$\left(\frac{1}{5} - \frac{2}{15}\delta_m \right) C_1^2 \delta_m^2 = 4 - \frac{\delta_m^2}{12} M^2 \quad (48)$$

Substituting the pressure P obtained from Eq.(28) into Eq.(29) and integrating the results over the boundary layer, the following equation is given:

$$\begin{aligned} & -C_1 \left(1 - \frac{2}{3}\delta_m + \frac{1}{6}\delta_m^2 \right)^2 \left(12 - 6\delta_m + \delta_m^2 \right) + \frac{7}{60}\delta_m^2 K^2 \\ & - \frac{19}{120}\delta_m^4 K^2 - \frac{1}{8}C_1 \delta_m^2 M^2 \left\{ (1-\delta_m)^2 + \frac{1}{3}\delta_m - \frac{1}{4}\delta_m^2 \right\} = 0 \quad (49) \end{aligned}$$

3.4.2 Resistance coefficient
defined as

The resistance coefficient of curved channels is

$$f_c = \left(-\frac{1}{R} \frac{\partial P}{\partial \theta} \right) \frac{2a}{(1/2)\rho W_m^2} = \frac{16}{R_z^2} \lambda \quad (50)$$

The following relation is obtained as the ratio of the resistance coefficient of curved channels f_c to that of straight ones f_s [$= -(dp/dz)(2a/(1/2)\rho W_m^2)$], by using Eqs.(45) and (46):

$$\frac{f_c}{f_s} = \frac{32}{f_s R_z \delta_m \left(1 - \frac{2}{3}\delta_m + \frac{1}{6}\delta_m^2 \right)} \quad (51)$$

δ_m in the above equation is determined from Eqs.(48) and (49) and the following equation is used in calculation of the resistance coefficient of straight channels⁽¹⁶⁾.

$$f_s R_z = \left(\frac{M^2}{4} \right) / \sum_{n=0}^{\infty} a_n \left(\frac{M}{4} \right) \sum_{j=1,3,5,\dots} (n+j) I_n^2 \left(\frac{M}{4} \right) \quad (52)$$

where

$$\begin{aligned} a_n \left(\frac{M}{4} \right) &= (-1)^n \left\{ I_{n+1} \left(\frac{M}{4} \right) + I_{n-1} \left(\frac{M}{4} \right) \right\} / I_n \left(\frac{M}{4} \right) \quad (n \neq 0) \\ a_0 \left(\frac{M}{4} \right) &= I_1 \left(\frac{M}{4} \right) / I_0 \left(\frac{M}{4} \right) \end{aligned}$$

I_n : modified Bessel function of the first kind

$C_1 \rightarrow 0$ ($K \rightarrow 0$) in Eq.(48) results in $\delta_m = 4\sqrt{3}/M$ and therefore the resistance coefficient of straight channels is obtained as

$$f_s R_z = \frac{8\sqrt{3}}{3} M / \left(1 - \frac{8\sqrt{3}}{3M} + \frac{8}{M^2} \right) \quad (53)$$

Fig.8 shows the comparison between the results of the above equation and the exact solution and Eq.(53) is found to be a good approximate formula of the resistance coefficient of straight channels. The resistance coefficient of the curved channels is shown in chapter 4 together with the experimental results.

Fig.8 Resistance coefficient of laminar flow in straight channels

4. Experiment

4.1 Experimental apparatus and procedure

The experimental apparatus is diagrammatically shown in Fig.9. In this system mercury is made to flow by a difference in head between the upper reservoir and the channel (mercury is used as the working fluid). The principal parts of the system are an upper reservoir, a flow-rate regulation valve, curved channels, a two liquid type manometer⁽¹⁷⁾, an electromagnet, a lower reservoir, and a mercury-purification device. In order to maintain a constant head during measurement, the upper reservoir has an extremely large sectional area in comparison with that of the channel (700mm × 700mm). The flow rate can be regulated continuously by a needle valve. The lower reservoir is connected to the

compressor via a magnetic valve and a reducing valve, and mercury in the lower reservoir is pushed up to the upper reservoir by compressed air. On the way to the upper reservoir, mercury passes through the mercury purification device which is filled with an aqueous solution of mercury nitrate and where all the impurities and mercury oxides are removed. The tip of the electromagnetic pole is rectangular in shape (350mm in length, 160mm in width) and the gap between the poles is 50mm. Rectified three-phase 200V AC is used as the power source. The magnetic flux density of the electromagnet can be changed continuously up to 1.48T by regulating the amperage of the current. The weighing method is used for measuring the flow-rate.

The testing channels consist of polyethylene pipes with inside diameters of 2.04mm and 3.45mm and their curved portions are formed by winding the polyethylene pipes three times on plastic pipes with different outside diameters. The shape and dimension of the curved channels are shown in Fig.10. In every channel, three static pressure taps of 0.5mm ID are provided on the straight portion at the intervals of 100mm and five pressure taps of 0.5mm ID provided on the curved portion at the intervals of 180°. The second and fourth pressure taps are used for measuring the pressure in the curved portion from the results of preliminary experiments. The front pressure tap on the straight pipe is located at the distance from the inlet of the channel $L_1 \approx 1700\text{mm}$ ($L_1/2a \approx 490 \sim 830$) and located at the distance measured backward from the leading edge of the electromagnet poles $L_2 \approx 130\text{mm}$ ($L_2/2a \approx 38 \sim 64$). The distance along the pipe axis from the leading edge of the electromagnet poles to the front pressure tap on the curved pipe is about $100 \sim 230$ times the inside diameter of the pipe. Therefore the pressure difference measured is considered a pressure difference of a fully developed flow.

① Upper reservoir	⑦ Stop valve
② Flow-rate regulation valve	⑧ Lower reservoir
③ Ventilator	⑨ Magnetic valve
④ Channel	⑩ Compressor
⑤ Manometer	⑪ Mercury purification equipment
⑥ Electromagnet	

Fig.9 Schematic diagram of experimental apparatus

4.2 Experimental results and discussion

4.2.1 Resistance coefficient of laminar flow

In Fig.11, the ratios of the measured resistance coefficient, denoted by f_c/f_s , are shown as a function of the Dean number K . From these results, it is found that an increase of the Hartmann number makes the value of f_c/f_s equal to unity, the flow pattern in the curved channels is similar to the one in the straight channels as mentioned in chapter 3, and the Dean number, at which the influence of the curvature of the channels on the resistance coefficient begins to appear, increases. The points at which the gradient of the curves connecting the experimental data begins to change (arrows in the figures) can be regarded as the points of transition from laminar flow to turbulent one⁽¹⁸⁾. Fig.12 shows the measurement in the laminar flow region together with the analytical results in chapter 3. The measurements agree with the analytical results pretty well. This fact indicates that a numerical analysis is effective for the evaluation of the resistance coefficient of the flow with low Dean numbers and an approximate analysis, backed by the boundary layer theory, is for the calculation of the flow with high Dean numbers.

4.2.2 Transition Dean number and resistance coefficient of turbulent flow

In Fig.13, the ratio of K_{tr} to $K_{tr,0}$ is shown as a function of the Hartmann number, where K_{tr} and $K_{tr,0}$ denote the transition Dean numbers of the flow with and without the magnetic field, respectively (see Fig.11). From the figure, it is seen that the transition Dean number increases with the intensity of the magnetic field but the ratio of the transition Dean number, $K_{tr}/K_{tr,0}$ depends hardly on the radius ratio.

Before the arrangement of the resistance coefficient of turbulent flow, the non-dimensional variables governing the phenomenon are induced by means of the dimensional analysis. The following relation must be satisfied by the non-dimensional variables having something to do with the phenomenon:

$$q(f_c, Re, D/2a, M, C, P_m) = 0 \dots\dots\dots (54)$$

where P_m is the magnetic Prandtl number and $C (= \sigma_w t / \sigma_f a, \sigma_w, \sigma_f$: conductivity of the wall and the fluid, t : thickness of the channel wall) is a non-dimensional num-

Fig.10 Shape and dimension of test channels

ber. Here P_m and C can be removed from the above equation because only mercury is used as the working fluid ($\sigma_f, \mu, \nu = \text{constant}$) and the wall of the channels is electrically insulated ($\sigma_w = 0$). Therefore Eq.(54) becomes

$$q(f_c, R_e, D/2a, M) = 0 \quad (55)$$

Now, by referring to the results of turbulent flow in curved channels without a magnetic field⁽¹⁹⁾

$$Q[f_c \sqrt{D/2a}, R_e(2a/D)^2] = 0 \quad (56)$$

the following relation is assumed:

$$Q[f_c \sqrt{D/2a}, R_e(2a/D)^2, M] = 0 \quad (57)$$

Fig.14 shows the experimental results arranged in conformity with Eq.(57). These figures indicate that for a fixed Hartmann number, $f_c \sqrt{D/2a}$ is a function of only $R_e \times (2a/D)^2$ and does not depend on the radius ratio, even in the MHD turbulent flow in the curved channels. Further, they show that f_c decreases by the action of a magnetic field in the turbulent flow with relatively low Reynolds numbers used in the present experiment.

Fig.11 Resistance coefficient

Fig.12 Comparison of resistance coefficients of laminar flow obtained experimentally and analytically

Fig.13 Transition Dean number

Fig.14 Resistance coefficient of turbulent flow

5. Conclusions

The internal mechanisms of the flow were made clear and an important resistance coefficient for engineering was presented by analyzing a steady, laminar flow through an electrically non-conducting curved channel under a transverse magnetic field. Further, the influence of the magnetic field on the resistance coefficient was examined experimentally by using mercury. Consequently the following conclusions were obtained:

(1) The secondary flow is suppressed by the magnetic field and the velocity profile comes up to that in straight channels. Similarly, the resistance coefficient comes close to the one in straight channels, too. This trend becomes gradually remarkable with an increase of the applied magnetic field.

(2) The Dean number at which the influence of the curvature of the channel begins to appear and the transition Dean number to turbulent flow increase with the application of the magnetic field.

(3) Two kinds of electric current, one flowing in the axial direction and the other circulating in the cross section of a channel, are induced by the interference of the magnetic field and the flow field.

(4) Numerical analyses are effective for calculation of the resistance coefficient in low Dean numbers and the application of the boundary layer theory is for calculation of the resistance coefficient in high Dean numbers.

(5) Even under the magnetic field, the resistance coefficient of turbulent flow can be arranged in the same manner as the resistance coefficient of hydrodynamic turbulent flow.

(6) The magnetic field reduces the resistance coefficient of turbulent flow with relatively low Reynolds numbers.

(7) An approximate formula was presented for the resistance coefficient of a laminar flow in straight channels which can be applied over a wide range of Hartmann numbers.

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