An Elastic Circular Finite Cylinder Under Torsion by a Pair of Annular Rigid Stamps*

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This paper deals with the so-called Reissner-Sagoci problem of an elastic circular finite solid cylinder or a circular thick plate twisted by a pair of annular rigid stamps. For the problem, a method of approach is developed by applying Green's functions for torsional body force and torsional surface force problems of an elastic thick infinite plate. That is, solutions to the problem of a finite solid cylinder are constructed by Green's functions for problems of an elastic thick infinite plate.

Stresses, displacements, torques necessary to twist a stamp and torsional forces on the contact plane under the bottom of a stamp for circular solid cylinders or circular thick plates are shown and compared with the results about the Reissner-Sagoci problems of an elastic half space and an elastic thick infinite plate. A comparison with a simple theory of torsion of a circular cylinder is made.

Key Words: Elasticity, Torsion, Rigid Stamp, Reissner-Sagoci Problem, Green's Function, Circular Cylinder, Circular Plate, Stress Analysis

1. Introduction

This paper deals with elastic problems of a finite solid circular cylinder or a circular thick plate subjected to torsional displacements by a pair of annular rigid stamps. Problems of this kind, that is, torsion of a circular rigid stamp adhered to the surface of an elastic solid, are a so-called Reissner-Sagoci problem⁽¹⁾ and have been investigated by many authors. For instance, Reissner and Sagoci⁽²⁾⁽³⁾ investigated the problem of an elastic half space subjected to torsional displacements by a circular rigid stamp and this problem in the case of an annular rigid stamp was investigated by Shibuya⁽⁴⁾. Similar problems of an infinite thick plate were considered by Shibuya⁽⁵⁾ in the case of a pair of annular rigid stamps of the same sizes and by Singh⁽⁶⁾ in the case of a pair of circular rigid stamps of different sizes. This problem of a nonhomogeneous infinite thick plate was investigated by Hassan⁽⁷⁾. Sneddon et al.⁽⁸⁾⁽⁹⁾ considered a long cylinder a pair of whose flat surface was subjected to torsion by a circular rigid stamp under the assumption that the curved surface was free from applied surface forces or rigidly fixed. Dhaliwal⁽¹⁰⁾ considered the torsion problem of a semi-infinite elastic cylinder embedded in an elastic half space. Kassir⁽¹¹⁾ considered the problem of a long cylinder of a nonhomogeneous solid.

For problems of a finite solid body, Aleksandrov et al.⁽¹²⁾ considered a case where the thickness of a circular plate was very small and the radius of the circular plate was very large compared with the radius

of a stamp. Chebakov⁽¹³⁾ considered a case where the ratio of the cylinder height to the radius of the stamp was sufficiently large or very small.

In previous papers^{(14)~(16)} methods of solution were developed for the problems of a finite solid cylinder twisted by a pair of rigid chucks and for the problems of an infinite thick plate twisted by a pair of rigid stamps (Shibuya's problem⁽⁵⁾ mentioned above), applying Green's functions⁽¹⁴⁾⁽¹⁷⁾ for torsional body force and torsional surface force problems of an elastic thick plate. By superposition of the two methods of solution, this paper shows a method of solution for the problem of a finite solid cylinder (or a finite circular plate) subjected to torsional displacements by a pair of annular rigid stamps.

The fundamental principle of the method of solution is to cut out an imaginary region of a finite solid cylinder (or a circular plate) from an infinite thick plate with the same elastic properties as those of a cylinder. That is, the method is to distribute torsional body forces near a cylindrical plane which is to become a cylindrical surface of the solid cylinder and also to distribute torsional surface forces on a plane corresponding to the bottom of an annular rigid stamp, and to determine the intensities of the body forces and surface forces thus distributed so as to satisfy the boundary conditions of the finite solid cylinder twisted by rigid stamps. Therefore, in the method of solution proposed here, a solution to the problem of a finite solid cylinder or a circular thick plate is expressed in terms of analytical solutions (the so-called Green's functions) for torsional body force and torsional surface force problems of an infinite thick plate.

Numerical calculations are performed in order to determine the components of stresses and displacements in the interior of a finite solid cylinder and to compare the present

* Received 1st October, 1982

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results with the results for the Reissner-Sagoci Problems of a half space⁽⁴⁾ and an infinite thick plate⁽⁵⁾ The torque necessary to twist the rigid stamp is shown and compared with the results for a half space, infinite thick plate and a simple theory of torsion of a circular shaft.

2. Boundary Conditions

Consider a finite solid cylinder of a radius C_3 and a height $2H$, subjected to torsional displacements by a pair of flat annular

$$\left. \begin{aligned} \text{(i)} \quad & Z = \pm H, \quad C_1 \leq r \leq C_2; \quad u_2 = \pm \omega r \\ \text{(ii)} \quad & Z = \pm H, \quad 0 \leq r < C_1, \quad C_2 < r \leq C_3; \quad \tau_{z\theta} = \sigma_z = \tau_{rz} = 0 \\ \text{(iii)} \quad & r = C_3, \quad |Z| \leq H; \quad \sigma_r = \tau_{r\theta} = \tau_{rz} = 0 \end{aligned} \right\} \dots\dots\dots (1)$$

where u_2 represents a torsional displacement about the z -axis. Note that the displacement components u_1 and u_3 in radial and axial directions are always equal to zero, therefore the nonzero components of stresses are $\tau_{r\theta}$ and $\tau_{z\theta}$, because the present problem is an axisymmetric torsional one.

3. Green's Functions

Here we shall prepare Green's functions for (I) torsional body force problems⁽¹⁴⁾ and (II) torsional surface force problems⁽¹⁷⁾ of an infinite thick plate, in order to apply

$$\left. \begin{aligned} u_2' &= \frac{1}{4\pi^2\mu\sqrt{ar}} \sum_{s=1}^{\infty} \left\{ \sum_{k=1}^4 P_k Q_{1/2}(x_k) \right\} \\ \tau_{r\theta}' &= \frac{1}{8\pi^2r\sqrt{ar}} \sum_{s=1}^{\infty} \left[\sum_{k=1}^4 P_k \left\{ \frac{r/a - x_k}{x_k^2 - 1} [x_k Q_{1/2}(x_k) - Q_{-1/2}(x_k)] - 3Q_{1/2}(x_k) \right\} \right] \\ \tau_{z\theta}' &= \frac{1}{8\pi^2ar\sqrt{ar}} \sum_{s=1}^{\infty} \left[\sum_{k=1}^4 P_k \frac{z_k}{x_k^2 - 1} (x_k Q_{1/2}(x_k) - Q_{-1/2}(x_k)) \right] \end{aligned} \right\} \dots\dots\dots (4)$$

where $\delta()$ is a Dirac delta function, μ is the shear modulus of elasticity, $Q_\nu(x_k)$ is a Legendre function of the second kind of the order ν , P_k means $P_k = 1$ for $k = 1, 2$ and $P_k = -1$ for $k = 3, 4$, and x_k is

$$x_k = \frac{r^2 + a^2 + z_k^2}{2ar} \dots\dots\dots (5)$$

where z_k represents

$$\begin{aligned} Z_1 &= Z - 2(S-1)H + (-1)^s h, & Z_2 &= Z + 2SH - (-1)^s h \\ Z_3 &= Z - 2SH + (-1)^s h, & Z_4 &= Z + 2(S-1)H - (-1)^s h \end{aligned} \dots\dots (6)$$

Green's functions (II) mentioned above are defined as a solution to the surface force problem of an infinite thick plate

rigid stamps with the same sizes of an inner radius C_1 and an outer radius C_2 as shown in Fig.1, where we used cylindrical coordinates (r, θ, z) . It is assumed that the bottoms of the stamps are perfectly bonded to the flat surfaces ($z = \pm H, C_1 \leq r \leq C_2$) of the cylinder and the remaining surfaces of the cylinder are free from applied surface forces. When the stamps are rotated through an angle ω about the z -axis in opposite directions to each other, the boundary conditions of the problem are expressed as follows:

them in the next chapter.

Green's functions (I) mentioned above are defined as a solution to the body force problem of an infinite thick plate ($0 \leq r < \infty, |z| \leq H$) subjected to the axisymmetric torsional body force

$$F_z = \frac{1}{2\pi r} \delta(r-a) \{ \delta(z-h) - \delta(z+h) \} \dots\dots\dots (2)$$

acting on two circles ($r = a, z = \pm h$) as shown in Figs. 3, 4 and satisfying the boundary conditions

$$Z = \pm H, \quad 0 \leq r < \infty; \quad \sigma_z = \tau_{rz} = \tau_{z\theta} = 0 \dots\dots\dots (3)$$

and are expressed as follows:

Fig. 2 A thick plate subjected to torsional body forces on two circles

Fig. 1 A solid cylinder or a circular plate twisted by rigid stamps

Fig. 3 A torsional force acting on a circle of a radius a

($0 \leq r < \infty$, $|z| \leq H$) subjected to the axisymmetric torsional surface force

$$\tau_{z\theta} = F_2' = \frac{1}{2\pi r} \delta(r-a) \dots \dots \dots (7)$$

acting on two circles ($r=a$, $z=\pm H$), that is, a solution satisfying the boundary conditions

$$Z = \pm H, \quad 0 \leq r < \infty; \quad \tau_{z\theta} = F_2' \\ (\sigma_r = \tau_{r\theta} = 0) \dots \dots \dots (8)$$

and are expressed as follows:

$$\begin{Bmatrix} u_2'' \\ \tau_{r\theta}'' \\ \tau_{z\theta}'' \end{Bmatrix} = \begin{Bmatrix} u_2' \\ \tau_{r\theta}' \\ \tau_{z\theta}' \end{Bmatrix}_{\lambda=H} \dots \dots \dots (9)$$

The right side terms in Eq.(9) mean the expressions obtained by putting $h=H$ in Eq.(4).

4. A Method of Solution

In this chapter, we shall show a method of solution to the problem of a finite solid cylinder or a circular thick plate subjected to torsional displacements by a pair of flat annular rigid stamps. Hereafter, we shall simply call the region of consideration a finite solid cylinder because the method of solution proposed here can be applied to the problems with any value of H/C_3 (the case of large values of H/C_3 is a finite solid cylinder and the case of small values of H/C_3 is a circular thick plate).

From the symmetry of the problem with respect to z , we may consider only the domain $0 \leq z \leq H$ of the cylinder. Figure 4 shows a quarter part of the region in consideration. The two-dot chain line of a radius C_3 is the plane which is to become a cylindrical surface of a solid cylinder ($0 \leq r \leq C_3$, $|z| \leq H$), the dotted line of a radius $a' = C_3 + \epsilon_1$, ($\epsilon_1/C_3 \ll 1.0$), represents the plane of distribution of the body forces of Eq.(2) of intensities $g(h)$ and the dotted lines at the plane ($z=H$, $C_1 \leq r \leq C_2$) represent the plane of distribution of the surface force of Eq.(7) of intensities $f(a)$. In the circumstances, the displacements and stresses at any point in the infinite thick plate can be expressed in terms of Green's functions (I) and (II) of Eqs.(4),(5) as follows:

$$\begin{Bmatrix} u_2 \\ \tau_{r\theta} \\ \tau_{z\theta} \end{Bmatrix} = \int_{C_1}^{C_2} f(a) \begin{Bmatrix} u_2'' \\ \tau_{r\theta}'' \\ \tau_{z\theta}'' \end{Bmatrix} da + \int_0^H g(h) \begin{Bmatrix} u_2' \\ \tau_{r\theta}' \\ \tau_{z\theta}' \end{Bmatrix}_{a=a'} dh \dots \dots \dots (10)$$

Fig. 4 A quarter part of a thick plate and the planes of distributions of body forces and surface forces

Since Green's functions (I) and (II) of Eqs.(4), (9) satisfy fundamental equations of elasticity and the boundary conditions (ii) of Eq.(1), Eq.(10) also satisfies those mentioned above regardless of the values of $f(a)$ and $g(h)$. Accordingly, if we determine the values of $f(a)$ and $g(h)$ so as to satisfy the boundary conditions (i) and (iii) of Eq.(10), then all the boundary conditions of Eq.(1) are satisfied.

Application of the boundary conditions (i) and (ii) of Eq.(1) to Eq.(10) yields the following dual integral equations for unknowns of $f(a)$ and $g(h)$:

$$(i) \quad C_1 \leq r \leq C_2;$$

$$\omega r = \int_{C_1}^{C_2} f(a) (u_2'')_{z=H} da + \int_0^H g(h) (u_2')_{a=a', z=H} dh \dots \dots \dots (11)$$

$$(ii) \quad 0 \leq Z \leq H;$$

$$0 = \int_{C_1}^{C_2} f(a) (\tau_{r\theta}'')_{r=C_3} da + \int_0^H g(h) (\tau_{r\theta}')_{a=a', r=C_3} dh \dots \dots \dots (12)$$

To solve the integral equations we shall use a numerical method of solution because it is difficult to solve it rigorously. To reduce the integral equation to a set of linear algebraic equations, we perform appropriate discretizations of the body force $g(h)$ and the surface force $f(a)$. For this purpose, we shall divide the plane subjected to surface forces from the bottom of a rigid stamp into N intervals ($a_n - a_{n-1}$, $C_1 \leq a_n \leq C_2$; $n=1 \sim N$, $a_0 = C_1$, $a_N = C_2$) and the plane of distribution of body forces into M intervals ($h_m - h_{m-1}$, $0 \leq h_m \leq H$; $m=1 \sim M$, $h_0 = 0$, $h_M = H$), and assume¹⁸ that the magnitudes of the surface forces and the body forces distributed are constant in all intervals divided. That is, the surface forces and the body forces are represented by N and M steps of distributed forces as follows:

$$\left. \begin{aligned} f(a) &= \sum_{n=1}^N f_n [S(a - a_{n-1}) - S(a - a_n)] \\ g(h) &= \sum_{m=1}^M g_m [S(h - h_{m-1}) - S(h - h_m)] \end{aligned} \right\} \dots \dots \dots (13)$$

where $S()$ is Heviside's step function, and f_n and g_m stand for N and M unknown constants representing the intensities of the surface and body forces distributed.

Substitution of Eq.(13) into the dual integral equations of Eqs.(11),(12) yields

$$(i) \quad C_1 \leq r \leq C_2;$$

$$\omega r = \sum_{n=1}^N f_n (u_2'')_{z=H} + \sum_{m=1}^M g_m (u_2')_{z=H} \dots \dots (14)$$

$$(ii) \quad 0 \leq Z \leq H;$$

$$0 = \sum_{n=1}^N f_n (\tau_{r\theta}'')_{r=C_3} + \sum_{m=1}^M g_m (\tau_{r\theta}')_{r=C_3} \dots \dots (15)$$

where $(u_2'')^n$ and so on represent the following expressions:

$$\left. \begin{aligned} \begin{Bmatrix} (u_2'')^n \\ (\tau_{r\theta}'')^n \\ (\tau_{z\theta}'')^n \end{Bmatrix} &= \int_{a_{n-1}}^{a_n} \begin{Bmatrix} u_2'' \\ \tau_{r\theta}'' \\ \tau_{z\theta}'' \end{Bmatrix} da \\ \begin{Bmatrix} (u_2')^m \\ (\tau_{r\theta}')^m \\ (\tau_{z\theta}')^m \end{Bmatrix} &= \int_{h_{m-1}}^{h_m} \begin{Bmatrix} u_2' \\ \tau_{r\theta}' \\ \tau_{z\theta}' \end{Bmatrix}_{a=a'} dh \end{aligned} \right\} \dots \dots \dots (16)$$

Since there are $(N+M)$ unknown constants of f_n and g_m in Eqs.(14),(15), we shall con-

sider $(N+M)$ points at the mid-point of $(N+M)$ intervals mentioned previously, and determine the values of f_n and g_m so as to satisfy the boundary conditions (i) and (iii) of Eq. (1) at these selected points ($C_1 < r_1 < C_2$, $i = 1 \sim N$; $0 < z_j < H$, $j = 1 \sim M$). It follows from the above that we can construct a set of simultaneous linear algebraic equations. Substituting these solutions into the expressions

$$\begin{pmatrix} u_2 \\ \tau_{r\theta} \\ \tau_{z\theta} \end{pmatrix} = \sum_{n=1}^N f_n \begin{pmatrix} (u_2'')^n \\ (\tau_{r\theta}'')^n \\ (\tau_{z\theta}'')^n \end{pmatrix} + \sum_{m=1}^M g_m \begin{pmatrix} (u_2')^m \\ (\tau_{r\theta}')^m \\ (\tau_{z\theta}')^m \end{pmatrix} \quad (17)$$

then stresses and displacements at any point can be obtained.

The torque T necessary to twist the rigid stamps is given by

$$T = 2\pi \int_{C_1}^{C_2} r^2 (\tau_{z\theta})_{z=H} dr = 2\pi \int_0^{C_2} r^2 (\tau_{z\theta})_{z=0} dr \quad (18)$$

It must be noted that Green's functions (II) of Eq. (9) have a singularity at the points on the plane surface where a point $r=a$ of acting surface forces coincides with an observation point $r=r$. In order to avoid the singularity, we shall take the surface $z/H = 1.0 - \varepsilon_2$, ($\varepsilon_2 \ll 1.0$), instead of the real surface $z/H = 1.0$, as the boundary of the plane surface of the cylinder. It seems that the results approach an exact solution when $\varepsilon_2 \rightarrow 0$.

5. Numerical Results

Based on the method of solution described above, numerical calculations were performed for cylinders with various sizes under the assumption of $C_1/C_2 = 0.5$. In the numerical calculations, we assumed that $N=80$, $M=5 \sim 15$, $\varepsilon_1/C_2 = 0.1$ and $\varepsilon_2/H = 0.0001 \sim 0.005$. The integrals in Eqs. (16), (18) were calculated by Simpson's formulas of integration.

Fig. 5 Distributions of displacements u_2

Fig. 6 Deformations of the plane surfaces

The results are shown in Figs. 5-12. These figures, except Fig. 12, refer to the case that the radius of the cylinder are $C_3/C_2 = 2.0$.

Figure 5 shows the distribution of displacements u_2 of a solid cylinder with the size of $H/C_2 = 2.0$. The displacements u_2 are nearly in proportion to r in the range of $z/H < 0.6$ and are close to the condition for the deformation of the simple theory of torsion of a shaft

$$u_2 = \frac{Z}{H} \omega r, \quad \tau_{z\theta} = \frac{\mu \omega}{H} r, \quad \tau_{r\theta} = 0 \quad (19)$$

It is said that in the cylinder with sizes of $H/C_2 < 1.0$ the deformations of this kind of the simple theory seldom appear regardless of the value z , although figures for explanation are omitted.

Figure 6 shows the distribution of displacements u_2 at the end face ($z/H = 1.0$) of the cylinder of the sizes of $H/C_2 = 0.5, 1.0$ and 2.0 . For comparison, the results for a half space ($C_3 \rightarrow \infty, H \rightarrow \infty$) and an infinite thick plate ($C_3 \rightarrow \infty$) are shown by black and white circles respectively. In the region of $r/C_2 > 1.0$, these results are very different from each other, whereas there are little differences in $r/C_2 < 1.0$. That is, in the region of $r/C_2 > 1.0$, (i) in the case of $H/C_2 = 2.0$, the

Fig. 7 Distributions of stresses $\tau_{r\theta}$

Fig. 8 Distributions of stresses $\tau_{r\theta}$ at the plane surface ($z = H$)

displacements u_2 of the solid cylinder are very large compared with those of an infinite thick plate and a half space, whereas the results of the infinite thick plate and the half space nearly agree with each other. This is due to the fact that a rotation of a rigid body appears in the displacements of the solid cylinder on account of the cylindrical surface being free from stresses; (ii) in the case of $H/C_2 = 1.0$, these results for a half space, an infinite thick plate and a finite solid cylinder are different from one another; (iii) in the case of $H/C_2 = 0.5$, the results for the infinite thick plate and the finite solid cylinder come close to each other since the solid cylinder in the case is regarded as a circular plate.

Figure 7 shows the distribution of stresses $\tau_{r\theta}$ of the finite solid cylinder with the size of $H/C_2 = 2.0$. The stress has similar character to those of a half space and an infinite thick plate and has singularities at the edges ($z/H = 1.0$, $r = C_1$ and

Fig. 9 Distributions of stresses $\tau_{r\theta}$ in the case of $H/C_2 = 2.0$

Fig. 10 Distributions of stresses $\tau_{r\theta}$ in the case of $H/C_2 = 0.5$

C_2) of the rigid stamp. Figure 8 shows the distributions of stresses $\tau_{r\theta}$ at the end surface ($z/H = 1.0$) of the cylinder, and the results for the half space (black circles) and the infinite thick plate (white circles, $H/C_2 = 1.0$) are also shown for comparison. It seems from Fig. 8 that the stress $\tau_{r\theta}$ has similar distributions although the height H/C_2 of the cylinder is varied. Note that $\tau_{r\theta} = 0$ at the cylindrical surface ($r/C_2 = 2.0$) of the cylinder whereas $\tau_{r\theta} \neq 0$ at the point ($r/C_2 = 2.0$) for the half space and the infinite thick plate.

Figures 9 and 10 show the distributions of stresses $\tau_{z\theta}$ for the cylinders with the sizes of $H/C_2 = 2.0$ and 0.5 respectively. These figures show that the stress $\tau_{z\theta}$ has singularities at the edges ($z/H = 1.0$, $r = C_1$ and C_2) of the rigid stamp. In the case of $H/C_2 = 2.0$, the stress $\tau_{z\theta}$ is fairly in proportion to r in the region $z/H < 0.4$ of the cylinder and has similar distributions to the simple theory of torsion of a shaft of Eq. (19). The values of the stress $\tau_{z\theta}$ at the vicinity of the cylindrical surface ($r = C_3$) are larger than the values for a half space and an infinite thick plate, due to the finiteness of the cross section of the cylinder. The case of $H/C_2 = 0.5$ of Fig. 10 is supposed to be the results for a circular plate since there are no regions of stress distributions such as in the simple theory

Fig. 11 Contact stress distributions at the bottom of a stamp

Fig. 12 Torque T to rotate a stamp

of torsion of a shaft of Eq.(19). It seems from Fig.10 that the stress $\tau_{z\theta}$ of a circular plate acts practically at the region of the bottom of a stamp and its vicinity.

The solid lines in Fig.11 show the distribution of contact stresses of $(\tau_{z\theta})_{z=H}$ transmitted to the end surface of the cylinder from a rigid stamp in the case of $H/C_2 = 0.5, 1.0$ and 2.0 . For comparison, the results for a half space and an infinite thick plate are shown by black and white circles respectively. In the case of $H/C_2 = 2.0$, the distributions for a half space and an infinite thick plate agree with each other, and are considerably larger than the results for a finite solid cylinder. The distribution for the cylinder is nearly equal to that of an infinite thick plate in the case of $H/C_2 = 1.0$ and agrees with the latter in the case of $H/C_2 = 0.5$.

Figure 12 shows the relation between the torque T required to rotate a stamp and the height H/C_2 of a cylinder. In this figure, the solid lines of $C_3/C_2 = 1.5$ and 2.0 are the present results for a finite solid cylinder, the dotted line indicates the results of a half space, the dot-chain line does the results of an infinite thick plate and the two dot-chain line does the results according to the simple theory of torsion of the shaft with a length $2H$ and a radius C_3 of Eq.(19) as follows:

$$T = \frac{\pi \mu \omega C_3^4}{2H} \dots \dots \dots (20)$$

The torque T for a finite solid cylinder is smaller than that for an infinite thick plate and is larger than that according to the simple theory of Eq.(20), and approaches the results of the simple theory of Eq.(20) when $H/C_2 \rightarrow \infty$. With a decrease of H/C_2 , these three results mentioned above come close to one another.

6. Concluding Remarks

This paper is concerned with the so-called Reissner-Sagoci problem of a finite solid cylinder or a circular plate. The principal results are summarized as follows:

(1) A method of solution is developed for the problem of a finite solid cylinder twisted by a pair of annular rigid stamp, Applying Green's functions for torsional body force and torsional surface force problems of an infinite thick plate.

(2) The distributions of stresses and displacements of the cylinder and the distributions of the contact stresses transmitted to the end surface of the cylinder from a rigid stamp are shown and compared with those of an infinite thick plate and a half space.

(3) The torque required to rotate a stamp is shown and compared with those of an

infinite thick plate, a half space and the simple theory of torsion of a circular shaft. It is shown that the torque for a finite solid cylinder, an infinite thick plate and the simple theory of torsion approach one another when the length (thickness) is very small.

(4) In this paper, we have investigated the case of a pair of stamps with the same sizes. We can similarly consider a case of stamps with different sizes.⁽¹⁹⁾

Acknowledgment

The author is indebted to Mr. Minami, M. of Sanken Electric Co. Ltd. for the numerical calculations.

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