An Elastic Stability Analysis for Spherical Shells*

Pape No. 224-2

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In this paper an elastic stability analysis has been developed for spherical shell segments with fixed or horizontally removable edges. The shells are subjected to external hydrostatic pressure or the combination load, namely, hydropressure and frictional resistance. And when these forces act nonconservatively on the surfaces of the shells, it is shown, the dynamic (flutter type) instabilities as well as static (divergence type) instabilities often take place. The authors consider that these analytical results give some contributions to the study of elastic shell stability.

Key Words: Spherical Shell, Elastic Stability, Buckling, Structural Analysis, Numerical Analysis

1. Notations

θ, ϕ	coordinates variables; circumferential and meridional, respectively
t	time variable
a	radius of curvature of the middle surface of the spherical shell
h	thickness of the shell
β	semi-opening angle of the shell segment
α	geometric parameter = $(a/h)\beta^2$ ($a/h = 100$ in this paper)
u, v, w	displacement components; meridional, circumferential and normal, respectively
E	Young's modulus
ν	Poisson's ratio
P	uniform normal pressure
q	$(1/2)(P/E)(a/h)^2$
$\bar{P}$	uniform tangential follower force
$\bar{q}$	$(1/2)(\bar{P}/E)(a/h)^2$
P_0	classical buckling pressure
q_0	$(1/2)(P_0/E)(a/h)^2$
Λ	$\bar{P}/P$
$N_\theta, N_\phi, N_{\phi\theta}$	in-plane stress resultants
$M_\theta, M_\phi, M_{\phi\theta}$	bending and twisting moments
Q_ϕ	shearing force
D	$Eh^3/12(1-\nu^2)$
Ω^2	eigenvalue
K	spring constant
i	$\sqrt{-1}$

Fig. 1 Nomenclature for Shell Segment

2. Introduction

More than twenty years have already passed since epockmaking works were presented by B. Budiansky, (1) H. J. Weinitschke (2) (3) and N. C. Huang (4) (5) on the buckling behaviour of elastic spherical shell segments subjected to an external hydrostatic pressure. But even now this subject is being studied actively because there have remained unexplored many important problems, though it is a typical application of classical (geometrically nonlinear) elasticity. From the historical aspect, the interest has been focussed on the disagreement between analytical and experimental results. This contradictory problem seemed to have been resolved by the above mentioned studies, being reduced to the following two points: (i) occurrence of asymmetric bifurcation modes and (ii) effect of initial imperfection on the critical value of buckling pressure.

However M. Sunakawa and K. Ichida (6) (1973) asserted that the deformation patterns of shells throughout the buckling process were observed to be axisymmetrical in their experimental studies. The latest experimental results by H. Yamada and S. Yamada (13) using a high-accuracy measuring system have many discrepancies with Sunakawa-Ichida's experimental ones. Furthermore K. Kokubo and Y. Yamamoto (12) (1981) obtained analytical results of spherical shells with initial imperfections at their edges and they suggested that axisymmetric snap-through instability was sensitive to these imperfections, but asymmetric bifurcation instability was not, and these results supported Sunakawa Ichida's experimental results.

In the present paper, the authors adopt more rigorous basic equations of spherical shells than so called Marguerre or Reissner type which have been used in the previous studies above mentioned. An elastic stability analysis of spherical shell segments subjected to an external uniform hydrostatic pressure or a combined load — the hydropressure (normal follower force) and small frictional resistance (tangential follower force) on their surfaces — is developed for the cases of clamped and simply supported edges, or horizontally removable edges. Finally the analytical results are compared with those of the previous studies by other authors.

3. Nonlinear basic equations

From the nonlinear shell theory established by one of the authors, the equations of spherical shells subjected to the hydrostatic pressure are given as follows:

$$N_\theta + 2N_\phi \cos\phi + N_{\phi\theta} \sin\phi + Q_\theta \sin\phi + P_\theta = pah \tilde{v} \sin\phi \quad (1)$$

$$N_\phi \sin\phi + (N_\theta - N_\phi) \cos\phi + N_{\phi\theta} + Q_\phi \sin\phi + P_\phi = pah \tilde{u} \sin\phi \quad (2)$$

$$Q_\phi \sin^2\phi + Q_\theta \sin\phi + Q_{\phi\theta} \cos\phi - (N_\phi + N_\theta) \sin^2\phi + P_r$$

$$- \frac{1}{a} N_\phi (w'' \sin^2\phi + w' \sin\phi \cos\phi - w \sin^2\phi + 2u' \sin^2\phi + u \sin\phi \cos\phi)$$

$$- \frac{1}{a} N_\theta (w'' - w \sin^2\phi + 2v' \sin\phi + u \cos\phi \sin\phi)$$

$$- \frac{1}{a} N_{\phi\theta} (2w' \sin\phi + 2v' \sin^2\phi + 2u' \sin\phi)$$

$$- \frac{1}{a} (N_\phi \sin^2\phi + N_{\phi\theta} \sin\phi) (w' + u)$$

$$- \frac{1}{a} (N_\theta \sin\phi + N_{\phi\theta}) (w' + v \sin\phi) = -pah \tilde{w} \sin\phi \quad (3)$$

$$\epsilon_\theta = \frac{1}{a} (v' \frac{1}{\sin\phi} + u \cot\phi - w)$$

$$+ \frac{1}{2a^2 \sin^2\phi} [(u' - v \cot\phi)^2 + (w' + v \sin\phi)^2 + (v' + u \cos\phi - w \sin\phi)^2] \quad (4)$$

$$\epsilon_\phi = \frac{1}{a} (u' - w) + \frac{1}{2a} ((u' - w)^2 + v'^2 + (w' + u)^2) \quad (5)$$

$$\epsilon_{\phi\theta} = \frac{1}{2a} (u' \frac{1}{\sin\phi} - v \cot\phi + v') + \frac{1}{2a^2 \sin\phi} [(u' - w)(u' - v \cos\phi) + v'(v' + u \cos\phi - w \sin\phi) + (w' + u)(w' + v \sin\phi)] \quad (6)$$

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$$\kappa_\phi = \frac{1}{a^2} (w'' + u') \quad (7)$$

$$\kappa_\theta = \frac{1}{a^2} \left[(u + w') \cot \phi + \frac{1}{\sin^2 \phi} (w'' + v' \sin \phi) \right] \quad (8)$$

$$\kappa_{\theta\theta} = \frac{1}{a^2} \left(\frac{1}{2} (v' - v \cot \phi + \frac{1}{\sin \phi} u') + \frac{1}{\sin \phi} (w'' - w' \cot \phi) \right) \quad (9)$$

$$N_\theta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_\phi) \quad (10)$$

$$N_\phi = \frac{Eh}{1-\nu^2} (\epsilon_\phi + \nu \epsilon_\theta) \quad (11)$$

$$N_{\theta\phi} = \frac{Eh}{1+\nu} \epsilon_{\theta\phi} \quad (12)$$

$$M_\theta = D(\kappa_\theta + \nu \kappa_\phi) \quad (13)$$

$$M_\phi = D(\kappa_\phi + \nu \kappa_\theta) \quad (14)$$

$$M_{\theta\phi} = D(1-\nu) \kappa_{\theta\phi} \quad (15)$$

$$Q_\theta = \frac{1}{a} (M_{\theta\phi} + M_{\phi\theta} \frac{1}{\sin \phi} + M_{\theta\theta} \cot \phi) \quad (16)$$

$$Q_\phi = \frac{1}{a} (M_{\phi\theta} + M_{\theta\phi} \frac{1}{\sin \phi} - M_{\theta\theta} \cot \phi + M_{\phi\phi} \cot \phi) \quad (17)$$

Here

$$(\quad)' = \partial(\quad)/\partial\phi, (\quad)'' = \partial^2(\quad)/\partial\phi^2, (\quad)^\circ = \partial(\quad)/\partial t$$

4. Analysis

In the present study the fundamental equilibrium states are considered to be also only axisymmetrical because of axisymmetric loading. Substituting Eqs. (4) ~ (8) into Eqs. (1) and (3) and nondimensionalizing by the use of Eq. (18), the following nonlinear ordinary differential equations with respect to $y(x)$ and $z(x)$ are obtained.

$$u = h\beta z, w = hy, \phi = \beta x \quad (18)$$

Applying Galerkin's method to the above basic equations, the following algebraic equations are obtained.

$$\sum_{j=0}^{2N} E_{ij} q_j + \sum_{j=0}^{2N} \sum_{k=0}^{2N} F_{ijk} q_j q_k + \sum_{j=0}^{2N} \sum_{k=0}^{2N} \sum_{l=0}^{2N} G_{ijkl} q_j q_k q_l = 0 \quad (19)$$

where $q_0, q_1, \dots, q_{2N}$ are coordinate variables. They are related to the approximate solutions $\bar{y}(x)$ and $\bar{z}(x)$ and the external pressure q as follows:

$$\bar{y}(x) = \sum_{k=1}^N q_k f^k(x), \quad (20)$$

$$\bar{z}(x) = \sum_{k=N+1}^{2N} q_k f^k(x), \quad (21)$$

$$q(x) = q_0. \quad (22)$$

Here $f^k(x)$ are coordinate functions which satisfy all or some of the boundary conditions. For example, in the case of clamped spherical shells, $f^k(x)$ is given by

$$f^k(x) = x^j (1 - x^2)^k, \quad (23)$$

$$f^l(x) = x (1 - x^m) \quad (24)$$

Table 1

i	1	2	3	4	5	6	7
j	0	2	4	6	12	2	4
k	2	2	2	2	2	8	8

Table 2

l	8	9	10	11	12	13	14
m	2	4	6	8	10	12	14

The stability of the respective equilibrium states on the equilibrium paths are examined. Taking account of no existence of coupled interactions between different modes, and applying Galerkin's method to the variational equations of Eqs. (1) (3), the following ordinary differential equations can be obtained.

$$\sum_k J_{nk}^{\text{ik}} \frac{d^2 \eta_{nk}}{d\tau^2} + \sum_k I_{nk}^{\text{ik}} \eta = 0 \quad (25)$$

$$\delta w = h \sum_{j=1}^N \eta_{nj} f^j \cos n\theta \quad (26)$$

$$\delta u = h\beta \sum_{j=N+1}^{2N} \eta_{nj} f^j \cos n\theta \quad (27)$$

$$\delta v = h\beta \sum_{j=2N+1}^{3N} \eta_{nj} f^j \cos n\theta \quad (28)$$

$$\tau = t/\omega, \omega = \sqrt{\rho a^2 (1 - \nu^2)} E$$

Equations (25) are satisfied if we set

$$\eta_{nk} = \Phi_{nk} \exp(i\Omega\tau) \quad (29)$$

where Φ_{nk} are constants. The frequencies Ω^2 can be obtained from the characteristic equation:

$$\det(I_{nk}^{jk} - \Omega^2 J_{nk}^{jk}) = 0 \quad (30)$$

5. Results

5.1 Clamped spherical shells under hydrostatic pressure

The components $\bar{P}_r$, $\bar{P}_\phi$, and $\bar{P}_\theta$ of hydrostatic pressure are given as follows:

$$\bar{P}_\theta = -P(w' + v \sin \phi) \quad (31)$$

$$\bar{P}_\phi = -P(w' + u) \sin \phi \quad (32)$$

$$\bar{P}_r = -P(a + u' + u \cot \phi - 2w + v' \frac{1}{\sin \phi}) \sin^2 \phi \quad (33)$$

Equilibrium paths for clamped spherical shells with $\alpha=5, 10, 30, 50$ are shown in Fig. 2. The nondimensional external pressure q and the nondimensional center deflection $y(0)$ are plotted as ordinate and abscissa, respectively. The stability of the respective equilibrium states on the equilibrium paths is indicated by the eigenvalue curves in Figs. 3, 4, and 5. The eigenvalues Ω^2 and the nondimensional external pressure q are plotted. All eigenvalues of the variational equations are purely real. Transition from stability to instability occurs at the value of $\Omega^2=0$; this case obviously corresponds to static instability. However, with increase in geometric parameter α , some of these eigenvalues approach extremely near each other. The details on this point are discussed later.

Fig. 2 q vs. $y(0)$ Curves for Clamped Shells under Hydrostatic Pressure

Fig. 3 Ω^2 vs. q Curves ($\alpha=5, n=0$)

Fig. 4 Ω^2 vs q Curves ($\alpha=50, n=0$) Fig. 5 Ω^2 vs q Curves ($\alpha=50, n=5$) Fig. 6 Ω^2 vs q Curves ($n \neq 0$)

Fig. 7 Comparison between Analytic Results

5.2 Influence of additional tangential follower force

It is well known that static instability is the only instability phenomenon occurring in clamped or simply supported spherical shells subjected to hydrostatic pressure which means conservative. Also in spherical shells with a relatively large geometric parameter the eigenvalues of variational equations approach each other on the equilibrium paths but no dynamic instability takes place. This is one of the cases when both the boundary condition and the loading condition are ideal. It is interesting to investigate the stability in the cases when the boundary and the loading conditions are slightly different from the ideal cases. Then, as a case of nonideal loading condition, shells subjected to combined hydropressure and additional follower forces are examined. The tangential follower force which is distributed along the tangent direction of a deformed surface does not have a potential. The normal component $\bar{P}_r$ and meridional component $\bar{P}_\phi$ of the tangential follower force are given as follows:

$$\bar{P}_r = -\bar{P}(w + u)\sin^2\phi \quad (34)$$

$$\bar{P}_\phi = -\bar{P}(a - w + u')\sin\phi \quad (35)$$

In Figs. 8 and 9 the equilibrium paths for shells with $\alpha=5$ and 50 are shown respectively. The magnitude and direction of the tangential follower force are given by parameter Λ in the figures, namely $|\Lambda|$ is ratio $|\bar{P}/P|$ and the positive sign of Λ means that the direction turns from the apex to the edge of the shell. Figures 10 and 11 indicate eigenvalue curves for shells with $\alpha=5$ and 50 respectively. As shown in Fig. 11, the eigenvalues approach each other as the pressure q increases, and some of them become complex conjugate and coalesce under the influence of the tangential follower force, therefore a dynamic instability occurs in this case. Further, in Fig. 12 the influence of the tangential follower force upon the critical value q_{cr} is compared between shells with geometric parameters $\alpha=5$ and $\alpha=50$. On a shell with $\alpha=50$, the influence is significant when $\Lambda > 0$, and q_{cr} significantly decreases according as Λ increases up to near $\Lambda=0.2$. In the region of $\Lambda > 0.2$, it seems that q_{cr} does not decrease so remarkably.

Fig. 8 q vs. $y(0)$ Curves under the Influence of Tangential Follower Force ($\alpha=5$)

Fig. 9 q vs. $y(0)$ Curves under the Influence of Tangential Follower Force ($\alpha=50$)

Fig. 10 Ω^2 vs. q Curves under the Influence of Tangential Follower Force ($\alpha=5$, $n=0$)

Fig. 11 Ω^2 vs. q Curves under the Influence of Tangential Follower Force ($\alpha=50$, $n=0$)

Fig. 12 q_{cr} vs. Λ Curves

5.3 Shells with removable edges

The dynamic method above mentioned can be also applied to a stability analysis of the spherical shells with removable edges which are supported by horizontal springs and subjected to hydropressure as shown in Fig. 13. In this case the following coordinate functions are taken in addition to those above mentioned in order to allow of any displacement at the edge.

$$f_1 N(x) = 1 \quad (36)$$

$$f_2 N(x) = x \quad (37)$$

And three boundary conditions are employed as follows:

$$\partial w / \partial \phi = 0, \quad (38)$$

$$Q \phi \sin \beta + N \phi \cos \beta = -k(v \cos \beta - w \sin \beta), \quad (39)$$

$$w \cos \beta + v \sin \beta = 0. \quad (40)$$

Equilibrium paths for the cases of $\alpha=5$ and 50 are shown in Fig. 13. In these cases no effect of springs appears for spring constant $K > 10^6$. In Fig. 14 the eigenvalue Ω^2 are shown for cases of $\alpha=30$, $K=0.2 \times 10^4 \sim 1.5 \times 10^4$ and it is clearly seen that a dynamic instability occurs.

Fig. 13 q vs. $y(0)$ Curves for Spring Supported Shells (--- $\alpha=5$, — $\alpha=30$)

Fig. 14 Ω^2 vs. q Curves for Spring Supported Shells ($\alpha=30$)

5.4 Simply supported spherical shells

In this case, the following functions are employed

$$f^k(x) = \sum_{j=1}^m a_{kj} x^{2k-2j+4} \quad (41)$$

Table 3							
k	1	2	3	4	5	6	7
m	3	3	4	4	4	5	6

where a_{kj} are constants determined as to satisfy the boundary conditions of $f^k(1)=f^{k''}(1) + \nu f^k(1)=0$. And the functions $h^k(x)$, $g^k(x)$ are employed same as in the case of clamped edges.

(1) The case of shells with ordinary simply supported edges under hydropressure

In Figs. 15 and 16, the variations of the axisymmetric and asymmetric critical pressures (q_{cr}/q_0) are shown. (A) being the results obtained by using the algebraic functions mentioned above and (B) those obtained by using Bessel functions. In these figures, it is seen that in the range of $10 < \alpha < 30$, the differences between the critical values obtained here and those of the previous works are not significant. But it is notable that in $\alpha=5$, a critical value for bifurcation instability exists and the result is different from that in Ref.(4).

Fig. 15 Comparison between Analytic Results for Simply Supported Shells (Symmetric)

Fig. 16 Comparison between Analytic Results for Simply Supported Shells (Asymmetric)

(2) Shells subjected to the combined force – hydropressure and small tangential force

Dynamic instabilities (with axisymmetric modes) occur in the range of comparatively large α as in the case of clamped edge condition.

(3) Shells with removable edges and subjected to hydropressure
As decreasing the spring constant K , both snap-through and bifurcation buckling pressure (static instabilities) become lower than those of the shells with immovable edges (ordinary simply supported), and especially the decrease of the former is more remarkable than that of the latter. Therefore, both criticals tend to approach each other with decrease of spring constant K .

(4) Shells subjected to the combined force – hydropressure and edge compression

The edge compression is distributed horizontally and uniformly. In Fig. 17, it is obvious that the snap-through pressure increases further more than that in the case of no compression, but in an early equilibrium state before the pressure reaches at such a high level, a dynamic instability phenomenon takes place.

Fig. 17 Ω^2 vs. q Curves ($\alpha=60$, —: $k=\infty$, ---: $k=0.3$)

6. Concluding remarks

The foregoing descriptions are summarized as follows:

(1) On the critical pressure for the static instabilities (both axisymmetric snap-through and asymmetric bifurcation) of spherical shells with fixed edges, there have been no remarkable differences between the results of the authors and those of the previous studies.

(2) However, in the nonconservative cases when the shells are subjected to hydropressure and tangential friction it can be seen that dynamic (flutter type) instabilities often occur at the early states of loading process.

(3) In the other nonconservative cases when the shells have removable edges similar phenomena can be observed and this tendency has been remarkable with increase of the geometric parameter α .

Finally the authors believe that the results obtained here give some important contributions to the study on the elastic shell stability and an explanation to the discrepancies between theoretical and experimental results.

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