Analysis of Vibration by Component Mode Synthesis Method *

(Part 4, Natural Frequency and Natural Mode (II))

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A component mode synthesis method is improved for analyzing the vibration of complex structures. The components are not classified into master or branch ones. The interface regions of the components are considered a component named \('\text{interface component}', whose displacement is represented with a linear combination of the natural modes calculated from the reduced mass and the reduced stiffness matrices of all other components. The displacements of the interior regions of all components except the \(\text{interface component} \) are represented with the natural modes of the interface region and the restrained natural modes of these \(\text{interface} \) regions. The equation of motion of the total structure is translated into a generalized coordinate of these \(\text{natural} \) modes. Numerical results of the \(\text{natural} \) frequencies and the \(\text{natural} \) modes of two \(\text{specimens} \) are shown.

**Key Words:** Vibration, Natural Frequency, Natural Mode, Numerical Analysis, Component Mode Synthesis Method, Finite Element Method, Generalized Coordinate

1. Introduction

In the previous parts (parts 1 \(^{(2)} \) , 2 \(^{(1)} \) and 3 \(^{(1)} \) of this series of research, the authors studied the component mode synthesis method proposed by Benfield \(^{(4)} \). First, a total structure or a \(\text{total} \) machine is divided into some \(\text{substructures} \) or \(\text{components} \), and all the \(\text{components} \) are classified into the master and the branch ones. The unconstrained \(\text{natural} \) modes of the \(\text{master} \) components and the \(\text{constrained} \) \(\text{natural} \) modes of the \(\text{branch} \) components are determined separately by the \(\text{finite} \) \(\text{element} \) method (FEM). The \(\text{natural} \) modes of all the \(\text{components} \) are \(\text{synthesized} \) to compose the \(\text{generalized} \) \(\text{system} \) coordinate. The \(\text{equation} \) of motion under these \(\text{system} \) coordinates is solved to know the \(\text{vibration} \) of the \(\text{total} \) structure.

This method is useful for \(\text{mechanical} \) structures so complex that their \(\text{vibration} \) can hardly be solved by the \(\text{usual} \) methods, for example \(\text{direct} \) application of FEM. But, this method has two \(\text{important} \) \(\text{defects} \) that all \(\text{the} \) \(\text{components} \) must be classified into \(\text{masters} \) and \(\text{branches} \), and that \(\text{direct} \) \(\text{connection} \) of \(\text{2 masters} \) is prohibited as \(\text{well} \) \(\text{direct} \) \(\text{connection} \) of \(\text{2 branches} \). In the present report, the authors propose a new component mode synthesis method in which \(\text{above} \) defects of the \(\text{previous} \) method are eliminated perfectly, and try to analyze the \(\text{natural} \) frequency and the \(\text{natural} \) mode by this method. Accuracy of analysis is examined for two \(\text{model structures} \) by comparing the calculated result by the \(\text{proposed} \) method with the \(\text{experimental} \) ones.

2. Basic Approach

A total structure shown in Fig.1 is divided into two \(\text{components} \). The internal regions of the \(\text{components} \) 1 and 2 are called the \(\text{regions} \) 1a and 2b respectively and the \(\text{connecting} \) region (interface) between the \(\text{components} \) 1 and 2 is called the \(\text{region} \) b.

The \(\text{equation} \) of motion of the \(\text{component} \) 1 is

\[
\begin{bmatrix}
M_{1a} & K_{1a} & K_{1b} \\
K_{1a} & K_{2a} & K_{2b} \\
K_{1b} & K_{2b} & K_{2b} \\
\end{bmatrix}
\begin{bmatrix}
\delta_{1a} \\
\delta_{2a} \\
\delta_{2b} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
\{F\} \\
\{0\} \\
\end{bmatrix}
\]

(1)

where \(\{F\} \) is the \(\text{internal} \) \(\text{force} \) acting from the \(\text{component} \) 2 to the \(\text{component} \) 1. The \(\text{equation} \) of motion of the \(\text{component} \) 2 is

\[
\begin{bmatrix}
M_{2a} & K_{2a} & K_{2b} \\
K_{2a} & K_{2a} & K_{2b} \\
K_{2b} & K_{2b} & K_{2b} \\
\end{bmatrix}
\begin{bmatrix}
\delta_{1a} \\
\delta_{2a} \\
\delta_{2b} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
\{F\} \\
\{0\} \\
\end{bmatrix}
\]

(2)

Guyan's method of static reduction is applied to the \(\text{component} \) 1. Neglecting the \(\text{inertia} \) \(\text{term} \) of the \(\text{upper} \) \(\text{half} \) of Eq. (1), we have

\[
[K_{1b}](\delta_{1a})+[K_{1a}](\delta_{2b})=0
\]

(3)

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Solving Eq. (3) for $\{\delta_{1a}\}$, we have

$$\{\delta_{1a}\} = [\mathbf{K}_{1a}^{-1} - \mathbf{K}_{1b}^{-1}] \{\delta_{2a}\} = [\mathbf{T}_1] \{\delta_{1b}\}$$

(4)

The internal freedom is eliminated, using Eq. (4).

$$\{\delta_{1a}\} = [\mathbf{T}_1] \{\delta_{1b}\}$$

(5)

The mass and the stiffness matrices of Eq. (1) are reduced using Eq. (5).

$$[\mathbf{M}_1] = [\mathbf{T}_1]^T \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} [\mathbf{T}_1]$$

$$= [\mathbf{T}_1] \mathbf{M}_{11} [\mathbf{T}_1] + [\mathbf{T}_1] \mathbf{M}_{12} [\mathbf{T}_2] + [\mathbf{T}_2] \mathbf{M}_{21} [\mathbf{T}_1] + [\mathbf{T}_2] \mathbf{M}_{22} [\mathbf{T}_2]$$

(6)

$$[\mathbf{K}_1] = [\mathbf{T}_1]^T \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} [\mathbf{T}_1]$$

$$= [\mathbf{T}_1] \mathbf{K}_{11} [\mathbf{T}_1] + [\mathbf{T}_1] \mathbf{K}_{12} [\mathbf{T}_2] + [\mathbf{T}_2] \mathbf{K}_{21} [\mathbf{T}_1] + [\mathbf{T}_2] \mathbf{K}_{22} [\mathbf{T}_2]$$

(7)

In the same way, the reduced mass and the stiffness matrices for the component 2 are obtained

$$[\mathbf{M}_2] = [\mathbf{T}_2]^T \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} [\mathbf{T}_2]$$

$$= [\mathbf{T}_2] \mathbf{M}_{11} [\mathbf{T}_2] + [\mathbf{T}_2] \mathbf{M}_{12} [\mathbf{T}_3] + [\mathbf{T}_3] \mathbf{M}_{21} [\mathbf{T}_2] + [\mathbf{T}_3] \mathbf{M}_{22} [\mathbf{T}_3]$$

(8)

$$[\mathbf{K}_2] = [\mathbf{T}_2]^T \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} [\mathbf{T}_2]$$

$$= [\mathbf{T}_2] \mathbf{K}_{11} [\mathbf{T}_2] + [\mathbf{T}_2] \mathbf{K}_{12} [\mathbf{T}_3] + [\mathbf{T}_3] \mathbf{K}_{21} [\mathbf{T}_2] + [\mathbf{T}_3] \mathbf{K}_{22} [\mathbf{T}_3]$$

(9)

At the connecting region b, we have

$$\{\delta_b\} = [\mathbf{K}_b]^{-1} \{\delta_{1b}\}$$

(10)

Adding together the equations of Eqs. (6), (7), (8), and (9), the equation of motion for the region b becomes

$$([\mathbf{K}_1 + \mathbf{K}_2] - \omega^2 [\mathbf{M}_1 + \mathbf{M}_2]) \{x\} = \{0\}$$

(11)

The modal matrix $[\mathbf{K}_b]$ is obtained from Eq. (11), and the displacement $\{\delta_b\}$ of the region b becomes

$$\{\delta_b\} = [\mathbf{K}_b]^{-1} [\mathbf{K}_1 \{\delta_{1b}\}]$$

(12)

Fixing the region b of the component 1, Eq. (1) becomes

$$([\mathbf{M}_{12} + \mathbf{M}_{22}] - \omega^2 [\mathbf{M}_{12}]) \{x\} = \{0\}$$

(13)

The constrained modal matrix $[\mathbf{K}_{1b}]$ for the component 1 is obtained from Eq. (13), and the internal displacement $\{\delta_{1b}\}$ of the component 1 is

$$\{\delta_{1b}\} = [\mathbf{T}_1] \{\delta_{1b}\} + [\mathbf{T}_1] \{\delta_{1a}\}$$

(14)

Substituting Eq. (12) into Eq. (14) yields

$$\{\delta_{1b}\} = [\mathbf{T}_1] \{\delta_{1b}\} + [\mathbf{T}_1] \{\delta_{1a}\}$$

(15)

In the same way, the internal displacement $\{\delta_{1a}\}$ of the component 2 is obtained

$$\{\delta_{1a}\} = [\mathbf{T}_1] \{\delta_{1a}\}$$

(16)

$$\{\delta_{2a}\} = [\mathbf{T}_2] \{\delta_{2a}\}$$

(17)

$$\{\delta_{2a}\} = [\mathbf{T}_2] \{\delta_{2a}\}$$

(18)

From Eqs. (12), (15), and (18), the displacement of the total structure is

$$\{\delta_{1b}\} = [\mathbf{T}_1] \{\delta_{1b}\} + [\mathbf{T}_1] \{\delta_{1a}\}$$

(19)

The equation of motion of the total structure is

$$[\mathbf{M}_{12} + \mathbf{M}_{22}] \{\ddot{x}\} = \{\omega^2 [\mathbf{M}_{12} \{x\} = \{0\} \}$$

(20)

Substituting Eq. (19) into Eq. (20), and multiplying by $[\mathbf{A}]$, we have

$$[\mathbf{M}_{12} + \mathbf{M}_{22}] \{\ddot{x}\} = \{\omega^2 [\mathbf{M}_{12} \{x\} = \{0\} \}$$

(21)

or as a simple expression of Eq. (21), we have

$$[\mathbf{M}] \{\ddot{x}\} + [\mathbf{K}] \{x\} = \{0\}$$

(22)

The freedom of Eq. (22) in the modal coordinates is much smaller than that of the original Eq. (20) in the physical coordinates. Thus, the natural frequencies and the natural modes of the total structure are obtained easily from

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Eq. (22).

According to this proposed method, all components are treated equally with no restriction in connection of components. Furthermore, all methods for the FEM eigenvalue problem are also valid in the present method.

Figure 2 is a flow chart of calculation by the present method.

![Diagram](image)

Fig. 1 Total structure with two components

--- Preparation ---

Division of structure into comps.  

[M] and [K] of each comp.  

Calculation  

of constrained natural modes  

Guyan's static reduction  

Calculation of natural modes  

of the connecting regions  

Composition of  

total equation of motion  

Eigenvalues calculation  

Transform of modes  

into physical coordinates  

Output

Fig. 2 Flowchart of calculation

3. Examination of Result

3.1 Way of Division

Accuracy of calculation by the present method is examined with a steel plate of 5mm thickness, 300mm length and 85mm width with one fixed side, changing the way of division into the components in 8 kinds as shown in Fig. 3. Finite element eigenvalue problem is compared with the experimental ones in Table 1. Seventeen modes for the connecting region and 20 modes for the internal regions of two components are adopted in these calculations.

![Diagram](image)

Fig. 3 Eight kinds of component divisions of plate

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<th>Div.2</th>
<th>Div.3</th>
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</table>

Table 1 Natural frequency of plate: 1st

Accuracy of calculation is very high in Div. 1, Div. 2 and Div. 7, followed by Div. 3. This fact implies that the calculation is accurate when the component division is symmetric and when the shape of the connecting region is simple, for instance a straight line. But, it is
known from Table 1 that accuracy of the lower modes than 10 is acceptable in all ways of component division. Thus, the way of component division does not have serious influence on accuracy of calculation.

3.2 Number of adopted modes
The same plate as in Fig.3 is divided into 2 components of equal shape, as shown in Fig.4. Each component is divided further into 48 finite elements. The calculated results of the natural frequencies are compared in Table 2 with the experimental results and with the calculated results by the reduced impedance method (RIM). In Table 2, M1, M2 and M3 denote the numbers of the adopted natural modes of the connecting region, of the internal region 1 and of the internal region 2 respectively.

<table>
<thead>
<tr>
<th>M1</th>
<th>Exp.</th>
<th>RIM</th>
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The number of the adopted component modes does not influence the accuracy of the natural frequencies lower than 3rd total modes, but on the whole, accuracy of the natural frequencies higher than 3rd total modes becomes better as the number of the adopted modes increases. Table 2 indicates that it is enough to adopt the component modes in the same number as that of the demanded total modes.

3.3 Analysis of Box model
The natural frequencies and the natural modes of a box model in Fig.6 are calculated by the present component mode synthesis method (CMS) and by the reduced impedance method (RIM). Twenty modes of the connecting region and 15 modes of the interior region of each component are adopted in CMS. The calculated results are compared in Table 3 with the experimental readings.
Fig. 7 Components of box model  
unit: mm  
Solid circle: Fixed nodal points  
Hollow circle: Connected nodal points

Table 3 Natural frequency of box model

<table>
<thead>
<tr>
<th>NO.</th>
<th>EXP</th>
<th>CMS</th>
<th>RIM</th>
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(Hz)

ones. The calculated results by CMS and by RIM agree well with each other, but small errors occur between the calculated and the experimental results. These errors seem to occur during the experiment. Figure 8 shows an example of the natural mode.

Fig. 8 Natural modes of box model

4. Conclusions

A new component mode synthesis method is proposed. No restriction exists in the component division, all components are treated equally, and the degree of freedom of the equation of motion of the total structure becomes considerably small in the presented CMS. The analytical technique of the natural frequency and the natural mode is explained. The calculated results by the proposed CMS are compared with the ones by RIM and with the experimental ones. All results agree well with one another, and it is clear that the present method is accurate and valid for the analysis of vibration.

References