Parametric Excitation in the Self-excited Vibration System
with Dry Friction
(1st Report, Parametric Resonance)*

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An application of solutions of linear Mathieu equation by a new method gives approximate solutions in a parametrically excited self-excitation system with dry friction. Steady solutions and the stability in the regions of parametric resonance of first and second orders are determined. Solutions outside the regions of parametric resonances are approximated by two limit cycles. One corresponds to a stable solution and the other corresponds to an unstable solution. Beyond a certain value of dry friction self-excited vibrations are completely suppressed and a parametric resonance of second order soon disappears, so that only a parametric resonance of first order remains within a certain interval of frequency.

Numerical results by the present analysis almost agree with those by Runge-Kutta method and it is seen that the proposed approximate methods have high accuracy.

Key Words: Theory of Vibration, Parametric Excitation, Self-excited Vibration, Dry Friction, Averaging Method, Limit Cycle, Phase Plane Trajectory

1. Introduction

In a system which consists of a beam subjected to a periodically varying axial force and consists of a pendulum with oscillating suspension in the direction of the longitudinal axis, parametric vibrations occur. When such a system is subjected to self-excitation by the influence of a flowing medium (for example lateral wind) and dry friction between structures, a parametric vibration occurs together with a self-excited vibration.⁽¹⁾ The occurrence of such vibrations was pointed out by Tondl⁽¹⁾. Vibrations of a rotor with asymmetric shaft rigidity supported by plain bearings and vibrations of a pipe conveying a gas-and-liquid two-phase flow subjected to self-excitation by the effect of a flowing medium are typical problems in the field of mechanical engineering. Actually such vibrations occur much less frequently than a forced vibration by external unbalanced force. Parametric excitation or self-excitation has been considered to cause such vibrations mainly, so that such vibrations are not dealt with as the problems of systems subjected to both self-excitation and parametric excitation. Therefore their vibration phenomena, excitation mechanisms and occurrence conditions are not clear and, what is worse,

they are not fully known even in the case of one-degree-of-freedom system. Very few studies have been made by Tondl⁽¹⁾, Kawai and Kotera⁽²⁾.

In the present paper excitation mechanisms of such a system are idealized by self-excitations of Van der Pol type and Coulomb's friction which are typical vibrational elements and by a parametric excitation of Mathieu type. Then such a system is described by an equation of motion of a one-degree-of-freedom and vibration phenomena of it are investigated in detail. By applying a new method of a parametric excitation system⁽³⁾, steady solutions in the regions of parametric resonance of first and second orders and solutions outside these regions are determined with high accuracy. The effects of dry friction are investigated in detail.

2. Equation of Motion

The motion of a system in which a parametric vibration occurs together with a self-excited vibration is quite complicated. So as a mathematical model we shall idealize the self-excitations by Coulomb's friction and the non-linear resistance of Van der Pol type which are respectively typical examples of hard and soft self-excitations and idealize the parametric excitation by Mathieu type. Then that system is described as the following non-dimensional equation⁽¹⁾:

$$\ddot{y} - (\beta - \delta y^2)\dot{y} + \partial \operatorname{sgn} \dot{y} + (1 + \gamma y^2 + \mu \cos 2\eta\tau)y = 0;$$

$$\dot{y} = dy/d\tau \quad \dots\dots\dots (1)$$

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where β , δ , $\mathcal{J}$, γ and μ are parameters smaller than unity.

Applying the method in Ref. (3) for linear Mathieu equation, steady solutions of Eq. (1) can be assumed as follows:

In the region of parametric resonance of first order

$$y = A_1 \cos \eta \tau + B_1 \sin \eta \tau \quad \dots\dots\dots (2)$$

and in the region of parametric resonance of second order

$$y = C_0 + A_2 \cos 2\eta \tau + B_2 \sin 2\eta \tau \quad \dots\dots\dots (3)$$

where A_i , B_i , C_0 ($i=1,2$) are unknown coefficients which depend upon initial conditions.

3. Steady Solution in the Region of Parametric Resonance of First Order and the Stability

In the region of parametric resonance of first order according to Eq. (2), a trajectory of Eq. (1) in the plane $y - \dot{y}/\eta$ is expected to be close to a circle rotating with an angular velocity η and then in the coordinate system $\xi - \zeta$ rotating clockwise with an angular velocity η it is expected to become a point.

Now we express Eq. (1) as ($y = y_1$)

$$\left. \begin{aligned} \dot{y}_1 &= \eta y_2 \\ \dot{y}_2 &= (\beta - \delta y_1^2) y_2 - \frac{\partial}{\eta} \operatorname{sgn}(\eta y_2) \\ &\quad - \frac{1}{\eta} (1 + \gamma y_1^2 + \mu \cos 2\eta \tau) y_1 \end{aligned} \right\} \quad \dots\dots\dots (4)$$

and introduce the following transformation

$$\left. \begin{aligned} y_1 &= \xi \cos \eta \tau + \zeta \sin \eta \tau \\ y_2 &= -\xi \sin \eta \tau + \zeta \cos \eta \tau \end{aligned} \right\}. \quad \dots\dots\dots (5)$$

Substituting Eq. (5) into Eq. (4) and assuming that parameters β , δ , $\mathcal{J}$, γ and μ are small enough compared with unity, it is considered that both ξ and ζ change slowly with time and then the averaging method can be applied. After arrangements we obtain the differential equations for ξ and ζ .

$$\left. \begin{aligned} \dot{\xi} &= \left(\beta - \frac{4\delta}{\pi A \eta} - \frac{\delta A^2}{4} \right) \frac{\xi}{2} + \left(1 - \eta^2 + \frac{3\gamma}{4} A^2 - \frac{\mu}{2} \right) \frac{\zeta}{2\eta} \\ \dot{\zeta} &= - \left(1 - \eta^2 + \frac{3\gamma}{4} A^2 + \frac{\mu}{2} \right) \frac{\xi}{2\eta} + \left(\beta - \frac{4\delta}{\pi A \eta} - \frac{\delta A^2}{4} \right) \frac{\zeta}{2} \end{aligned} \right\} \quad \dots\dots\dots (6)$$

where $A^2 = \xi^2 + \zeta^2$. In the polar coordinate system (A , φ)

$$\left. \begin{aligned} A &= \left(\beta - \frac{4\delta}{\pi A \eta} - \frac{\delta A^2}{4} \right) \frac{A}{2} - \frac{\mu A}{4\eta} \sin 2\varphi \\ \dot{\varphi} &= - \frac{1}{2\eta} \left(1 - \eta^2 + \frac{3\gamma}{4} A^2 \right) - \frac{\mu}{4\eta} \cos 2\varphi \end{aligned} \right\}. \quad \dots\dots\dots (7)$$

Steady non-trivial solutions (resonance curves) are given by

$$\left. \begin{aligned} \eta^4 + \left\{ \left(\beta - \frac{\delta A^2}{4} \right)^2 - 2 \left(1 + \frac{3\gamma}{4} A^2 \right) \right\} \eta^2 - \frac{8\delta}{\pi A} \left(\beta - \frac{\delta A^2}{4} \right) \eta \\ + \left(1 + \frac{3\gamma}{4} A^2 \right)^2 - \left(\frac{\mu}{2} \right)^2 + \left(\frac{4\delta}{\pi A} \right)^2 = 0, \end{aligned} \right\} \quad \dots\dots\dots (8)$$

$$\frac{\zeta}{\xi} = - \eta \frac{\beta - \frac{4\delta}{\pi A \eta} - \frac{\delta A^2}{4}}{1 - \eta^2 + \frac{3\gamma}{4} A^2 - \frac{\mu}{2}}. \quad \dots\dots\dots (9)$$

Then using the following relation (the phase φ)

$$W = - \eta \frac{\beta - \frac{4\delta}{\pi A \eta} - \frac{\delta A^2}{4}}{1 - \eta^2 + \frac{3\gamma}{4} A^2 - \frac{\mu}{2}} = \tan \varphi, \quad \dots\dots\dots (10)$$

ξ_s and ζ_s components are determined by

$$\xi_s = \pm \frac{A_s}{\sqrt{1+W^2}}, \quad \zeta_s = \pm \frac{A_s W}{\sqrt{1+W^2}}. \quad \dots\dots\dots (11)$$

Subscripts s stand for steady solutions.

Let us analyze the stability of steady solutions (ξ_s , ζ_s). Introducing small disturbances x and y as

$$\xi = \xi_s + x, \quad \zeta = \zeta_s + y \quad \dots\dots\dots (12)$$

and substituting Eq. (12) into Eq. (6), we obtain a characteristic equation. According to the theorem of Routh-Hurwitz, steady solutions are stable when both of the following conditions are satisfied.

$$\left. \begin{aligned} \text{i)} \quad & - \frac{2\delta}{\pi \eta A_s} + \beta - \frac{\delta}{2} A_s^2 < 0 \\ \text{ii)} \quad & \left\{ \left(\frac{2\delta}{\pi \eta A_s^2} - \frac{\delta}{4} \right) \left(\beta - \frac{4\delta}{\pi \eta A_s} - \frac{\delta}{4} A_s^2 \right) \right. \\ & \left. + \frac{3\gamma}{(2\eta)^2} \left(1 - \eta^2 + \frac{3\gamma}{4} A_s^2 \right) \right\} A_s^2 > 0 \end{aligned} \right\} \quad \dots\dots\dots (13)$$

The trivial solution (the origin) is stable because

$$\frac{dA}{d\tau} = - \frac{2\delta}{\pi \eta} < 0 \quad \dots\dots\dots (14)$$

when A approaches zero.

4. Steady Solution in the Region of Parametric Resonance of Second Order and the Stability

In this region according to Eq. (3), a trajectory of Eq. (1) in the plane $y - \dot{y}/2\eta$ is considered to be close to a circle of which the centre is $y = C_0$ and then in the coordinate system $\xi - \zeta$ rotating clockwise around $y = C_0$ with an angular velocity 2η it becomes a point.

We shall express Eq. (1) as

$$\left. \begin{aligned} \dot{y}_1 &= 2\eta y_2 \\ \dot{y}_2 &= (\beta - \delta y_1^2) y_2 - \frac{\partial}{2\eta} \operatorname{sgn}(2\eta y_2) \\ &\quad - \frac{1}{2\eta} (1 + \gamma y_1^2 + \mu \cos 2\eta \tau) y_1 \end{aligned} \right\} \quad \dots\dots\dots (15)$$

and introduce the following transformation

$$\left. \begin{aligned} y_1 &= C_0 + \xi \cos 2\eta \tau + \zeta \sin 2\eta \tau \\ y_2 &= -\xi \sin 2\eta \tau + \zeta \cos 2\eta \tau \end{aligned} \right\}. \quad \dots\dots\dots (16)$$

Substituting Eq. (16) into Eq. (15) and applying the averaging method, we obtain two differential equations and an algebraic equation.

$$\left. \begin{aligned} \dot{\xi} &= \left\{ \beta - \frac{4\delta}{\pi A 2\eta} - \frac{\delta}{4} (A^2 + 4C_0^2) \right\} \frac{\xi}{2} \\ &\quad + \left\{ 1 - (2\eta)^2 + \frac{3\gamma}{4} (A^2 + 4C_0^2) \right\} \frac{\zeta}{4\eta}, \end{aligned} \right\} \quad \dots\dots\dots (17)$$

$$\left. \begin{aligned} \dot{\zeta} &= - \left\{ 1 - (2\eta)^2 + \frac{3\gamma}{4} (A^2 + 4C_0^2) \right\} \frac{\xi}{4\eta} \\ &\quad + \left\{ \beta - \frac{4\delta}{\pi A 2\eta} - \frac{\delta}{4} (A^2 + 4C_0^2) \right\} \frac{\zeta}{2} - \frac{\mu}{4\eta} C_0, \end{aligned} \right\} \quad \dots\dots\dots (18)$$

$$C_0 \left\{ 1 + \frac{3\gamma}{2} \left(A^2 + \frac{2}{3} C_0^2 \right) \right\} = - \frac{\mu}{2} \xi. \quad \dots\dots\dots (19)$$

where $A^2 = \xi^2 + \zeta^2$. Steady non-trivial solutions are given by the points of intersection of the following two curves.

$$\begin{aligned} & \{(2\eta\delta)^2 + (3\gamma)^2\}C_s^2 + \left[-2\delta(2\eta)^2\left(\beta - \frac{4\theta}{\pi A 2\eta}\right) - \frac{\delta}{4}A^2\right] + 6\gamma\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\} - \left(\frac{\mu}{A}\right)^2\right]C_s^2 \\ & + (2\eta)^2\left(\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}A^2\right)^2 \\ & + \left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\}^2 = 0, \quad \dots\dots\dots (20) \end{aligned}$$

$$2C_s(1 + \gamma C_s^2) = A^2\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\} \quad \dots\dots (21)$$

The ratio between ξ and ζ is given by

$$\frac{\xi}{\zeta} = -2\eta \frac{\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}(A^2 + 4C_s^2)}{1 - (2\eta)^2 + \frac{3\gamma}{4}(A^2 + 4C_s^2)} \quad \dots\dots (22)$$

When steady solutions A^2 (A^2) and C_s^2 (C_s^2) are obtained by Eqs. (20) and (21) (see Fig. 3), ξ_s and ζ_s components are determined by

$$\xi_s = \pm A_s / \sqrt{1 + W^2}, \quad \zeta_s = \pm A_s W / \sqrt{1 + W^2} \quad (23)$$

where

$$W = -2\eta \frac{\beta - \frac{4\theta}{\pi A_s 2\eta} - \frac{\delta}{4}(A_s^2 + 4C_s^2)}{1 - (2\eta)^2 + \frac{3\gamma}{4}(A_s^2 + 4C_s^2)} \quad \dots\dots (24)$$

Figure 3 proves that $A^2 \gg C_s^2$, so that Eqs. (17) to (19) reduce to

$$\left. \begin{aligned} \xi &= \left(\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}A^2\right) \frac{\xi}{2} \\ &+ \left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\} \frac{\xi}{4\eta} \\ \xi &= -\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\} \frac{\xi}{4\eta} \\ &+ \left(\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}A^2\right) \frac{\xi}{2} - \frac{\mu}{4\eta} C_s \\ C_s \left(1 + \frac{3\gamma}{2}A^2\right) &= -\frac{\mu}{2} \xi \end{aligned} \right\} \quad \dots\dots (25)$$

Steady non-trivial solutions of Eqs. (25) are given by the following equations instead of Eqs. (20) and (21).

$$\begin{aligned} & (2\eta)^4 + \left\{\left(\beta - \frac{\delta}{4}A^2\right)^2 - 2\left(1 + \frac{3\gamma}{4}A^2\right) + \frac{\mu^2}{2} \frac{1}{1 + \frac{3\gamma}{2}A^2}\right\}(2\eta)^2 - \frac{8\delta}{\pi A} \left(\beta - \frac{\delta}{4}A^2\right)(2\eta) \\ & + \left(1 + \frac{3\gamma}{4}A^2\right)^2 - \frac{\mu^2}{2} \frac{1 + \frac{3\gamma}{4}A^2}{1 + \frac{3\gamma}{2}A^2} + \left(\frac{4\theta}{\pi A}\right)^2 = 0 \quad (26) \end{aligned}$$

The ratio between ξ and ζ is given by

$$\frac{\xi}{\zeta} = -2\eta \frac{\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}A^2}{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2} \quad \dots\dots\dots (27)$$

Then using the following relation (the phase φ)

$$W = -2\eta \frac{\beta - \frac{4\theta}{\pi A_s 2\eta} - \frac{\delta}{4}A_s^2}{1 - (2\eta)^2 + \frac{3\gamma}{4}A_s^2} = \tan \varphi, \quad (28)$$

ξ_s and ζ_s components are determined by

$$\xi_s = \pm A_s / \sqrt{1 + W^2}, \quad \zeta_s = \pm A_s W / \sqrt{1 + W^2} \quad (29)$$

The constant component C_s is given by

$$C_s^2 = \frac{1}{2} \frac{1 - (2\eta)^2 + \frac{3\gamma}{4}A_s^2}{1 + \frac{3\gamma}{2}A_s^2} A_s^2 \quad \dots\dots\dots (30)$$

The sign of C is determined by the third equation of (25). Then resonance curves are given by $A + C$ and $A - C$.

In the same manner as in Chapt. 3 steady solutions (ξ_s, ζ_s, C_s) are stable when both of the following conditions are satisfied.

$$\begin{aligned} \text{i) } & \beta - \frac{2\theta}{\pi 2\eta A_s} - \frac{\delta}{2}A_s^2 \\ & + \frac{\mu}{4\eta} \frac{3\gamma C_s}{1 + \frac{3\gamma}{2}A_s^2} \frac{A_s W}{\sqrt{1 + W^2}} < 0 \\ \text{ii) } & \frac{3\gamma}{8} \frac{A_s^2}{(2\eta)^2} \left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A_s^2\right\} \\ & \times \left\{1 + \left(\frac{\mu}{1 + \frac{3\gamma}{2}A_s^2}\right)^2\right\} + \frac{1}{2} \left(\frac{2\theta}{\pi 2\eta A_s} - \frac{\delta}{4}A_s^2\right) \left(\beta - \frac{4\theta}{\pi 2\eta A_s} - \frac{\delta}{4}A_s^2\right) \left[2 - \frac{3\gamma}{4} \frac{\beta - \frac{4\theta}{\pi 2\eta A_s} - \frac{\delta}{4}A_s^2}{\left(\frac{2\theta}{\pi 2\eta A_s} - \frac{\delta}{4}\right) \left(1 - (2\eta)^2 + \frac{3\gamma}{4}A_s^2\right)}\right] > 0 \end{aligned} \quad \dots\dots\dots (31)$$

The trivial solution (the origin) is stable because $dA/d\tau < 0$ when A approaches zero, where

$$\frac{dA}{d\tau} = \left(\beta - \frac{4\theta}{\pi A 2\eta} - \frac{\delta}{4}A^2\right) \frac{A}{2} + \frac{\mu^2 A \sin 2\varphi}{16\eta \left(1 + \frac{3\gamma}{2}A^2\right)} \quad \dots\dots\dots (32)$$

5. Solutions outside the Regions of Parametric Resonance of First Order

In the regions of η outside the regions of parametric resonance of first order, the vibration with beat character occurs according to Eq. (1) of Runge-Kutta (R.K.) method. This vibration is expected to correspond to the limit cycle in the phase plane except for converging the origin. Let us determine the maximum and minimum amplitudes of a beat by approximately obtaining an equation of the limit cycle.

Equation (7) yields in the polar coordinate (A^2, φ)

$$\frac{dA^2}{d\tau} = \left(\beta - \frac{4\theta}{\pi A \eta} - \frac{\delta}{4}A^2\right) A^2 - \frac{\mu A^2}{2\eta} \sin 2\varphi, \quad (33)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{2\eta} \left(1 - \eta^2 + \frac{3\gamma}{4} A^2 \right) - \frac{\mu}{4\eta} \cos 2\varphi. \quad (34)$$

Now let us put

$$A^2 = A_0^2 + \Delta A^2 \quad \dots\dots\dots (35)$$

where A_0 are positive roots of the following equation.

$$\beta - \frac{4\vartheta}{\pi A \eta} - \frac{\delta}{4} A^2 = 0 \quad \dots\dots\dots (36)$$

Assuming that $A_0^2 \gg \Delta A^2$ and ignoring small terms of higher order, Eqs.(33) and (34) yield

$$\frac{d(\Delta A^2)}{d\tau} = \left(\frac{2\vartheta}{\pi \eta A_0^3} - \frac{\delta}{4} \right) A_0^2 \Delta A^2 - \frac{\mu A_0^2}{2\eta} \sin 2\varphi, \quad (37)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{2\eta} \left(1 - \eta^2 + \frac{3\gamma}{4} A_0^2 \right) - \frac{\mu}{4\eta} \cos 2\varphi. \quad (38)$$

In Eq.(38) assuming that

$$\left| 1 - \eta^2 + \frac{3\gamma}{4} A_0^2 \right| \gg \frac{\mu}{2} \quad \dots\dots\dots (39)$$

and putting

$$\Omega_0 = -\frac{1}{2\eta} \left(1 - \eta^2 + \frac{3\gamma}{4} A_0^2 \right) \quad \dots\dots\dots (40)$$

we obtain $\varphi = \Omega_0 \tau$. Then Eq.(37) yields a differential equation of first order with respect to ΔA^2 .

$$\frac{d(\Delta A^2)}{d\tau} + \left(\frac{\delta}{4} - \frac{2\vartheta}{\pi \eta A_0^3} \right) A_0^2 \Delta A^2 = -\frac{\mu A_0^2}{2\eta} \sin 2\Omega_0 \tau \quad \dots\dots\dots (41)$$

A steady solution of Eq.(41) is given by

$$\Delta A^2 = a \cos 2\Omega_0 \tau + b \sin 2\Omega_0 \tau \quad \dots\dots\dots (42)$$

where

$$a = \frac{\mu \Omega_0 A_0^2}{\eta(4\Omega_0^2 + D^2)}, \quad b = -\frac{\mu A_0^2 D}{2\eta(4\Omega_0^2 + D^2)},$$

$$D = \left(\frac{\delta}{4} - \frac{2\vartheta}{\pi \eta A_0^3} \right) A_0^2.$$

Consequently limit cycles corresponding to each A_0 are approximately given as

$$\left. \begin{aligned} \xi &= A \cos \Omega_0 \tau, \quad \zeta = A \sin \Omega_0 \tau, \\ A^2 &= A_0^2 + a \cos 2\Omega_0 \tau + b \sin 2\Omega_0 \tau \end{aligned} \right\}. \quad (43)$$

The maximum and minimum amplitudes of a beat given by

$$\sqrt{A_0^2 + \sqrt{a^2 + b^2}} \quad \text{and} \quad \sqrt{A_0^2 - \sqrt{a^2 + b^2}}. \quad \dots\dots (44)$$

6. Numerical Results

Approximate solutions in each region are compared with numerical solutions of equation of motion (1) by R.K. method in positive and negative times, which are respectively denoted by the marks $\circ$ and $\bullet$ in the figures. Then the extreme values of y are denoted by symbol $[y]$. We abbreviate numerical solutions of Eq.(1) by R.K. method as numerical solutions simply. Prior to an examination of approximate solutions, we shall consider the trajectories of parametric resonances in the phase plane. Parameters are as follows:

Alternative I: $\beta = 0.05$, $\delta = 0.01$, $\gamma = 0.01$, $\mu = 0.2$, $\vartheta = 0.01$

6.1 Stroboscopic phase plane

Since the phase trajectory of Eq.(1) with the periodic coefficient is much complicated, it is simplified by connecting the reference points in a period of vibration. This trajectory is called the trajectory in the stroboscopic phase plane which is distinguished as $(y) - (\dot{y})$. In the figures the following symbols are employed: F - focus; N - node; SP - saddle point; L - limit cycle. Subscript s stands for stable and u for unstable.

Trajectories in the stroboscopic phase plane are shown in Figs.1(a) and (b) in the region of parametric resonance of first order and in Figs.2(a) and (b) in the region of parametric resonance of second order. Figures (a) respectively show trajectories in the unstable region of resonance and a magnified figure in the neighborhood of the origin is shown in the right and low side. The trajectories in the plane $\xi - \zeta$ are shown in Fig.1(c) by Eq.(6) and in Fig.2(c) by Eq.(25). These trajectories are respectively quite similar to ones in the stroboscopic phase plane in Fig.1(b) and Fig.2(b). The

Fig.1 Trajectories of parametric resonance of first order in the phase plane

Fig.2 Trajectories of parametric resonance of second order in the phase plane

Fig.3 Resonance curve (parametric resonance of first order)

Fig.4 Frequency characteristic of resonance vector (parametric resonance of first order)

Fig.5 Diagram $A^2 - C_0^2$

Fig.6 Diagram $[y] - \eta$

origins in Figs.1 and 2 are always stable.

6.2 Examination of approximate solutions

Figure 3 shows a resonance curve of parametric resonance of first order. Figure 4 shows a frequency characteristic of resonance vector by Eq.(11), where a counterpart with respect to the origin is drawn. The solid lines stand for stable solutions and the broken lines for unstable ones.

Steady solutions in the region of parametric resonance of second order are given by the points of intersection of Eq.(20) and Eq.(21) as in Fig.5. Then a diagram $[y] - \eta$ where $A + C_0$ and $A - C_0$ are plotted is shown in Fig.6. On the other hand diagrams $A - \eta$ and $C_0 - \eta$ by Eqs.(26) and (30) are shown in Fig.7. Then Fig.8 shows a frequency characteristic by Eq.(29) and Fig.9 shows a diagram $[y] - \eta$ (resonance curve). The results in Fig.9 coincide with those in Fig.6 and numerical solutions.

Trajectories in the coordinate system $\xi - \zeta$ rotating with an angular velocity η and waveforms for η outside the region of parametric resonance are shown in Fig.10 by Eq.(1) and in Fig.11 by Eq.(43). In Fig.10(i) the outer trajectory is stable and corresponds to a beat vibration. The inner closed one, which is drawn in nega-

tive time, is unstable and it forms a separatrix which divides the regions of initial conditions leading to the stable origin. It is seen that the limit cycles by Eq.(43) for η under and over the regions of parametric resonance of first order respectively correspond to the inner and outer envelopes of the trajectory in Fig.10. Figures (ii) and (iii) show waveforms in positive and negative times. Waveforms in Fig.10 and Fig.11 are not necessarily similar to each other but they coincide well as to the maximum and minimum amplitudes.

Figure 12 shows diagram $[y] - \eta$ in the regions of parametric resonances and outside the regions of those by Eq.(44). In Fig.12 an unstable solution in the region of parametric resonance of second order is omitted. The maximum and minimum amplitudes by Eq.(44) agree well with numerical solutions for intervals of η satisfying a condition by Eq.(39).

(a) Diagram $C_0 - \eta$

(b) Diagram $A - \eta$

Fig.7 Diagrams $C_0 - \eta$ and $A - \eta$

Fig.8 Frequency characteristic of resonance vector (parametric resonance of second order)

Fig.9 Diagram $[y] - \eta$

6.3 Effects of dry friction and negative damping

Now we consider the influence of dry friction. Only parameter η is varied from Alternative I as follows:

Fig.12 Diagram $[y] - \eta$ (Alternative I)

Alternative II: $\eta = 0.07$ (Fig.13)

Alternative III: $\eta = 0.1$ (Fig.14)

The larger the value of η , the larger becomes the region of initial conditions leading to the origin outside the region of resonance. After self-excited vibration disappears and parametric resonances of first and second orders remain as shown in Fig.13, soon only a parametric resonance of first order occurs in the system (1). It is seen that the amplitude of resonance curve of first order is not so variable.

Next we consider the effect of negative linear damping β . Only parameter β is

Fig.13 Diagram $[y] - \eta$ (Alternative II; $\eta = 0.07$)

(i) Trajectories

(ii) Trajectories

(i) Trajectories by Eq. (43)

(ii) Trajectories by Eq. (43)

(iii) In positive time

(iv) In positive time

(iii) In positive time

(iv) In positive time

(v) In negative time
(a) $\eta = 0.8$

(vi) In negative time
(b) $\eta = 1.2$

(v) In negative time
(a) $\eta = 0.8$

(vi) In negative time
(b) $\eta = 1.2$

Fig.10 Trajectories and waveforms by Eq. (1),

Fig.11 Trajectories and waveforms

varied from Alternative I.

Alternative IV: $\beta = 0.2$ (Fig.15)
Resonance curve of first order is separated into two regions and the amplitude in the stable region becomes larger than in Alternative I.

7. Conclusions

By an investigation of Eq.(1) which is idealized by typical vibrational elements the following properties of a system in which a parametric vibration occurs together with a self-excited vibration are revealed.

(1) In a vibration system subjected to a self-excitation and a parametric excitation, parametric resonances of first and second orders with frequencies η and 2η corresponding to the regions of instability in a linear Mathieu equation occur. Solutions of parametric resonances can be determined by the transformation to the rotating coordinate systems and the averaging method in an application of a new method of linear Mathieu equation. These approximate solutions have high accuracy.

(2) Solutions outside the regions of parametric resonances are complicated beat vibrations and the amplitudes of a beat can be simply estimated by Eq.(44). Equation (44) is obtained by determining approximate limit cycles in the phase plane and it agrees well with numerical solutions of Eq.(1) except for the neighborhood of parametric resonance of first order. The larger one of two limit cycles

Fig.14 Diagram $[y] - \eta$
(Alternative III; $\gamma = 0.1$)

is stable and it represents a beat vibration. The smaller one is unstable and it forms a separatrix which divides the region of initial conditions leading to the origin.

(3) In a system (1) a dry friction produces the region of initial conditions leading to rest. When γ exceeds a certain value, dry friction completely suppresses the self-excited vibration and each parametric resonance remains. Soon in a system (1) only a parametric resonance of first order occurs within a certain interval of frequency. It is seen that a dry friction affects the unstable regions of parametric resonances but does not affect the stable ones so much as in a system (1) without a dry friction (2).

References

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Fig.15 Resonance curve in the region of parametric resonance of first order (Alternative IV; $\beta = 0.2$)