Vibration of Shell of Revolution* (Improved Theory)

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Vibration of shell of revolution is studied by improved theory. The meridian of the middle surface is a concave circular arc. Under the assumptions of displacements and stresses proposed by Mirsky, the equations of vibration and the boundary conditions are introduced. Rigorous and approximate solutions are obtained and the relations between the frequencies and each parameter or boundary conditions are discussed.

Key Words: Vibration, Shell, Improved Theory, Shell of Revolution, Mirsky Cylinder

I. Introduction

There are many papers published on vibrations of shells having the form of a surface of revolution, but almost all of them concern the vibrations of cylindrical and conical shells. Kalnins⁽¹⁾ and Cohen⁽²⁾ have investigated vibrations of various shells of revolution by solving eight simultaneous linear ordinary differential equations by the iteration method. There are other papers by Webster⁽³⁾ (the finite element method), Bacon and Bert⁽⁴⁾ (Ritz method), Hamada and Miyata⁽⁵⁾ (the finite difference method) and Hoppmann⁽⁶⁾ who has suggested a concise method.

These studies are done using the classical theory of Bacon and Bert. As an improved theory, Mirsky's research on the vibration of a cylindrical shell is famous. Using Mirsky's improved theory, authors have studied the vibrations of cylindrical and conical shells with variable thickness. Now they study the vibration of a shell of revolution, of which the generator of surface is a concave circular arc. This kind of vibration is very interesting in operating centrifugal separator.

Except Bacon and Bert's paper on the vibration of a paraboloidal shell, the authors suppose there is no research using the improved theory on the vibration of shells, of which the generator of middle surface are curves.

In this paper, the vibrations of a shell with a concave circular surface mentioned above are studied on assumptions that displacements are linear in thickness direction and relations between stresses and strains obey Mirsky's theory. Authors obtain Lagrangian of vibration, equations of vibration and boundary conditions for this shell. They also obtain a rigorous solution in such a special case that the thickness varies in a specific manner. They carry out numerical

calculations on the case that the shell is symmetric about the smallest cross section and show relations of frequencies with many boundary conditions and with various parameters. They also solve approximately the vibration problem by Ritz method and compare the result with the rigorous solution.

2. Coordinate System and Strains

We consider the vibration of a shell, which has a middle surface obtained by rotating a concave circular arc about an axis. We use the Cartesian coordinates $O-xyz$ as shown in Fig.1. We denote an arbitrary point in the shell of revolution by Q , the generator of the middle surface (circular arc) by PA and the center of arc by O' . The center O' is on xy plane, $O'Q$ and $O'O$ intersect the middle surface at P and A , and the plane containing point Q and arc PA is perpendicular to xy plane. We denote the radius of generator by R and the length OA by a . We use R as the representative length and θ , ϕ and z' in the figure as a new coordinate system to indicate the point. We denote the foot of perpendicular from Q to z axis by B and the length QB by r , where $r = R + a - (R' + z') \cos \theta$. The linear elements at Q in θ , ϕ and z' directions are $(R + z')d\theta$, $rd\phi$ and dz' , respectively.

We denote the circular frequency and time by p' and t , and displacements in θ , ϕ and z directions by $(u, v$ and $w) \sin p't$, respectively.

Fig.1

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Naturally, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are functions of θ , ϕ and z' , but we assume that $\bar{u}$ and $\bar{v}$ are approximately linear in z' direction according to Mirsky and that $\bar{w}$ is independent of z' . From these assumptions, we obtain the following relations, where u , v , w , u_0 and v_0 are functions of θ and ϕ only.

$$\bar{u} = u + z' u_0 / R, \quad \bar{v} = v + z' v_0 / R, \quad \bar{w} = w. \quad (1)$$

The strains denoted by θ , ϕ and z' coordinates are

$$\left. \begin{aligned} A &= (\partial u / \partial \theta + z' \partial u_0 / R \partial \theta + w) / (R + z'), \\ B &= (\partial v / \partial \phi + z' \partial v_0 / R \partial \phi) / r, \\ C &= (\partial u / \partial \phi + z' \partial u_0 / R \partial \phi) / r \\ &\quad + (\partial v / \partial \theta + z' \partial v_0 / R \partial \theta) / (R + z'), \\ D &= (-u + \partial w / \partial \theta + u_0) / (R + z'), \\ E &= (\partial w / \partial \phi) / r + v_0 / R, \\ F &= (u + z' u_0 / R) / r, \\ G &= (v + z' v_0 / R) / r, \quad H = w / r. \end{aligned} \right\} \quad (2)$$

$$\varepsilon_\theta = A, \quad \varepsilon_\phi = B + F \sin \theta - H \cos \theta, \quad \varepsilon_{z'} = D,$$

$$\gamma_{\theta\phi} = E + G \cos \theta, \quad \gamma_{z'\theta} = C,$$

$$\gamma_{\theta z'} = C - G \sin \theta. \quad (3)$$

It is easy to see that these strains coincide with ones of Mirsky's cylindrical shell⁽⁸⁾ in the following special case. We use the variable x' , for θ , where $x' = R\theta/a$ and Ru_0/a and Rv_0/a for u_0 and v_0 . The middle surface of the shell of revolution is cylindrical when R is very large and θ is very small. In this case, $\cos \theta \approx 1$, $R \sin \theta \approx ax'$, $r \approx a(1 - z'/a)$, $R \pm z' \approx R$, $\partial / (R + z') \approx \partial / a \partial x'$ and so we obtain $A = (\partial u / \partial x' + z' \partial u_0 / a \partial x') / a$, $B = (\partial v / \partial \phi + z' \partial v_0 / a \partial \phi) / a(1 - z'/a)$, etc. from Eq. (2). Substituting these A , B etc. into Eq. (3), we obtain equations of ε_x , ε_ϕ , $\gamma_{\theta z}$, $\gamma_{z\theta}$, $\gamma_{x\theta}$ similar to those in the paper mentioned above. On calculating ε_ϕ and $\gamma_{\theta\phi}$, we neglect terms $F \sin \theta$ and $G \sin \theta$ because θ is very small. Thus, we find that

$$\begin{aligned} -\delta L / (E' / (1 - \nu^2)) &= \iint (E_1 \delta u + E_2 \delta v + E_3 \delta w + E_4 \delta u_0 + E_5 \delta v_0) d\theta d\phi \\ &\quad - \int [T_1 \delta u + T_2 \delta v + T_3 \delta w + M_1 \delta u_0 + M_2 \delta v_0] d\theta \\ &\quad - \int [T_4 \delta u + T_5 \delta v + T_6 \delta w + M_3 \delta u_0 + M_4 \delta v_0] d\theta = 0 \end{aligned} \quad (7)$$

where

$$\begin{aligned} \zeta &= (1 - \nu) / 2, \quad k = 1 + a / R, \quad \alpha' = \gamma p'^2 R^2 (1 - \nu^2) / g E', \\ T_1 &= \left\{ h (\cos \theta - k) - k \frac{h^3}{12 R^2} \right\} \left(\frac{\partial u}{\partial \theta} + w \right) \\ &\quad - h (1 - 2\zeta) \left(\frac{\partial v}{\partial \phi} + u \sin \theta - w \cos \theta \right) + k \frac{h^3}{12 R^2} \frac{\partial u_0}{\partial \theta}, \\ T_2 &= \zeta \left[-h \left(\frac{\partial u}{\partial \phi} - v \sin \theta \right) + \left\{ h (\cos \theta - k) - k \frac{h^3}{12 R^2} \right\} \frac{\partial v}{\partial \theta} + k \frac{h^3}{12 R^2} \frac{\partial v_0}{\partial \theta} \right], \\ T_3 &= k' \zeta \left\{ h (\cos \theta - k) - k \frac{h^3}{12 R^2} \right\} \left(-u + \frac{\partial w}{\partial \theta} + u_0 \right), \\ M_1 &= \frac{h^3}{12 R^2} \left\{ k \left(\frac{\partial u}{\partial \theta} + w \right) + (\cos \theta - k) \frac{\partial u_0}{\partial \theta} - (1 - 2\zeta) \left(\frac{\partial v_0}{\partial \phi} + u_0 \sin \theta \right) \right\}, \\ M_2 &= \zeta \frac{h^3}{12 R^2} \left\{ k \frac{\partial v}{\partial \theta} - \frac{\partial u_0}{\partial \phi} + v_0 \sin \theta + (\cos \theta - k) \frac{\partial v_0}{\partial \theta} \right\}, \\ T_4 &= \zeta \left[\left\{ \frac{h}{\cos \theta - k} + \frac{h^3}{12 R^2} \frac{k \cos \theta}{(\cos \theta - k)^3} \right\} \left(\frac{\partial u}{\partial \phi} - v \sin \theta \right) \right. \\ &\quad \left. - h \frac{\partial v}{\partial \theta} - \frac{h^3}{12 R^2} \frac{k}{(\cos \theta - k)^2} \left(\frac{\partial u_0}{\partial \phi} - v_0 \sin \theta \right) \right], \\ T_5 &= -h (1 - 2\zeta) \left(\frac{\partial u}{\partial \theta} + w \right) + \left\{ \frac{h}{\cos \theta - k} + \frac{h^3}{12 R^2} \frac{k \cos \theta}{(\cos \theta - k)^3} \right\} \left(\frac{\partial v}{\partial \phi} + u \sin \theta - w \cos \theta \right) \\ &\quad - \frac{h^3}{12 R^2} \frac{k}{(\cos \theta - k)^2} \left(\frac{\partial v_0}{\partial \phi} + u_0 \sin \theta \right), \\ T_6 &= k' \zeta \left\{ \frac{h}{\cos \theta - k} + \frac{h^3}{12 R^2} \frac{k \cos \theta}{(\cos \theta - k)^3} \right\} \left(\frac{\partial w}{\partial \phi} + v \cos \theta - v_0 (\cos \theta - k) \right), \\ M_3 &= \zeta \frac{h^3}{12 R^2} \left\{ \frac{-k}{(\cos \theta - k)^2} \left(\frac{\partial u}{\partial \phi} - v \sin \theta \right) + \frac{1}{\cos \theta - k} \left(\frac{\partial u_0}{\partial \phi} - v_0 \sin \theta \right) - \frac{\partial v_0}{\partial \theta} \right\}, \\ M_4 &= \frac{h^3}{12 R^2} \left\{ \frac{-k}{(\cos \theta - k)^2} \left(\frac{\partial v}{\partial \phi} + u \sin \theta - w \cos \theta \right) \right. \\ &\quad \left. - (1 - 2\zeta) \frac{\partial u_0}{\partial \theta} + \frac{1}{\cos \theta - k} \left(\frac{\partial v_0}{\partial \phi} + u_0 \sin \theta \right) \right\}. \end{aligned} \quad (8)$$

$$\quad (9)$$

Mirsky's cylindrical shell is a special case of the shell of revolution and that the report in this paper concerns the case of Mirsky's theory being applied to a shell of revolution.

3. Lagrangian, Equations of Vibration and Boundary Conditions

Denoting density by γ/g and normal and tangential stresses by σ and τ respectively, we obtain Lagrangian of vibration in a period as follows

$$\begin{aligned} L &= (1/2) \iiint ((\gamma p'^2 / g) (\bar{u}^2 + \bar{v}^2 + \bar{w}^2) \\ &\quad - \sigma_\theta \varepsilon_\theta - \sigma_\phi \varepsilon_\phi - \sigma_{z'} \varepsilon_{z'} - \tau_{\theta\phi} \gamma_{\theta\phi} \\ &\quad - \tau_{\theta z'} \gamma_{\theta z'} - \tau_{z'\theta} \gamma_{z'\theta}) \\ &\quad \times r (R + z') d\theta d\phi dz' \end{aligned} \quad (5)$$

We denote Young's modulus, Poisson's ratio and shear coefficient by E' , ν and k' , respectively. According to Mirsky's improved theory, we assume stresses-strain relations as follows

$$\left. \begin{aligned} \sigma_\theta &= E' (\varepsilon_\theta + \nu \varepsilon_\phi) / (1 - \nu^2), \\ \sigma_\phi &= E' (\nu \varepsilon_\theta + \varepsilon_\phi) / (1 - \nu^2), \\ \tau_{\theta\phi} &= E' \gamma_{\theta\phi} / 2(1 + \nu), \\ \tau_{\theta z'} &= k' E' \gamma_{\theta z'} / 2(1 + \nu), \\ \tau_{z'\theta} &= k' E' \gamma_{z'\theta} / 2(1 + \nu). \end{aligned} \right\} \quad (6)$$

We substitute Eqs. (2), (3), and (6) into Eq. (5).

We denote the thickness at an arbitrary point on the middle surface by h and assume that it is very small compared with other dimensions. Expanding $A \sqrt{H}$ in Eq. (2) into a series of z' , we integrate the Lagrangian L with respect to z' from $(-h/2)$ to $(h/2)$ and neglect terms higher than h^5 . After the variational calculus and the partial integral of Lagrangian, we obtain the following equations

$$\left. \begin{aligned} E_1 &= \frac{\partial T_1}{\partial \theta} + \frac{\partial T_4}{\partial \phi} + T_3 - T_3 \sin \theta + \alpha^4 \left[hu(\cos \theta - k) + \frac{h^3}{12R^2} \{u \cos \theta + u_0(2 \cos \theta - k)\} \right], \\ E_2 &= \frac{\partial T_2}{\partial \theta} + \frac{\partial T_5}{\partial \phi} + T_4 \sin \theta - T_4 \cos \theta + \alpha^4 \left[hv(\cos \theta - k) + \frac{h^3}{12R^2} \{v \cos \theta + v_0(2 \cos \theta - k)\} \right], \\ E_3 &= \frac{\partial T_3}{\partial \theta} + \frac{\partial T_6}{\partial \phi} - T_1 + T_1 \cos \theta + \alpha^4 \left\{ hu(\cos \theta - k) + \frac{h^3}{12R^2} w \cos \theta \right\}, \\ E_4 &= \frac{\partial M_1}{\partial \theta} + \frac{\partial M_2}{\partial \phi} - T_3 - M_4 \sin \theta + \alpha^4 \frac{h^3}{12R^2} \{u(2 \cos \theta - k) + u_0(\cos \theta - k)\}, \\ E_5 &= \frac{\partial M_2}{\partial \theta} + \frac{\partial M_4}{\partial \phi} + T_4(\cos \theta - k) + M_3 \sin \theta + \alpha^4 \frac{h^3}{12R^2} \{v(2 \cos \theta - k) + v_0(\cos \theta - k)\}. \end{aligned} \right\} \dots (10)$$

The variation δL must be zero and so we obtain equations of vibrations: $E_1=E_2=E_3=E_4=E_5=0$ and boundary conditions: $T_1 u=T_2 v=T_3 w=M_1 u_0=M_2 v_0=0$ at $\theta=\theta_1$ and θ_2 , where θ_1 and θ_2 are the lower and the upper limits of angular coordinate θ . It is supposed that the shell is complete and that the upper and the lower limits of ϕ are $\pm\pi$, and so the third integrand in Eq.(7) is zero. By the way, we consider a shell element on the middle surface. We can easily show that $T_1 E'/(1-\nu^2)$ is a force acting on the side of $\theta=\text{const}$ along displacement u and M is a quantity proportional to moment about the side, because u_0 is an angular displacement about it. In above equations, h is a function of θ and ϕ .

2. Solution

We denote the representative thickness by $H(=\text{const})$. We shall solve equations of vibration in the case that $h=H\psi$, where $\psi=(1-\cos\theta/k)^2$, because it is the easiest case to solve and when θ is small, the results obtained almost coincide with those a shell with uniform thickness. We assume that displacements are separable as follows, where $2n$ is the number of nodal generators:

$$u, w, u_0 = \{u(\theta), w(\theta), u_0(\theta)\} \sin n\phi, \quad v, v_0 = \{v(\theta), v_0(\theta)\} \cos n\phi.$$

We use the notation that $\beta=12R^2/H^2$. Substituting above equations of $h, u, v, \dots, v_0$ into equations of vibration and multiplying them by ψ^3 , we can reduce them as follows:

$$\begin{aligned} & (1+1/\beta)(G_A d^2 u/d\theta^2 + G_B du/d\theta) + I_A u + I_B dv/d\theta + I_C v + I_D dw/d\theta + I_E w \\ & - (1/\beta)(G_A d^2 u_0/d\theta^2 + G_B du_0/d\theta) + I_F u_0 + I_G v_0 = 0, \\ & -I_B du/d\theta - H_A u + \zeta(1+1/\beta)(G_A d^2 v/d\theta^2 + G_B dv/d\theta) + H_B v + H_C w + I_G u_0 \\ & - (\zeta/\beta)(G_A d^2 v_0/d\theta^2 + G_B dv_0/d\theta) + H_D v_0 = 0, \\ & -I_D du/d\theta + J_A u + H_C v + k'\zeta(1+1/\beta)(G_A d^2 w/d\theta^2 + G_B dw/d\theta) + J_B w \\ & + \{k'\zeta(1+1/\beta) + 1/\beta\} G_A du_0/d\theta + J_C u_0 + J_D v_0 = 0, \\ & (-1/\beta)(G_A d^2 u/d\theta^2 + G_B du/d\theta) + I_F u + I_C v - \{k'\zeta(1+1/\beta) + 1/\beta\} G_A dw/d\theta + K_A w + (1/\beta) \\ & \times (G_C d^2 u_0/d\theta^2 + G_D du_0/d\theta) + K_B u_0 + \{n(1-\zeta)/k\beta\} G_A dv_0/d\theta + \{n(1-8\zeta)/3k\beta\} G_B v_0 = 0, \\ & I_C u - (\zeta/\beta)(G_A d^2 v/d\theta^2 + G_B dv/d\theta) + H_D v + J_D w - \{n(1-\zeta)/k\beta\} G_A du_0/d\theta \\ & - \{n(2+5\zeta)/3k\beta\} G_B u_0 + \{ \zeta/\beta \} (G_C d^2 v_0/d\theta^2 + G_D dv_0/d\theta) + K_C v_0 = 0 \dots \dots \dots (11) \end{aligned}$$

The notations $G_A \sim K_C$ in above equations are as follows, where

$$\begin{aligned} \Phi &= (1-1/\beta)\psi^2/k^2 + 1/k^2\beta, \quad \Phi' = \{(1-2\zeta)/k\}\psi^2, \quad \Phi'' = \{(1-2\zeta)/2k^2\}\psi^2 \\ G_A &= \Phi^2, \quad G_B = (3/2k)\Phi^2 \sin \theta, \quad G_C = \Phi^2, \quad G_D = (5/2k)\Phi^2 \sin \theta, \\ I_A &= \alpha^4 \{ (1-1/\beta)\Phi^2 + \Phi^2/\beta \} - k'\zeta(1+1/\beta)\Phi^2 + \Phi' \cos \theta + \Phi'' \sin^2 \theta - (n^2\zeta + \sin^2 \theta)\Phi, \\ I_B &= \{n(1-\zeta)/k\}\Phi^2, \quad I_C = n \sin \theta \{ \Phi'' - (1+\zeta)\Phi \}, \\ I_D &= (1+k'\zeta)(1+1/\beta)\Phi^2 - \Phi' \cos \theta, \quad I_E = (1+1/\beta)G_B + (\Phi - \Phi'') \sin \theta \cos \theta, \\ I_F &= \alpha^4 (2\Phi^2 - \Phi^2/\beta) + k'\zeta(1+1/\beta)\Phi^2 - (n^2\zeta + \sin^2 \theta)\Phi^2/k^2\beta, \\ I_G &= -\{n(1+\zeta)/k^2\beta\}\Phi^2 \sin \theta, \quad H_A = n \sin \theta \{ \zeta\Phi^2/2k^2 + (1+\zeta)\Phi \}, \\ H_B &= \alpha^4 \{ (1-1/\beta)\Phi^2 + \Phi^2/\beta \} - (\zeta/k)\Phi^2 \cos \theta - (\zeta/2k^2)\Phi^2 \sin^2 \theta \\ & - (n^2 + \zeta \sin^2 \theta + k'\zeta \cos^2 \theta)\Phi, \\ H_C &= n(1+k'\zeta)\Phi \cos \theta - n\Phi', \\ H_D &= \alpha^4 (2\Phi^2 - \Phi^2/\beta) - (n^2 + \zeta \sin^2 \theta)\Phi^2/k^2\beta - k'\zeta k\Phi^2 \cos \theta, \\ J_A &= -k'\zeta(1+1/\beta)G_B - \Phi' \sin \theta + \Phi \sin \theta \cos \theta, \\ J_B &= \alpha^4 \{ (1-1/\beta)\Phi^2 + \Phi^2/\beta \} - (1+1/\beta)\Phi^2 + 2\Phi' \cos \theta - (n^2 k'\zeta + \cos^2 \theta)\Phi, \\ J_C &= k'\zeta(1+1/\beta)G_B + (\Phi^2/k^2\beta) \sin \theta \cos \theta, \quad J_D = n\Phi^2 \{ k'\zeta\Phi + (1/k^2\beta) \cos \theta \}, \\ K_A &= -G_B/\beta + (\Phi^2/k^2\beta) \sin \theta \cos \theta, \\ K_B &= \alpha^4 \Phi^2/\beta + \{(1-2\zeta)/k\beta\}\Phi^2 \cos \theta - k'\zeta(1+1/\beta)\Phi^2 \\ & + \{3(1-2\zeta)/2k^2\beta\}\Phi^2 \sin^2 \theta - (n^2\zeta + \sin^2 \theta)\Phi^2/k^2\beta, \\ K_C &= \alpha^4 \Phi^2/\beta - (\zeta\Phi^2/k\beta) \cos \theta - \{3\zeta\Phi^2/2k^2\beta\} \sin^2 \theta - (n^2 + \zeta \sin^2 \theta)\Phi^2/k^2\beta - k'\zeta k\Phi^2 \cos \theta. \end{aligned} \dots \dots \dots (12)$$

To solve these equations of vibration, we expand each term into a power series of θ . Take for example, the first term of the first equation in Eq.(11). Expanding G_A and u into a series of θ , we can obtain a series from the product of G_A and $d^2 u/d\theta^2$. Collecting the same power terms into a group, which must be zero, we successively obtain equations to determine coefficients of series solutions of u, v etc.

We denote expansions of each equation in Eq.(12) as follows. For the sake of convenience, we use suffixes from 1 to infinity except expansions of G_A and G_C .

$$\left. \begin{aligned} G_A &= G_{A0} + \sum_{m=1}^{\infty} G_{Am} \theta^{2m}, \quad G_C = G_{C0} + \sum_{m=1}^{\infty} G_{Cm} \theta^{2m}, \\ G_B &= \sum_{m=1}^{\infty} G_{Bm} \theta^{2m-1}, \quad G_D = \sum_{m=1}^{\infty} G_{Dm} \theta^{2m-1}, \\ I_A &= \sum_{m=1}^{\infty} I_{Am} \theta^{2m-2}, \quad I_B = \sum_{m=1}^{\infty} I_{Bm} \theta^{2m-2}, \quad I_C = \sum_{m=1}^{\infty} I_{Cm} \theta^{2m-1}, \dots \end{aligned} \right\} \dots \dots \dots (13)$$

The coefficients G_{Am} and G_{Bm} , for example, are as follows:

$$G_A = \psi^4 = (1-1/k)^2 + \sum_{m=1}^{\infty} \{(-1)^m / (2m)!\} \{-3^{2m} / 4k^2 + 3(2^{2m} / 2k^2) - 3/k - 3/4k^3\} \theta^{2m},$$

$$G_B = (3/2k) \psi^4 \sin \theta = \sum_{m=1}^{\infty} \{(-1)^{m-1} / (2m-1)!\} \{3/2k(3^{2m-1} / 4k^2 - 2^{2m-1} / k + 1 + 1/4k^3) \theta^{2m-1}\}$$

The general solution of equations of vibration is a linear combination of solutions of the symmetric and antisymmetric vibrations. We first show a solution on the antisymmetric vibration. We assume that power series of u , v etc. are as follows.

$$u, u_0 = \sum_{m=1}^{\infty} (A_m, D_m) \theta^{2m-2}; v, w, v_0 = \sum_{m=1}^{\infty} (B_m, C_m, E_m) \theta^{2m-1} \quad \dots (14)$$

where A_m, B_m are constants to be determined. Substituting Eqs. (13) and (14) into equations of vibration and collecting the same power of θ , we obtain

$$\left. \begin{aligned} \sum_{m=2}^{\infty} [(2m-2)(2m-3)G_{A0}\{(1+1/\beta)A_m - D_m/\beta\} + f_{1m}] \theta^{2m-4} &= 0, \\ \sum_{m=2}^{\infty} [-(2m-2)I_{B1}A_m + \zeta(2m-1)(2m-2)G_{A0}\{(1+1/\beta)B_m - E_m/\beta\} + f_{2m}] \theta^{2m-3} &= 0, \\ \sum_{m=2}^{\infty} [(2m-2)\{-I_{D1}A_m + [k'\zeta(1+1/\beta) + 1/\beta]G_{A0}D_m\} \\ + k'\zeta(1+1/\beta)(2m-1)(2m-2)G_{A0}C_m + f_{3m}] \theta^{2m-3} &= 0, \\ \sum_{m=2}^{\infty} [(1/\beta)(2m-2)(2m-3)(-G_{A0}A_m + G_{C0}D_m) + f_{4m}] \theta^{2m-4} &= 0, \\ \sum_{m=2}^{\infty} [\zeta/\beta(2m-1)(2m-2)(-G_{A0}B_m + G_{C0}E_m) \\ - \{n(1-\zeta)/k\beta\}(2m-2)G_{A0}D_m + f_{5m}] \theta^{2m-3} &= 0, \end{aligned} \right\} \dots (15)$$

$$\left. \begin{aligned} f_{1m} &= \sum_{p=1}^{m-1} \{[(1+1/\beta)(2p-2)((2p-3)G_{A(m-p)} + G_{B(m-p)}) + I_{A(m-p)}]A_p \\ &+ [(-1/\beta)(2p-2)((2p-3)G_{A(m-p)} + G_{B(m-p)}) + I_{E(m-p)}]D_p\} + (2m-3)\{I_{B1}B_{m-1} + I_{D1}C_{m-1}\} \\ &+ \sum_{p=1}^{m-1} \{[(2p-1)I_{B(m-p)} + I_{C(m-p-1)}]B_p + [(2p-1)I_{D(m-p)} + I_{E(m-p-1)}]C_p + I_{G(m-p-1)}E_p\}, \\ f_{2m} &= \sum_{p=1}^{m-1} \{[\zeta(1+1/\beta)(2p-1)((2p-2)G_{A(m-p)} + G_{B(m-p)}) + H_{B(m-p)}]B_p \\ &+ [(-\zeta/\beta)(2p-1)((2p-2)G_{A(m-p)} + G_{B(m-p)}) + H_{D(m-p)}]D_p \\ &+ [(-2p-2)I_{B(m-p-1)} - H_{A(m-p)}]A_p + H_{C(m-p)}C_p + I_{G(m-p)}D_p\}, \\ f_{3m} &= \sum_{p=1}^{m-1} \{[k'\zeta(1+1/\beta)(2p-1)((2p-2)G_{A(m-p)} + G_{B(m-p)}) + J_{B(m-p)}]C_p \\ &+ [(-2p-2)I_{D(m-p-1)} + J_{A(m-p)}]A_p + H_{C(m-p)}B_p \\ &+ [k'\zeta(1+1/\beta) + 1/\beta](2p-2)G_{A(m-p)} + J_{C(m-p)}D_p + J_{D(m-p)}E_p\}, \\ f_{4m} &= \sum_{p=1}^{m-1} \{[(-1/\beta)(2p-2)((2p-3)G_{A(m-p)} + G_{B(m-p)}) + I_{E(m-p)}]A_p \\ &+ [(1/\beta)(2p-2)((2p-3)G_{C(m-p)} + G_{D(m-p)}) + K_{B(m-p)}]D_p\} \\ &+ (2m-3)\{-[k'\zeta(1+1/\beta) + 1/\beta]G_{A0}C_{m-1} + \{n(1-\zeta)/k\beta\}G_{A0}E_{m-1}\} \\ &+ \sum_{p=1}^{m-1} \{I_{C(m-p-1)}B_p + [-[k'\zeta(1+1/\beta) + 1/\beta](2p-1)G_{A(m-p-1)} + K_{A(m-p-1)}]C_p \\ &+ [\{n(1-\zeta)/k\beta\}(2p-1)G_{A(m-p-1)} + \{n(1-8\zeta)/3k\beta\}G_{B(m-p-1)}]E_p\}, \\ f_{5m} &= \sum_{p=1}^{m-1} \{[(-\zeta/\beta)(2p-1)((2p-2)G_{A(m-p)} + G_{B(m-p)}) + H_{D(m-p)}]B_p \\ &+ [\zeta/\beta(2p-1)((2p-2)G_{C(m-p)} + G_{D(m-p)}) + K_{C(m-p)}]E_p + I_{G(m-p)}A_p \\ &+ J_{D(m-p)}C_p - \{n(1-\zeta)/k\beta\}(2p-2)G_{A(m-p)} + \{n(2+5\zeta)/3k\beta\}G_{B(m-p)}]D_p\}. \end{aligned} \right\} \dots (16)$$

It is clear that $p \leq m-1$ and that f_{1m}, f_{2m}, f_{3m} in Eq. (16) are linear combinations of A_p, E_p . So, using the relation that $G_{C0} = (1-1/k)G_{A0}$, we obtain A_m and D_m from the first and the fourth equations in Eq. (15) as follows

$$A_m = ((1-1/k)f_{1m} + f_{4m})/D_E, \quad D_m = (f_{1m} + (\beta+1)f_{4m})/D_E,$$

$$D_E = (2m-2)(2m-3)G_{A0}\{1/\beta - (1-1/k)(1+1/\beta)\},$$

That is, we can express A_m and D_m by linear equations of $A^{\vee}E$, of which suffixes are equal to or smaller than $(m-1)$. Substituting these A_m and D_m into the second, the third and the fifth equations in Eq. (15), we obtain B_m, C_m and E_m .

Suffix m of A_m, B_m etc. obtained from Eq. (15) starts from 2 and A_1, B_1, C_1, D_1 and E_1 are arbitrary. Putting $A=1$ and $B_1=C_1=D_1=E_1=0$, we obtain one of five independent solutions and similarly we can obtain other solutions. We use the notations u_1, v_1 etc. to denote the solution obtained by putting $A=1$ and the other zero and so on. As the general solution of the antisymmetric vibration, we adopt the following linear combination of five independent solutions.

$$\left. \begin{aligned} u &= C_1 u_1 + C_2 u_2 + C_3 u_3 + C_4 u_4 + C_5 u_5, \\ v &= C_1 v_1 + C_2 v_2 + C_3 v_3 + C_4 v_4 + C_5 v_5, \end{aligned} \right\} \dots (17)$$

where C_1, C_5 are arbitrary constants.

Likewise, we can obtain the solution for the symmetric vibration, in which case we must use the following u, v etc. instead of Eq. (14)

$$u, u_0 = \sum_{m=1}^{\infty} (A_m, D_m) \theta^{2m-1}, \quad v, w, v_0 = \sum_{m=1}^{\infty} (B_m, C_m, E_m) \theta^{2m-2},$$

where A, B, C, D and E are arbitrary. After similar calculations mentioned above, we obtain the following equations for $m=2$.

$$\begin{aligned} B_m &= ((1-1/k)f_{2m} + f_{5m})/\zeta G_K G_M, \quad E_m = (f_{2m} + (\beta+1)f_{5m})/\zeta G_K G_M, \\ C_m &= -f_{3m}/k'\zeta(1+1/\beta)G_K, \quad A_m = ((1-1/k)f_{1m} + f_{4m})/G_L G_M, \\ D_m &= (f_{1m} + (\beta+1)f_{4m})/G_L G_M, \quad G_K = (2m-2)(2m-3)G_{A0}, \quad G_L = (2m-1)(2m-2)G_{A0}, \\ G_M &= 1/\beta - (1-1/k)(1+1/\beta), \quad f_{1m} = f_{1m} + (2m-2)(I_{B1}B_m + I_{D1}C_m), \\ f_{4m} &= f_{4m} + (2m-2)\{-[k'\zeta(1+1/\beta) + 1/\beta]C_m + \{n(1-\zeta)/k\beta\}E_m\}G_{A0}. \end{aligned}$$

where

$$\begin{aligned}
 f_{1m} = & \sum_{p=1}^{m-1} \{ [(1+1/\beta)G_p + I_{A(m-p)}]A_p + [(-1/\beta)G_p + I_{F(m-p)}]D_p \\
 & + [(2p-2)I_{B(m-p-1)} + I_{C(m-p)}]B_p + [(2p-2)I_{D(m-p-1)} + I_{E(m-p)}]C_p + I_{G(m-p)}E_p \}, \\
 f_{2m} = & \sum_{p=1}^{m-1} \{ [\zeta(1+1/\beta)G_p + H_{B(m-p)}]B_p + [(-\zeta/\beta)G_p + H_{D(m-p)}]E_p + H_{C(m-p)}C_p \} \\
 & - (2m-3)I_{B_1}A_{m-1} + \sum_{p=1}^{m-2} \{ -[(2p-1)I_{B(m-p)} + H_{A(m-p-1)}]A_p + I_{C(m-p-1)}D_p \}, \\
 f_{3m} = & \sum_{p=1}^{m-1} \{ [k'\zeta(1+1/\beta)G_p + J_{B(m-p)}]C_p + H_{C(m-p)}B_p + J_{D(m-p)}E_p \} \\
 & - (2m-3)I_{D_1}A_{m-1} + [k'\zeta(1+1/\beta) + 1/\beta](2m-3)G_{A0}D_{m-1} + \sum_{p=1}^{m-2} \{ [-(2p-1)I_{D(m-p)} + J_{A(m-p-1)}]A_p \\
 & + [k'\zeta(1+1/\beta) + 1/\beta](2p-1)G_{A(m-p-1)} + J_{C(m-p-1)}D_p \}, \\
 f_{4m} = & \sum_{p=1}^{m-1} \{ [(-1/\beta)G_p + I_{F(m-p)}]A_p + I_{C(m-p)}B_p \\
 & + [-(k'\zeta(1+1/\beta) + 1/\beta)(2p-2)G_{A(m-p)} + K_{A(m-p)}]C_p \\
 & + [(1/\beta)(2p-1)(2p-2)G_{C(m-p)} + G_{D(m-p)}] + K_{B(m-p)}D_p \\
 & + [n(1-\zeta)/k\beta](2p-2)G_{A(m-p)} + [n(1-8\zeta)/3k\beta]G_{B(m-p)}E_p \}, \\
 f_{5m} = & \sum_{p=1}^{m-1} \{ [(-\zeta/\beta)G_p + H_{D(m-p)}]B_p + J_{D(m-p)}C_p \\
 & + [(\zeta/\beta)(2p-2)(2p-3)G_{C(m-p)} + G_{D(m-p)}] + K_{C(m-p)}E_p \} \\
 & - [n(1-\zeta)/k\beta](2m-3)G_{A0}D_{m-1} + \sum_{p=1}^{m-2} \{ I_{C(m-p-1)}A_p \\
 & + [-(n(1-\zeta)/k\beta)(2p-1)G_{A(m-p-1)} - \{n(2+5\zeta)/3k\beta\}G_{B(m-p-1)}]D_p \}, \\
 G_p = & (2p-1)\{(2p-2)G_{A(m-p)} + G_{B(m-p)}\}, \quad G_0 = (2p-2)\{(2p-3)G_{A(m-p)} + G_{B(m-p)}\}.
 \end{aligned}
 \tag{18}$$

5. Numerical Calculations

Numerical calculations are carried out on the case that the shell of revolution is symmetric about xy plane in Fig.1. The numerical results obtained from the rigorous solution mentioned in the previous chapter is shown in tables and figures, and the one obtained by Ritz method is also shown when the upper and the lower circles of the shell are built-in. The values of ζ and k' used in the calculations, are 0.35 (Poisson's ratio=0.3) and 0.822 (the value given by Mirsky), respectively.

It seems that the following procedure is easiest to use to obtain frequencies from the rigorous solution. After we calculate the coefficients A_m, B_m etc. of displacements u, v etc. for given values of n, β, k and α , we can obtain the frequency equation by substituting these displacements into boundary conditions. From the frequency equation, we can obtain θ_0 i.e. the value of θ which makes it equal to zero. It is the value of θ at the upper circle of the shell. Considering the symmetric shell we ought to calculate the frequency equation of the symmetric vibration or the antisymmetric one, separately. In both case, the frequency equation is a determinant consisting of five rows and columns. When k is small the shell is slender and when β is small it is thick.

To obtain frequencies, 70 or 110 terms of coefficients A_m-B_m are used. while these coefficients are 1 or zero when $m=1$, their absolute values are maximums when $m=6$ and 4 and some of them are about 10^{-10} . When m is nearly 70, they are about 10^{-40} . The convergencies of displacements u, v etc. are very good. Even when $\theta=2$, equations of vibration, which consist of the second derivative of θ , are satisfied to more than 10 decimals. But the convergency of a

α	$T_1T_2M_1M_2$	$T_1T_2M_1v_0$	$uT_2M_1v_0$	$uT_2M_1M_2$	$uT_2u_0M_2$	$uT_2u_0v_0$	$T_1T_2u_0v_0$	$T_1T_2u_0M_2$
2.0	0.23417	0.23418	0.23246	0.23244	0.13276	0.13281	0.13355	0.13350
1.6	0.36903	0.36912	0.36334	0.36324	0.17474	0.17491	0.17664	0.17646
1.4	0.48653	0.48674	0.47573	0.47547	0.20424	0.20458	0.20740	0.20705
1.2	0.61080	0.61130	0.61099	0.61042	0.24316	0.24385	0.24881	0.24810
1.0	0.74735	0.74869	0.74862	0.74723	0.29729	0.29885	0.30883	0.30716
2.0	$T_1v_0u_0M_2$	$T_1v_0u_0v_0$	uvu_0v_0	uvu_0M_2	uvM_1M_2	uvM_1v_0	$T_1v_0M_1v_0$	$T_1v_0M_1M_2$
2.0	0.13350	0.13355	0.13282	0.13277	0.33286	0.33287	0.33304	0.33304
1.6	0.17646	0.17664	0.17495	0.17478	0.43855	0.43862	0.43872	0.43865
1.4	0.20705	0.20739	0.20468	0.20434	0.51294	0.51312	0.51318	0.51301
1.2	0.24810	0.24882	0.24412	0.24342	0.61107	0.61152	0.61155	0.61110
1.0	0.30718	0.30885	0.29976	0.29814	0.74762	0.74890	0.74890	0.74763
2.0	$T_1T_2M_1M_2$	$T_1T_2M_1v_0$	$uT_2M_1v_0$	$uT_2M_1M_2$	$uT_2u_0M_2$	$uT_2u_0v_0$	$T_1T_2u_0v_0$	$T_1T_2u_0M_2$
2.0	0.23411	0.23412	0.23220	0.23219	0.23227	0.23227	0.23412	0.23411
1.6	0.34921	0.34928	0.34402	0.34398	0.36097	0.36102	0.36829	0.36823
1.4	0.40857	0.40869	0.40282	0.40273	0.46072	0.46074	0.47571	0.47570
1.2	0.48658	0.48676	0.47809	0.47795	0.56854	0.56854	0.58141	0.58139
1.0	0.59504	0.59530	0.58014	0.57994	0.70107	0.70108	0.71681	0.71679
2.0	$T_1v_0u_0M_2$	$T_1v_0u_0v_0$	uvu_0v_0	uvu_0M_2	uvM_1M_2	uvM_1v_0	$T_1v_0M_1v_0$	$T_1v_0M_1M_2$
2.0			0.29941	0.29938	0.26394	0.26399	0.26571	0.26565
1.6	0.41026	0.41031	0.40656	0.40652	0.34683	0.34693	0.34990	0.34980
1.4	0.48611	0.48616	0.48162	0.48157	0.40482	0.40496	0.40914	0.40900
1.2	0.58555	0.58560	0.57994	0.57990	0.48091	0.48110	0.48740	0.48721
1.0	0.72240	0.72244	0.71525	0.71520	0.58591	0.58619	0.59654	0.59625

Table 1 Relation between boundary conditions and θ_0 . Antisymmetric vibration of shell of revolution in which $n=2, \beta=500$ and $k=4.5$. The boundary condition is $T_3=0$ in the upper half of table and $w=0$ in the lower half. The notation $T_1T_2M_1M_2$ denotes the boundary condition that $T_1=T_2=M_1=M_2=0$.

frequency equation which consists of only the first derivative is not good. When $\theta = \pi/2$, the value of θ obtained is doubtful. The reason is that preceding decimals disappear in calculation of the determinant. When θ_0 is smaller than or nearly equal to 0.8, its value obtained when $m=70$ is different from the one obtained when $m=110$ at the fifth decimal by 1, or they are almost equal to each other.

The effects of parameters on the convergency of solution are as follows. When β becomes larger, the convergency becomes worse. The largest value of β is 500 in rigorous solutions of this paper. When k is smaller, it becomes worse. When k is smaller than 4, the obtainable values of θ_0 are less than 0.3. A change of n has little effect and the convergency is similar to the case of $n=2$ to 10. Moreover, when α becomes larger than 3, or when the mode is higher, the convergency is worse. The authors are only able to calculate to the second mode of both symmetric and antisymmetric vibrations. The computer used is ACOS series 77, NEAC system 900.

Frequencies of the shell with both built-in ends are also calculated by Ritz method. For trial functions, the following equations are used in the antisymmetric vibration, where $x = \theta/\theta_0$,

Fig.2 Relation between α and β . (α is a nondimensional frequency and θ_0 is the value of θ at both ends. This is the same in following figures.)

Fig.3 Relation between α and n .

$$\begin{aligned} u, u_0 &= [(A_0, D_0)(1 - \bar{x}^2) \\ &\quad + (A_1, D_1)(-\bar{x}^2 + \bar{x}^4)] \sin n\phi, \\ v, v_0 &= [(B_0, E_0)(\bar{x} - \bar{x}^3) \\ &\quad + (B_1, E_1)(-\bar{x}^3 + \bar{x}^5)] \cos n\phi, \\ w &= [C_0(\bar{x} - \bar{x}^3) + C_1(-\bar{x}^3 + \bar{x}^5)] \sin n\phi. \end{aligned} \quad (19)$$

In the symmetric vibration, odd powers of $\bar{x}$ in equations of u and u_0 must be used instead of even powers and on the contrary, even powers in other displacements. It is clear that Eq.(19) satisfies boundary conditions and so we obtain the frequency equation of 10×10 determinant by calculating

$$L_1 = \iint (E_1 u + E_2 v + \dots + E_5 v_0) d\theta d\phi. \quad (20)$$

The integration of Eq.(20) about θ is carried out by Simpson's method with 128×2 equal divisions. In Ritz method, α is calculated for the given θ_0 .

The calculation of frequencies of the shell with the variable thickness by Ritz method takes about equal time to the time taken for the uniform shell. After the calculation about the antisymmetric vibration, we can easily obtain frequencies in the symmetric one by changing some statements of the computer program. The frequencies of the following two shells are almost equal to each other: one of them is uniform and the other with the variable thickness as mentioned in the previous chapter, and their thicknesses at xy plane are equal to each other. When $k=4.5$ and $\theta_0 \leq 1$, their difference is about 2~3%.

k	4.5	6.5	8.5	10.5
1st	0.385163	0.389085	0.391106	0.392335
2nd	0.698176	0.702815	0.705978	0.708031

Table 2 Value of θ_0 : relation between k and θ_0 . ($\alpha=2$, antisymmetric vibration. $k=2$, $\beta=200$)

Fig.4 Comparison of the approximate solution with the rigorous one. (both built-in ends)

Fig.5 Mode shapes (symmetric vibration with both built-in ends)

Fig.6 Mode shapes (antisymmetric vibration with both built-in ends)

The effects of boundary conditions on the value of θ_0 are shown in Table 1, where $n=2$, $\beta=500$ and $k=4.5$ and the vibration is antisymmetric in the first mode. According to the boundary condition that $T_3 w=0$, the table is divided into the upper half and the lower one. The former is the value when $T_3=0$ and the latter, when $w=0$. The first row denotes other boundary conditions and $TTMM$, for example, denotes $T_1=T_2=M_1=M_2=0$ at $\theta=\pm\theta_0$. It is clear from the Table that the boundary conditions $T_3 w=0$ has a great effect on the value of θ_0 , that is, on the frequency. The condition $M_2 v=0$ or $T_1 u=0$ has a little effect on θ_0 . Comparing two modes of vibration where $M_2=0$ or $v=0$ and the other boundary conditions are same, we can recognize little difference between them. The effect of boundary conditions is similar to the case of other values of n , and k and of the symmetric vibration.

The effect of k on θ_0 is shown in Table 2, where $k \geq 4.5$. J_c is an antisymmetric vibration, where $n=2$, $\beta=200$ and $u=v=w=u_0=v_0=0$ (built-in at both ends). The upper row shows the first mode and the lower, the second. When k is large, the effect of k on the value of θ_0 is little.

In Fig.2, the effect of β on the frequency α is shown. They are the first and the second modes of antisymmetric vibrations with both built-in ends, where $n=2$, $k=4.5$ and $\beta=200 \sim 500$. As β becomes large, α becomes small.

In Fig.3, the relation between n and α is shown, where n is the number of nodal generators. Because three curves of $n=2 \sim 4$ are almost the same in the figure, only the curve of $n=2$ is illustrated for these curves. They are the first and the second modes of antisymmetric vibrations with both free ends ($T_1=T_2=T_3=M_1=M_2=0$), where $\beta=200$ and $k=4.5$. Generally α becomes large as n does. Though the curves of $n=3$ and 4 are neglected in the figure, those curves intersect that of $n=2$ when $\theta_0 < 0.3$ in the first mode. In this case α becomes slightly smaller as n changes from 2 to 4.

The comparison of frequencies obtained from the rigorous solution with those obtained by Ritz method is made in Fig.4, where $n=2$, $\beta=500$ and $k=4.5$. Frequencies by Ritz method are 2~3% larger than those from the rigorous solution, when the shell vibrates in the mode of the symmetric or the antisymmetric vibration. In the second mode, both frequencies differ extraordinarily from each other. As the authors think, the approximation by 10 terms seems to be insufficient. Similar results are obtained in the case that n and β take other values and $k > 4.5$. When $k=2.5$, though the obtainable lower limits of θ_0 from the rigorous solution are $0.25 \sim 0.3$, frequencies obtained by Ritz method agree with the rigorous ones within this limit of θ_0 in the same order mentioned above.

Displacements in the first and the second modes of the symmetric or the antisymmetric vibration of the shell with both built-in ends are shown in Figs.5 and 6, where $n=2$, $\beta=500$ and $k=4.5$. These are the case of $\alpha=2$ and so the value of θ_0 is different according to orders and modes.

Fig.7 Relation between α and k (for small k , both built-in ends)

Fig.8 Comparison of α between a cylindrical shell and a shell of revolution (uniform thickness, both built-in ends)

For small k , its effect on frequencies are studied. It is shown in Fig.7 where $n=2$, $\beta=500$ and both ends of the shell are built-in. When $k=2.5$, the frequency curves are roughly similar to one of $k=4.5$, but when $k<2.5$, they deviate from it as shown in the figure, and this trend is remarkable in the antisymmetric vibration.

The frequencies of a shell of revolution and a cylindrical shell⁽⁹⁾ are compared in Fig.8. They are of same height and uniform thickness. The former is obtained by Ritz method and the latter, from the rigorous solution. The thickness of shells is denoted by H and β_0 in the cylindrical shell is $12a^2/H^2$. When the value of k is determined, we can obtain the value of a/R and hence the value of $\beta=(12R^2/H^2)$ for the given value of β_0 . Thus, we can calculate the numerical values of β in the figure, as $\beta=4000$. The height of the cylindrical shell is similarly denoted by θ_0 . In the symmetric vibration, frequencies of both shells are nearly equal when θ_0 is smaller than 0.2 or 0.25, but to avoid the complication, the plotting of the cylindrical shell is omitted. When θ_0 is larger than 0.2 or 0.25, the frequency of the cylindrical shell is smaller than that of the shell of revolution. In the antisymmetric vibration, the range of calculations is narrow as shown in the figure. The authors have not been able to obtain α in a wider range, but they think that both frequencies are nearly equal to each other when θ_0 is small. When θ_0 is large, α of the shell of revolution is smaller unlike in the symmetric vibration.

6. Conclusions

The authors have rigorously studied the vibration of a shell of revolution, of which the wall is a concave circular arc. They have

also studied it approximately by Ritz method to cover a wider variety of shells. Based on these solutions, they have discussed the effects of boundary conditions and some parameters on the frequency. Parameters used in this paper are thickness, slenderness, number of nodal generators, etc. They have also made the comparison of frequencies between the shell of revolution and the cylindrical shell.

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