Determination of Yield Surface by means of Proportional Loading Tests *

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Stress strain curves of proportional loading paths under combined axial tension and internal pressure were obtained for aluminium pipe specimens.

These curves were unified to a single curve through the equivalent stress and strain defined in this paper.

The surface of iso-equivalent-stress loci is considered to be the yield surface, which was verified by some experiments.

Key Words : Plasticity, Material Testing, Forming, Yield Surface, Proportional Loading

1 Introduction

Establishment of a constitutive law of plastic deformation is prerequisite to reliable solutions of plastic working process. The most popular law is a flow theory proposed by Prandtl and Reuss, which has been developed into a plastic potential theory through the maximum plastic work principle by Bishop & Hill ⁽¹⁾ or Drucker ⁽²⁾. Identifying the plastic potential with the yield function enables us to get a constitutive law of anisotropic material, and many investigators have made experiments on the shape of yield surface. Ikegami ⁽³⁾ and Hecker ⁽⁴⁾ have reviewed widely these papers.

However, the ambiguity of definition of yielding point is confounding for us to get rigorous shape of yield surface especially when Bauschinger effect is involved. For example, some studies propose a yielding surface which does not contain the origin of stress space, but it is incompatible with the definition of plastic strain as remaining strain. In order to avoid these confusions some new theories which are not necessarily based on the concept of yield surface have been proposed; general theory by Il'iusin ⁽⁵⁾, rational continuum mechanics by Truesdell ⁽⁶⁾ and endochronic theory by Valanis ⁽⁷⁾.

But the flow theory is still dominant in practical uses on account of its simplicity. Tozawa ⁽⁸⁾ obtained the iso-strain loci by proportional loading tests.

The iso-strain loci can be determined with no ambiguity and they always contain the origin of stress space. But any discussion

has not yet been made on the question whether or not the iso-strain loci can be regarded as the yield surface.

The present study also obtains stress strain curves of proportional loading paths for aluminium pipe specimens. These curves are unified to a single curve through the equivalent stress and strain defined in this paper. And some experiments assure the surface of iso-equivalent-stress loci can be regarded as the yield surface.

2 Experiments

2-1 Experimental method

Aluminium pipe specimens of $40\phi \times 2t \times 230l$ are just commercial drawn pipes, which assure us of availability of many uniform specimens. The annealing condition is $410^\circ\text{C} \times 3\text{ hr}$.

The measuring apparatus of axial and circumferential elongations is shown in Fig.1.

Fig.1 Strain measuring apparatus.

One pair of loop springs is used for measuring axial elongation of gauge length 30mm and three pairs are for diametral changes. Strain gauges are attached to both faces of a loop spring and make a bridge for each pair.

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Axial tension and internal pressure are applied by manual oil pumps. Both ends of specimen are clamped by four-pieces split chucks.

2-2 Stress and strain

Bulging deformation was slightly observed at the gauge length. The deviation of diameter was less than 0.5% and its effect on the stress value is less than 2%. In this paper uniform deformation is assumed and the axial and circumferential stresses are calculated by

$$\sigma_z = \frac{pr}{2t} + \frac{W}{2\pi r t}, \sigma_t = \frac{pr}{t}, (\sigma_r = 0) \quad (1)$$

where p : internal pressure, W : axial load, r : mean radius, t : thickness.

The value of t is calculated from the condition of volume conservation. Elastic strains are neglected.

It is convenient to introduce a five dimensional space of stress or strain deviator for description of plastic behavior under combined loadings. In the present experiment it is enough to consider a two dimensional space, i.e. π plane, whose components are defined as

$$\left. \begin{aligned} \sigma_1 &= \sigma_z - \frac{1}{2}(\sigma_t + \sigma_r) = \frac{W}{2\pi r t}, \\ \sigma_2 &= \frac{\sqrt{3}}{2}(\sigma_z - \sigma_r) = \frac{\sqrt{3}pr}{2t}, \end{aligned} \right\} \quad (2)$$

$$\varepsilon_1 = \varepsilon_z, \varepsilon_2 = \frac{1}{\sqrt{3}}(\varepsilon_t - \varepsilon_r) = \frac{1}{\sqrt{3}}(2\varepsilon_t + \varepsilon_z). \quad (3)$$

The magnitude of stress or strain vector is defined as

$$\sigma = (\sigma_1^2 + \sigma_2^2)^{\frac{1}{2}}, d\varepsilon = (d\varepsilon_1^2 + d\varepsilon_2^2)^{\frac{1}{2}}. \quad (4)$$

The direction of stress vector is represented as

$$\tan \theta = \frac{\sigma_1}{\sigma_2} = \frac{W}{\sqrt{3}\pi r^2 p} \quad (5)$$

In the proportional loading we have to keep the value of θ constant, which was attained by the manual control of the ratio W/p changing with the radius r .

2-3 Experimental results

Fig.2 shows the stress strain curves for various loading directions. The strain paths are straight as shown in Fig.3 but

Fig.2
Stress strain curves

Fig.3 Strain paths

Fig.4 Iso strain loci.

their directions are somewhat different from the stress paths (dotted lines).

Fig.4 shows the iso-strain loci. The direction of strain vector is denoted by an arrow which seems approximately normal to the surface.

3 Determination of Yield Surface

3-1 Equivalent stress and strain

Various definitions of the equivalent stress and strain have been proposed in order that stress strain relations for different loading paths may be unified into a single curve. This paper also tries to find such equivalent stress σ_{eq} and strain ε_{eq} .

The curves of Fig.2 are represented as

$$\sigma = F_{\theta}(\varepsilon \cos \varphi) \quad (6)$$

where φ is the angle between $\vec{\sigma}$ and $\vec{d\varepsilon}$. Denote the ratio σ/σ_{eq} by M_{θ} .

$$\sigma = M_{\theta} \cdot \sigma_{eq} \quad (7)$$

Define the equivalent strain by the following.

$$\begin{aligned} \sigma_{eq} \cdot d\varepsilon_{eq} &= \sigma \cdot d\varepsilon \cdot \cos \varphi \quad \text{or} \\ d\varepsilon_{eq} &= M_{\theta} \cdot d\varepsilon \cdot \cos \varphi \end{aligned} \quad (8)$$

Assuming the value of M_{θ} to be constant during the proportional loading, we have

$$\varepsilon_{eq} = M_{\theta} \cdot \varepsilon \cdot \cos \varphi \quad (9)$$

Here we suppose some relation between σ_{eq} and ε_{eq} as

$$\sigma_{eq} = F(\varepsilon_{eq}) \quad (10)$$

where this function is specific to the material and independent of the loading paths. If we prescribe $M_{\theta}=1$ for simple tension ($\theta=90^{\circ}$), then the function $F(x)$ is identical with $F_{90^{\circ}}(x)$. Now we consider how to determine M_{θ} for the other loading directions. From Eqs.(6)~(10) we obtain

$$F_{\theta}(\varepsilon \cos \varphi) = M_{\theta} \cdot F(M_{\theta} \cdot \varepsilon \cos \varphi)$$

Multiplication on both sides by $\varepsilon \cos \varphi$ gives

$$G_{\theta}(x) = G(M_{\theta} \cdot x) \quad (11)$$

where $G(x) = x \cdot F(x)$, $G_{\theta}(x) = x \cdot F_{\theta}(x)$ and $x = \varepsilon \cos \varphi$. The function $G(x)$ is shown in Fig.5 and $G_{\theta}(x)$ is the same as $G_{90^{\circ}}(x)$.

Fig.5 Function $G_{\theta} = \sigma \cdot \varepsilon \cos \varphi$

Then the value of M_{θ} can be determined by the ratio of their inverse functions, i.e. $M_{\theta} = G_{\theta}^{-1}/G^{-1}$. Fig.6 shows thus determined values of M_{θ} , which are scattered under small strain but converge to a constant value as the deformation

Fig.6 Stress ratio M_{θ}

Fig.7 Transformation from $\theta=90^{\circ}$ to $\theta=60^{\circ}$

proceeds ($G > 5$ MPa or $\varepsilon > 0.08$).

Now we can derive the stress strain relation of an arbitrary loading direction from the simple tension curve $\sigma_1 \sim \varepsilon_1$ through the following transformation.

$$\left. \begin{aligned} \sigma &= M_{\theta} \sigma_1 \\ \varepsilon &= \varepsilon_1 \cos \varphi_1 / M_{\theta} \cos \varphi \end{aligned} \right\} \quad (12)$$

where the angle φ is given in Fig.3 and $\varphi_1 = \varphi|_{\theta=90^{\circ}} = 6.5^{\circ}$. Fig.7 shows an example of $\theta=60^{\circ}$, where we can see good agreement between the experiment and the transformation.

3-2 Meaning of equivalent stress

Now we discuss the meaning of the equivalent stress considering plastic slip in crystals. The plastic work increment dW^P of a crystal is given by

$$dW^P = \sum_r \tau_r d\gamma_r = \sum_r |\tau_r| \cdot |d\gamma_r|$$

where τ_r and γ_r are the shearing stress and strain respectively for the r -th slip system. Assume the isotropic hardening and denote the yielding stress by k , then we have

$$|\tau_r| = k \quad \text{and} \quad dW^P = k \cdot d\Gamma$$

where $d\Gamma = \sum_r |d\gamma_r|$.

Suppose the average of dW^P gives the macroscopic work done by a polycrystal, i.e.

$$\overline{k \cdot d\Gamma} = \sigma \cdot d\varepsilon \cdot \cos \varphi$$

Approximation of $\overline{k \cdot d\Gamma} = \overline{k} \cdot \overline{d\Gamma}$ and introduction of $m = \sigma/\overline{k}$ give

$$\overline{d\Gamma} = m \cdot d\varepsilon \cdot \cos \varphi \quad (13)$$

Comparing the above with Eq.(8), we can see that $\overline{k}$ and $\overline{d\Gamma}$ correspond to σ_{eq} and ε_{eq} respectively. Thus the equivalent

stress σ_{eq} means the macroscopic yielding stress as a degree of work hardening. When the Bauschinger effect is not involved, we can assume Taylor's work hardening law, i.e. $k = H(I')$, which is equivalent to Eq.(10).

The theoretical value of m has been obtained for F.C.C. metal by Bishop & Hill⁽¹⁾ from which M_θ can be calculated as the ratio of $m(\theta)$ to $m(90^\circ) = 3.06$.

Thus determined value of M_θ is compared with the present experiment and the two hypotheses, Mises and Tresca, in Fig.8.

Fig.9 shows the yield surface which correspond to Fig.8. It is interesting that the point of $\theta = 30^\circ$ is a round corner while the point of $\theta = 90^\circ$ seems to be a pointed one. The arrows show the directions of strain vector and good normality can be seen except for $\theta = 0^\circ$ and $\theta = 75^\circ$.

Fig.8 Comparison of M_θ

Fig.9 Yield surfaces

3-3 Experimental examination of yield surface

In the preceding section the surface of $\sigma_{eq} = \text{const.}$ has been supposed to be the yield loci as far as the Bauschinger effect is not involved. In this section we try to examine this supposition by two experiments.

In the first experiment the pipe specimens are stretched to various amounts of prestrain ($\epsilon_1 = 3, 5, 8, 12\%$) and then subjected to hydrostatic bulging, i.e. pure shear $\theta = 0^\circ$. From Eq.(14) we have

$$\epsilon_2 = \epsilon_1 \cos 6.5^\circ / 0.92 \cdot \cos 1^\circ = 1.078 \epsilon_1 \quad (14)$$

which means the prestrain ϵ_1 is equivalent to ϵ_2 of Eq.(14) if it is given by pure shear. In Fig.10 the stress strain curves are shifted to the right by the amount of Eq.(14).

The dotted line and solid line denote the stress strain curves with no prestrain of simple tension and of pure shear respectively. The curves of pure shear show good agreement between the annealed material and the prestrained one.

In the second experiment the prestrain is only $\epsilon_1 = 5\%$ and the proportional loadings for various directions ($\theta = 10^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$) follow.

As examples the stress strain curves of $\theta = 30^\circ$ and 60° are shown in Fig.11, where these are shifted by the following amounts.

$$\left. \begin{aligned} \epsilon &= 1.037 \epsilon_1 & \text{for } \theta = 30^\circ \\ \epsilon &= 1.117 \epsilon_1 & \text{for } \theta = 60^\circ \end{aligned} \right\} \quad (15)$$

Fig.10 Stress strain curves of pure shear.

Fig.11 Stress strain curves of proportional loading

The chain lines denote the curves for the annealed material and show good agreement with the prestrained ones. Fig.12 shows the strain paths. The chain lines denote the strain direction of annealed material.

These experimental results convince us that the iso-equivalent-stress loci can be regarded as the yield surface.

If we take the iso-strain loci as the yield surface, the equivalent prestrains for the various directions are always equal to ϵ_1 .

Fig.12 Strain paths.

4 Conclusions

The equivalent stress and strain are defined to unify the proportional loading curves of various directions into a single curve. Some experiments verify that the iso-equivalent-stress loci can be regarded

as a yield surface.

In this paper the yield stress is considered not to be a threshold from elastic to plastic but to be a degree of work hardening.

It is left to a future study how to generalize this definition of the equivalent stress and strain for arbitrary loading paths involving Bauschinger effect.

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References

- (1) Bishop, J.F.W. & Hill, R., *Phil. Mag.*, Vol.42 (1951), p.414 & 1298
- (2) Drucker, D.C., *Proc. 1st U.S. Natl. Cong. Appl. Mech.*, (1951), p.487.
- (3) Ikegami, K., *J. Soc. Mat. Sci. (in Japanese)*, Vol.24 (1975), p.491 & p.709.
- (4) Hecker, S.S., *ASME Appl. Mech. Symposia Ser.*, AMD-Vol.20 (1976), p.1.
- (5) Il'iushin, A.A., *Prikl. Mat. Mekh.*, Vol.24, No.3 (1960), p.399.
- (6) Truesdell, C., *Continuum Mechanics*, Gordon & Breach Inc., New York, (1966).
- (7) Valanis, K.C., *Archives of Mech.*, Vol.23 (1971), p.399.
- (8) Tozawa, Y. & Shirai, H., *J. Japan S. Tech. Plas. (in Japanese)*, Vol.16 (1975), p.550, p.645 & p.1041.