Computational Procedure For Large Scale GERT Network Using Graph Theory

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This paper presents an algorithm to analyze the calculation of a large scale GERT network by means of the method of graph theory. The detection of path is made by applying latin square multiplication and the detection of circuit and loop is made by applying the Boolean matrix, reachability matrix and strongly connected matrix. A method is shown to obtain the determinant of network $\Delta(s)$ by the matrix L of touching circuit and loop as well as latin multiplication, and further to obtain $\Delta_k(s)$ by the calculated HP_j of the matrix of touching path, circuit and loop by using the coefficient of $(1 - Y_{ij})$. A numerical investigation demonstrates the applicability of the described methods.

Key Words: Production Systems, Systems Engineering,
Graph Theory

2. Definition of the Problem

1. Introduction

In the design of production systems, many of the matters of the system as the object of analysis include the probabilistic element. For the analysis of these matters, we have GERT (Graphical Evaluation & Review Technique)(1). As examples of analysis of production systems applying GERT, we can mention production control(2), quality control(3), facility layout(4) and so on(5). On the contrary, GERT network contains the feed-back-loop, and accordingly when the production system as the object of analysis gets bigger, the network gets proportionally more large-scaled, and consequently the calculation by the conventional method of analysis of GERT becomes considerably complicated, posing a big obstacle in the practical use. This fact has also restricted the scope of its application. On the other hand, regarding PERT and CPM, an algorithm, which can deal with many nodes, has been developed and practically used.

Under these circumstances, this study will seek a method to find, by introducing the idea of graph theory, a calculating procedure, by which a large scale GERT net can be calculated very efficiently and a computerized analysis can be made.

The node in the GERT network is composed of a combination of input function and output function. The input function comprises the Exclusive-OR, Inclusive-OR and And (logical product), and output function comprises Deterministic case and Probabilistic case. From the combination of input and output, we have six symbols of logics node, which we call logical nodes. When we consider that the logical node used in GERT is the vertex of graph in general, GERT network consists of directed graph. When we consider that the graph is concerned with denumerable set, $X = \{x_1, x_2, \dots, x_n\}$, and multivocal mapping Γ from X to X , and the graph is to be made of the pair of X and Γ , $G = (X, \Gamma)$. When the vertex is a point that corresponds to each element of X and $x_i \in \Gamma x_j$, the two points $x_i, x_j \in X$ are joined by an arrow from x_i to x_j . We call this an arc. Put $u = (x_i, x_j)$, then the graph is also represented as $G = (X, U)$, where the set of arcs is U . By a path we mean a sequence of arcs $(u_1, u_2, \dots, u_n)$ where the terminal extremity of each arc coincides with the initial extremity of the next arc. By a circuit we mean a path where the initial extremity u_1 coincides with the terminal extremity u_k in the sequence of arcs, and by a loop we mean a circuit which comprises a vertex and an arc and in which the initial vertex and the terminal vertex of an arc are the same. Generally, the analysis by GERT will follow the under mentioned procedures:

* Received 6th May, 1982

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Table 1 Six Types of Decision Boxes

Input Output	And	Inclusive OR	Exclusive OR
Determini stic
Probabili stic

- 1) Represent the objective system or problem by a GERT network.
- 2) Collect the data to be given to the branches of the network.
- 3) Make the conversion of equivalence of the network and express the equivalent function by a branch function.
- 4) Convert one or more equivalent functions into a performance measure of the system.
- 5) Using the rating measure obtained from 4), make a rating of the system.

About the above procedures, we shall in this report make an examination mainly of the procedures 3) and 4) out of the networks which have been fixed in the procedures 1) and 2). Here, the basis of calculation is the conditional probability P_a to take a branch between the two vertexes and the W -function of formula (1) defined by the product of the moment generating function $M_t(s)$ of various kinds of parameters attached to the branch.

$$W(s) = P_a \cdot M_t(s) \quad (1)$$

Provided that the moment generating function concerning the parameters becomes the formula (2), where $f(t)$ is a density function of the parameter and s is the operator of Laplace transform.

$$M_t(s) = \begin{cases} \int_0^{\infty} e^{-st} f(t) dt & (t: \text{continuous, variable}) \\ \sum_t e^{-st} f(t) & (t: \text{discrete, variable}) \end{cases} \quad (2)$$

when t is not a density function but a constant,

$$M_t(s) = e^{-st_0}$$

When we use the W -function as the characteristic value of arc, we can find the GERT network as a set of characteristic values of W -function. The rating measure of the system from the initial vertex to the terminal vertex of GERT can be obtained from the equivalent function $W_E(s)$, and the expected value μ is expressed by equation (3), according to whose size the rating of the system will be come possible.

$$\mu = \frac{\partial [W(s)]}{\partial s [W(0)]} \bigg|_{s=0} \quad (3)$$

Regarding $W_E(s)$ which constitutes equation (3), detect the passage, circuit and loop are detected and applying the result $W_E(s)$ is calculate.

3. Detection of Path

For detecting the path from the initial vertex to the terminal vertex of GERT, apply the following theorem 1.

[Theorem 1] In the graph of GERT which has the initial vertex u_s and the terminal vertex u_e , when we have n vertexes, the length of the longest path from u_s to u_e is at most $n-1$. (The proof is omitted.)

The path is to be found by the application of this theorem, and from among various methods to find the path, we shall here apply the modification of the method of latin multiplication by Kaufmann⁽⁶⁾. Now, if the graph has n vertexes, form a matrix of $n \times n$, and if there is any arc from the vertex i to the vertex j for each element of the matrix, plot $i \cdot j$ on the row i column j , and if there is no arc, plot zero. The latin matrix thus formed is represented by $[M]^{(1)}$.

$[M]^{(1)}$ of the graph of PERT indicated in Fig. 1 will be expressed by formula (4).

Fig. 1

$$[M]^{(1)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1.2 & 0 & 0 \\ 0 & 0 & 2.3 & 0 \\ 0 & 3.2 & 0 & 3.4 \\ 4 & 1 & 0 & 0 \end{bmatrix} \end{matrix} \quad (4)$$

$$[\tilde{M}]^{(1)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (5)$$

Let the matrix formed by removing the first figure from the matrix $[M]^{(1)}$ be $[\tilde{M}]^{(1)}$, and consequently formula (4) becomes formula (5). According to Kaufmann, when $[M]^{(2)}$ is obtained multiplying $[M]^{(1)}$ by $[\tilde{M}]^{(1)}$, the figures of $[\tilde{M}]^{(1)}$ are arranged after the figures (i, j) of $[\tilde{M}]^{(1)}$. In this case, if there is not repeat of figures, continue to put the figures of the element of the matrix $[M]$, and if there is repeat of figure (s) , make the element zero. Kaufmann represents this calculation by the signal L . The following shall be calculated in conformity with $[M]^{(2)} = L[\tilde{M}]^{(1)} = [M]^{(3)}$. Let $[M]^{(k)}$ be the path of the length k , then we get formula (6).

$$[M]^{(k+1)} = [M]^{(k)} L[\tilde{M}]^{(1)} \quad (6)$$

Accordingly, when we find the path up to $[M]^{(n-1)}$ by formula 1, we shall get all the paths. By the way, in the graph of GERT, the vertex numbers for the initial vertex and the terminal vertex are known beforehand. Accordingly, let the initial vertex be i and the terminal vertex be j , then the (i, j) element of the matrix $[M]^{(\lambda)}$; $\lambda = 1, 2, \dots, n-1$ will indicate the path from i to j in case where the path from i to j exists. For this reason, we represent the vector of the i -throw out of $[M]^{(\lambda)}$ by $[F]_i^{(\lambda)}$, which will

indicate that this vector has the initial vertex i . Consequently, by equation (7), the path from the initial vertex i to the terminal vertex j can be obtained. The calculation shall be made up to $[F]_i^{(n-1)}$.

$$[F]_i^{(\lambda+1)} = [F]_i^{(\lambda)} L[\tilde{M}]^{(1)} \quad (7)$$

In the graph of Fig. 1, let the initial vertex be 1 and the terminal vertex be 4. Then $[F]_1^{(1)}$, becomes formula (8), where the path from the initial vertex to the terminal vertex will at most be $\lambda = 3$, and so from equation (10) it is found that the path from the initial vertex 1 to the terminal vertex 4 is (1, 2), (2, 3), (3, 4).

$$[F]_1^{(1)} = [0 \ 1 \cdot 2 \ 0 \ 0] \quad (8)$$

$$[F]_1^{(2)} = [F]_1^{(1)} L[\tilde{M}]^{(1)} = [0 \ 0 \ 1 \cdot 2 \cdot 3 \ 0] \quad (9)$$

$$[F]_1^{(3)} = [F]_1^{(2)} L[\tilde{M}]^{(1)} = [0 \ 0 \ 0 \ 1 \cdot 2 \cdot 3 \cdot 4] \quad (10)$$

As mentioned above, regarding the path to be found, a definition of the path vector P_k is given to the path R_k where the graph has n vertexes.

$$P_k = [\beta_{k1}^1 \ \beta_{k2}^2 \ \dots \ \beta_{kn}^n] \quad (11)$$

$$\beta_{kj} = \begin{cases} 1 & \text{where the path } P_k \text{ includes} \\ & \text{the vertex } j \\ 0 & \text{where the path } R_k \text{ does not} \\ & \text{include the vertex } j \end{cases}$$

4. Detection of Circuit and Loop

4.1 Detection of circuit and loop by Boolean matrix

In case there are plural circuits and loops in the graph, to detect all the circuits and loops, the following existing Theorem 2 will be applied.

[Theorem 2] Suppose the graph includes a circuit and a loop. When the length of the circuit s is given in the Boolean matrix M_b , the calculation of M_b^{vs} ($v=1,2,\dots$) will indicate that only s pieces of 1 appears in the diagonal elements.

In case of detecting the existence of a circuit and a loop, since the length of the circuit is at most n when the number of the vertex of the graph is n , if n pieces of 1 appear in the diagonal elements as the result of the calculation of $M_b^1, M_b^2, \dots, M_b^n$, the vertex corresponding to 1 represents the circuit with length s . A loop exists at the vertex corresponding to 1 in the diagonal elements of M_b^1 . On the other hand, M_b^3 of the graph of GERT (hereinafter called graph) represented in Fig. 2 is obtained in the form of equation (12).

Fig. 2

$$M_b^3 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (12)$$

The matrix in which 1 appears in the diagonal elements has $v = 3$, while vertexes 2, 3, 4, 5 appear in the diagonal elements, and there is no coincidence with the value of v . This is because of the appearance of two vertexes in the diagonal elements due to the fact that there exist in M_b^3 two circuits, vertexes 2, 3, 4 and vertexes 3, 4, 5. In case there are plural circuits of the same length as seen here, there appear plural 1 in the diagonal elements of Boolean matrix, so that it is insufficient to detect the circuit only by Boolean matrix. For this reason, it is necessary to examine the contents of the vertex constituting the circuit.

4.2 Detection of circuit applying strong connection

In order to find the circuit in the graph, make use of the property that the vertex included in the circuit is strongly connected. Regarding the strong connection of graph, find R^* from equation (13) by the reachability R and the transpose of matrix, and the (i, j) element of matrix will indicate whether or not the vertex i and vertex j belong to the same strongly connected portion.

$$R^* = R \cap R^T \quad (13)$$

Therefore, make use of the property that the removal of arc included in a circuit composing a strong connection will get rid of the circuit, and then detect the circuit. For this reason, a definition is given on the entrance of circuit.

[Definition 1] Let S be a set of vertexes composing a strong connection in the graph. If any arc comes into a vertex of S from the vertexes other than S , define this vertex as the entrance. Namely, that is y which satisfies $x \notin S, y \in S \Rightarrow (x, y)$.

Since the entrance is included in the strongly connected portion, at least one arc enters the entrance from any of the vertexes in the set S composing the strongly connected portion. The removal of this arc will result in loss of one circuit. Since a strongly connected portion appears on the diagonal elements of the reachability matrix, now represent the strongly connected portion by equation (14), in which the vertexes composing $S_1, S_2, \dots, S_n$ are on the Boolean matrix. In order to find the entrance out of the Boolean matrix M_b of $n \times n$, just like equation (14) adopt w_i , which is the number of the vertexes composing the strongly connected portion S_i , as the number of column of matrix, and concerning the submatrix formed out of the submatrix $(n \times w_i)$ exclusive of the strongly connected portion S_i ($w_i \times w_i$ matrix), represent the submatrix on the higher rows than S_i by U_i and the

lower, by L_i , and then 1 existing in U_i or L_i should be chosen from among w_i columns composing the strongly connected portion. Provided that U_1 does not exist in S_1 which is located first in the diagonal and L_n does not exist in S_n which is located last, so that 1 should be found either in L_1 or in U_n . Secondly, regarding the vertex coming into the entrance from the vertex of the strongly connected portion, namely the vertex composing the arc to be deleted, find 1 which exists in the same row as the vertex of the entrance obtained in Boolean matrix M_b , exists at the same time in the strongly connected S_i , and excludes the arc forming the path from the initial extremity to the terminal extremity, and then find the corresponding row. In the example of equation (14), 1 in U_i lies in row h_α column h_β , so the entrance is in h_β , and if h_γ corresponds to 1 that does not form the path among the strongly connected S_i even in the same column, the arc exists from h_γ to h_β . For that reason, if 1 of row h_γ column h_β is removed out of Boolean matrix, one circuit is to be eliminated. In the same way, remove the paths composing the strongly connected portion in the rest of the Boolean matrix until all the strongly connected portions are deleted completely. For the judgement of the above, there are two methods; one is to make sure that 1 of the diagonal elements no longer exists according to Theorem 2; the other is to make confirmation by equation (13). When there is no more circuit in the strongly connected S_i , take out the submatrix of S_i (square matrix $w_i \times w_i$) out of Boolean matrix and call it $M_{b(i)}^0$. In the next place, in order to check the circuit, let the removed arc be $u_1 = (h_1, h_2)$, $u_2 = (h_3, h_4)$, ..., $u_{c/2} = (h_{c-1}, h_c)$ in order from the last, and let the submatrix without the circuit be $M_{b(1)}^0$ combining the vertex h_1 coming from u_1 into the entrance h_2 , and then let the element of row h_2 column h_1 of $M_{b(1)}^0$ be 1 to form a new $M_b(1)$. From theorem 2 by checking the vertex appearing in the diagonal elements, the vertex of cycle can be found. In the following, in order as $u_2, u_3, \dots$, add one arc u_i , once removed, to the matrix $M_{b(i)}^0$; i.e. S_j to form a new Boolean matrix $M_b(i)$, and then find the circuit by theorem 2. Apply the above mentioned procedure to S_i ; $i=1, 2, \dots, n$ which constitutes the strongly connected portion.

[Example 1] In Fig. 2 the strongly connected portion is composed of the vertexes 2, 3, 4, 5. The line of vertexes of the arc from the initial vertex 1 to the terminal vertex 6 is $U = [1, 2, 3, 4, 5, 6]$ from formula (7), so cir-

cular marks are put to the matrix of equation (15). Let the vertexes 2, 3, 4, 5 of the

$$M_b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & \textcircled{1} & 0 & 0 & 0 \\ 2 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 3 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 4 & 0 & 1 & 0 & 0 & \textcircled{1} \\ 5 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

strongly connected portion be S_1 . Then, 1 in U_1 and L_1 is in the column 2. Secondly, when 1, exclusive of the vertex composing the arc in the strongly connected S_1 is chosen from column 2, it corresponds to row 4, and so the arc coming into the entrance 2 is 4. In the matrix M_b excluding $u_1 = (4, 2)$, the strongly connected portion is found as the vertexes 3, 4, 5. Let them be S_2 , and 1 in U_2 and L_2 lies in the column 3. Secondly, 1 excluding the vertex composing the path in the column 3 of the strongly connected S is 1 in the row 5, so $u_2 = (5, 3)$ is obtained. Further, in the submatrix S of the strongly connected portion, form a

$$M_{b(1)} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (16)$$

$$M_{b(1)}^3 = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (17)$$

matrix $M_{b(1)}^0$ without the arcs u_1, u_2 . Add the removed arc u_2 to the matrix $M_{b(1)}^0$, and the equation (16) is obtained. By calculating $M_{b(1)}^3$ from theorem 2, the equation (17) is obtained. Thus, the circuit (3, 4, 5, 3) is found. Likewise the arc is also found as (2, 3, 4, 2).

4.3 Detection of circuit applying the strongly connected portion and latin multiplication

To the detection of circuit, the method of latin square multiplication used to detect the path and the method of using the strongly connected portion are also applicable. First of all, take out only the submatrix portion composing the strongly connected portion in Boolean matrix, and find Kaufmann's matrix $[M](1)$, $[\tilde{M}](1)$ by the method mentioned in chapter 3. In order to find the circuit from the initial vertex i indicating the entrance of the circuit to the terminal vertex i from $[M](1)$, take out the row where lies the vertex i of matrix and name it Vector $[F]_i$. Hereupon, in the calculation of equation (7), the repeat of the same figure leads to zero, but in the circuit same vertex is common to the initial vertex and the terminal vertex, so the calculation shall be continued until the same vertex appears at first and at last in the combination of letters of latin multiplication. The times calculation for this trial is done up to

$[F]_i^\lambda$ when the number of the vertexes composing the circuit found in the section 4,2 is λ . The calculation of latin square multiplication allowing the repeat of figures like this is represented by the sign U as mentioned in equation (18).⁽⁴⁾

$$[F]_i^{(\lambda)} = [F]_i^{(\lambda-1)} [M]^{(1)} \quad (18)$$

In Fig. 2, when the entrance is 2, the number of vertexes of circuit is $\lambda = 3$, the strongly connected portion of equation (15) will lead to equation (19). The circuit becomes (2,3,4,2). From the above, the calculation of equation (18) will indicate the sequence from the initial vertex to the terminal vertex.

$$\begin{aligned} [F]_2^{(1)} &= [0 \ 2 \cdot 3 \ 0 \ 0], \\ [F]_2^{(2)} &= [0 \ 0 \ 2 \cdot 3 \cdot 4 \ 0] \\ [F]_2^{(3)} &= [0 \ 0 \ 0 \ 2 \cdot 3 \cdot 4 \cdot 2] \end{aligned} \quad (19)$$

5. Detection of Circuit and Loop in Contact

To calculate formula (3) Mason's rule is applied, in which case the touching situation of circuit and loop shall be detected. Accordingly, a definition is given to the contact between circuit and loop.

[Definition 2] In case 2 or more circuits and loops have one or more vertexes in common, the circuits and loops are in contact.

Now, represent the vertexes composing circuits and loops by the vector C_i of equation (20), and let the circuits and loops corresponding to C_i be C_i . Here, let the number of vertexes be n .

$$C_i = [\lambda_{i1} \ \lambda_{i2} \ \dots \ \lambda_{in}] \quad (20)$$

Provided that: $\lambda_{ij} = \begin{cases} 1: \text{The circuit and loop } C_i \text{ include the vertex } j. \\ 0: \text{The circuit and loop } C_i \text{ do not include the vertex } j. \end{cases}$

Whether the circuit and the loop C_i and C_j are in contact or not is found by the Boolean algebra (21). T denotes the transpose of matrix.

$$f_{ij} = C_i \cdot C_j^T \quad (21)$$

Provided that: $f_{ij} = \begin{cases} 1: \text{The circuit and loop } C_i \text{ and } C_j \text{ are in contact.} \\ 0: \text{The circuit and loop } C_i \text{ and } C_j \text{ are not in contact.} \end{cases}$

Now, if p circuits and loops are given, by applying the value of f_{ij} , matrix L can be suggested whose diagonal element is 1 as seen in equation (22).

$$L = \begin{bmatrix} C_1 & C_2 & \dots & C_p \\ 1 & f_{12} & \dots & f_{1p} \\ C_2 & f_{21} & 1 & f_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ C_p & f_{p1} & f_{p2} & \dots & 1 \end{bmatrix} \quad (22)$$

If p vectors C_i of the circuit C_i are given in order to find matrix L , equation (23) shall be defined.

$$H = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p1} & \lambda_{p2} & \dots & \lambda_{pn} \end{bmatrix} \quad (23)$$

Accordingly, matrix L of equation (22) is found from the Boolean algebra of equation (24).

$$L = H \cdot H^T \quad (24)$$

6. Detection of Touching Path, Circuit and Loop

It is necessary to find the contact between path and circuit in order to get an equivalent function. Accordingly, a definition is given to the contact among the path, circuit and loop.

[Definition 3] In case the path R_j and the circuit C_i have one or more vertexes in common, the path P_j and the circuit C_i are in contact.

The vector of path P_j is defined as P_j as mentioned in equation (11). Therefore, regarding the contact between the path P_j and the circuit C_i , if there exist p circuits and loops, the equations (23) and (11) can lead to equation (25). Accordingly, by γ_{ji} calculated from equation (25), contact among the path, circuit and loop can be detected.

$$H P^T = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p1} & \lambda_{p2} & \dots & \lambda_{pn} \end{bmatrix} \begin{bmatrix} \beta_{j1} \\ \beta_{j2} \\ \vdots \\ \beta_{jn} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \begin{bmatrix} r_{j1} \\ r_{j2} \\ \vdots \\ r_{jn} \end{bmatrix} \quad (25)$$

Provided that: $\gamma_{ji} = \begin{cases} 1: \text{The path } R_j \text{ contacts the circuit and the loop } C_i \\ 0: \text{The path } R_j \text{ is not contacting the circuit and the loop } C_i \end{cases}$

7. Calculation of Equivalent Function $W_E(s)$

7.1 The W function of the path, circuit and loop

For the calculation of $W_E(s)$, W function attendant on each arc is applied, and in this case the W function of each arc is given as a parameter matrix.

[Definition 4] The parameter matrix $\tilde{W}(s)$ is, concerning the GERT network with n vertexes, a W function which is attendant on the arc with vertexes joining each other and is described in a $n \times n$ matrix.

The W function attendant on the path, circuit and loop is defined as follows:

[Definition 5] Regarding a sequence of arc $(u_1, u_2, \dots, u_k)$ where the terminal extremity of each arc coincides with the initial extremity of the next arc, when the paths, in each of which the initial extremity coincides with the terminal extremity, comprise K , W functions attendant on each arc $\prod_{j=1}^K W_j(s)$ becomes a W function of the circuit; and when the number of arc is one, the said path composed of one piece of W function becomes a W function of the loop.

[Definition 6] When the path from the initial extremity to the terminal extremity comprises a W function attendant on d pieces of sequence of arc $(u_1, u_2, \dots, u_d)$, $\prod_{j=1}^d W_j(s)$ becomes a W function of a path.

7.2 Calculation of the equivalent function

The calculation of equation (26) representing the equivalent function is made by Mason's rule. Among $\Delta(s)$, $\Delta_k(s)$, $R_k(s)$, constituting the equation, $R_k(s)$ is a W function of the K th path from the initial extremity to the terminal extremity of GERT network, and is obtained by the detection of the path.

$$W_E(s) = \frac{\sum_{k=1}^n R_k(s) \cdot \Delta_k(s)}{\Delta(s)} \quad (26)$$

Accordingly, the calculation of $\Delta(s)$, $\Delta_k(s)$ shall be examined. i) Regarding $\Delta(s)$ the following is known from Mason's rule (7). Generally, $\Delta(s)$ is called matrix of network, the solution of which is obtained by the determinant as equation (27).

$$\Delta(s) = |I_n - \tilde{W}(s)| \quad (27)$$

where I_n is unit matrix.

On the other hand the determinant of equation (27) can be developed as equation (28).

$$\Delta(s) = 1 + \sum_i C_{i1}(s) + \sum_{i_1, i_2} C_{i_1 1}(s) C_{i_2 2}(s) \dots \dots \dots (-)^f \sum_{i_1 i_2 \dots i_f} C_{i_1 1}(s) C_{i_2 2}(s) \dots C_{i_f f}(s) \quad (28)$$

The total $i_1, i_2, \dots, i_f$ shall be obtained as the total of j items circuits and loops $C_i(s)$ which do not have a vertex in common each other, where f is the maximum value of j. This calculation can also be performed as follows: In the matrix L of equation (22), check up 0 located in the higher row than the diagonal elements. Of the element fc_{ij} of matrix L is 0 and at the same time located above the diagonal elements, write $C_i C_j$ the product of C_i and C_j in row C_i column C_j , and if the element fc_{ij} is 1, write 0 likewise. Form a matrix $[D]^{(2)}$ in which the elements below 1 of diagonal elements and the diagonal elements are all 0. Each element of matrix which appears in $[D]^{(2)}$ represents the product of a non-contacting pair of circuit and loop. Form a matrix $[\tilde{D}]^{(1)}$ excluding the leftside letters of the product of pair of circuit and loop, C_i and C_j in $[D]^{(2)}$. In the following, perform the calculation of equation (29) by the same method as the latin square multiplication mentioned in chapter 3.

$$[D]^{(K+1)} = [D]^{(K)} L[\tilde{D}]^{(1)} \quad (K \geq 2) \quad (29)$$

When all the elements of the matrix $[D]^{(\lambda)}$ become 0, stop the calculation, and write down each element of the matrix $[D]^{(j)}$, $j=1, 2, \dots, \lambda-1$ according to equation (30). Then the same result as equation (28) is obtained.

$\Delta(s) = 1 -$ (total of the circuit and loop $C_i(s)$ existing in GERT network) + $(-)^j \prod_{j=2}^{\lambda-1}$ (total of the products of j circuits

and loops, $C_{i1} \cdot C_{i2} \dots C_{ij}$ appearing in the elements of matrix $[D]^{(j)}$).

ii) $\Delta_k(s)$ is a determinant of network obtained from $\Delta(s)$ exclusive of the vertexes and arcs included in the K th path. When $\Delta_k(s)$ is calculated according to the determinant of equation (27), due to the fact that the circuit and loop are expressed as the product of W function, express the circuit and loop C_i in W function by means of equation (23) and parameter matrix $\tilde{W}(s)$, and then convert the product of W function developed by equation (27) with C_i to give an expression like equation (38). When equation (30) is applied, use the result directly obtained by C_i . Let the coefficient of γ_{ji} , which is obtained from HP_j^T calculated by equation (25) using C_j obtained by either the way mentioned above, be $(1 - \gamma_{j1})C_1(s)$, $(1 - \gamma_{j2})C_2(s)$, $\dots$, $(1 - \gamma_{jn})C_n(s)$ and then the value of γ_{ji} , either 0 or 1, should be substituted in all the circuits and loop $C_i; i=1, 2, \dots, n$ which are found by $\Delta(s)$.

8. Calculation of the Rating

For the calculation of the rating, the differentiated value of $W_E(s)$ and value of $W_E(0)$ should be substituted in to equation (3). In this calculation, if the Moment generating function has a density function, the differentiation calculation becomes much complicated. Accordingly, the calculation of the numerator of equation (3) will be made using equation (31).

$$\frac{\partial W_E(s)}{\partial s} \Big|_{s=0} = \frac{\sum_{i=1}^n (R_i'(s) \Delta_i(s) + R_i(s) \Delta_i'(s))}{\Delta(s)} = \frac{\sum_{i=1}^n R_i(s) \cdot \Delta_i(s) \Delta_i'(s)}{\Delta^2(s)} \Big|_{s=0} \quad (31)$$

Here, the calculation of each of the variations of equation (31) is made using the following equations from i) to vi). Provided that the symbol $M_{ij}(s)$ represents a Moment generating function given to the j-th arc of i-th path, and the j-th arc of the i-th circuit and loop; and P_{ij} represents the probability of being relaized on the arc of the same path circuit and loop.

$$\begin{aligned} \text{i) } R_i(s) \Big|_{s=0} &= \prod_{j=1}^{\lambda_i} W_{ij}(0) = \prod_{j=1}^{\lambda_i} P_{ij}; i=1, 2, \dots, n \quad (32) \\ \text{ii) } R_i'(s) \Big|_{s=0} &= \left(\prod_{j=1}^{\lambda_i} P_{ij} \right) \left(\prod_{j=1}^{\lambda_i} M_{ij}(s) \Big|_{s=0} \right); i=1, 2, \dots, n \quad (33) \end{aligned}$$

The equations (32) and (33) are composed of λ_i W function in the i-th path, and indicate the case where there are n lines of paths in all.

iii) The calculation of $\Delta(s) \Big|_{s=0}$ is done using the determinant of equation (27) and it yields.

$$\Delta(s)|_{s=0} = |I_n - \tilde{W}(0)| \quad (34)$$

By the way, regarding equation (34), from equation (1) which represents the property of the W function, each element of the determinant of parameter matrix $\tilde{W}(0)$ becomes $W(0) = Pa$, so that equation (34) is composed only of the probability arising in each arc. On the other hand, the calculation by equation (30) leads to equation (35). Here, due to the fact that C_i is composed of the W function of circuit and loop, represent C_i is composed of the W function of circuit and loop, represent C_i is represented as $C_i(s)$.

$\Delta(s)|_{s=0} = 1 - (\text{total of the circuit and loop } C_i(0) \text{ existing in GERT network}) + (-)^j \sum_{j=2}^n (\text{total of the products of } j \text{ circuits and loops, } C_{i1}(0) \cdot C_{i2}(0) \cdots C_{ij}(0) \text{ appearing in the elements of matrix } [D]^{(j)}).$ (35)

$$\text{Provided that } C_i(0) = \prod_{j=1}^{\lambda_i} W_{ij}(0) = \prod_{j=1}^{\lambda_i} P_{ij} \quad (36)$$

Equation (36) indicates that the i-th circuit and loop are composed of λ_i W functions. iv) The calculation of $\Delta'(s)|_{s=0}$ by the determinant of equation (27) yields equation (37).

$$\Delta'(s)|_{s=0} = \frac{\partial}{\partial s} |I_n - \tilde{W}(s)|_{s=0} \quad (37)$$

On the other hand, when calculated by equation (30), it becomes equation (38).

$\Delta'(s)|_{s=0} = - (\text{sum of circuit and loop } C_i'(s)|_{s=0} \text{ existing in GERT network}) + (-)^j \sum_{j=2}^n (\text{total of the products of } j \text{ circuits and loops } C_{i1}(s) \cdot C_{i2}(s) \cdots C_{ij}(s)|_{s=0} \text{ appearing in the elements of matrix } [D]^{(j)}).$ (38)

$$\text{Provided that } C_i'(s)|_{s=0} = \left(\prod_{j=1}^{\lambda_i} P_{ij} \right) \left(\sum_{j=1}^{\lambda_i} W_{ij}'(s)|_{s=0} \right) \quad (39)$$

$$(C_{i1}(s)C_{i2}(s) \cdots C_{ij}(s))|_{s=0} = C_{i1}'(s)|_{s=0} \cdot C_{i2}(0) \cdots C_{ij}(0) + (C_{i1}(0) \cdot C_{i2}'(s)|_{s=0} \cdots C_{ij}(0))|_{s=0} + \cdots + (C_{i1}(0) \cdot C_{i2}(0) \cdots C_{ij}'(s)|_{s=0})|_{s=0} \quad (40)$$

Equation (39) indicates that the i-th circuit and loop are composed of λ_i W functions. v) Regarding the calculation of $\Delta_k(s)|_{s=0}$, in $\Delta(s)|_{s=0}$ of iii) by the application of the coefficient of equation (25) to the circuit and loop, $C_1(0), C_2(0), \cdots, C_n(0)$ are all replaced with $(1 - \gamma_{j1})C_1(0), (1 - \gamma_{j2})C_2(0), \cdots, (1 - \gamma_{jn})C_n(0)$. Provided that the same equation as equation (36) will be applied to $C_i(0)$.

vi) Regarding $V1) \Delta_k(s)|_{s=0}$, in $\Delta_k(s)|_{s=0}$ of item v) replace $C_1(0), C_2(0), \cdots, C_n(0)$ with $(1 - \gamma_{j1})C_1(0), (1 - \gamma_{j2})C_2(0), \cdots, (1 - \gamma_{jn})C_n(0)$, differentiate the equation, and then substitute $s = 0$ in to the said equation. Here, the calculation is carried out according to equations (39) and (40). In this case, the value of equation (25) is substituted in to γ_{ji} of v) and vi).

9. GERT Procedural Steps

The methods mentioned up to now are arranged as the GERT procedural steps as follows.

Step 1: Regard the GERT network expressed in logical node as a graph and represent it in the Boolean matrix.

Step 2: Detect the path according to Theorem 1 and equation (7) of chapter 3, and obtain the path vector P_k .

Step 3: 1 which appears in the diagonal elements of the Boolean matrix M_b represents the loop, so find the corresponding vertex.

Step 4: Find the reachability matrix from the Boolean matrix M_b and find the vertex composing the strong connected by equation (13). Depending on detection of the circuit by applying the strongly connected portion in section 4.2, find the entrance, and then find the vertex of circuit using jointly Theorem 2. Or applying the strongly connected portion and latin multiplication, find the vertex circuit.

Step 5: Find the contact circuit and loop by equation (24) in chapter 5.

Step 6: Find the contact path and circuit by equation (25) in chapter (26).

Step 7: Find the equivalent W_E function according to chapter 7.

Step 8: Find $W_E(0)$ by the equivalent W_E function obtained in chapter 7, find $\partial W_E(s)/\partial s$ according to the method of chapter 8, and then substitute the result in to equation (3) to make the rating.

10. Example of Numerical Calculation

To make the rating regarding the GERT network (P_i is the probability that the arc is realized) indicated in Fig. 2, in case W function is given as Table 2. In order to detect the path, find latin matrix $[M]^{(1)}$ (16×16) from Fig. 3, and find $[F]_1^{(1)}$ as equation (41).

Fig. 3 GERT network

Table 2 W Function of GERT network

$W_1 = 0.3 \cdot e^{10S}$	$W_2 = 0.7 \cdot e^{15S}$	$W_3 = e^{(20S + 25S)}$
$W_4 = 0.4 \cdot 1/(1 - 10S)$	$W_5 = 0.6 \cdot 1/(1 - 20S)$	$W_6 = 0.2 \cdot e^{8S}$
$W_7 = 0.8 \cdot e^{10S}$	$W_8 = e^{(10S + 2S)}$	$W_9 = 0.1 \cdot 1/(1 - 15S)$
$W_{10} = 0.3 \cdot 1/(1 - 10S)$	$W_{11} = 0.6 \cdot 1/(1 - 12S)$	$W_{12} = 0.2 \cdot e^{20S}$
$W_{13} = 0.8 \cdot e^{15S}$	$W_{14} = 0.4 \cdot e^{(30S + 35S)}$	$W_{15} = 0.6 \cdot e^{(20S + 25S)}$
$W_{16} = 0.1 \cdot e^{6S}$	$W_{17} = 0.9 \cdot e^{8S}$	$W_{18} = e^{20S}$
$W_{19} = 0.3 \cdot 1/(1 - 10S)$	$W_{20} = 0.7 \cdot 1/(1 - 20S)$	$W_{21} = e^{10S}$
$W_{22} = 0.7 \cdot e^{20S}$	$W_{23} = 0.3 \cdot e^{15S}$	$W_{24} = e^{(20S + 25S)}$

$$[F]_1^{(1)} = [0 \ 1.2 \ 0 \ 0 \ 0 \ 1.6 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \quad (41)$$

The path from vertex 1 to vertex 15 can be found from $[F]_1^{(5)}$: the path R_1 as 1-6-7-8-9-15 and the path R_2 as 1-6-10-11-13-15. The path from vertex 1 to vertex 16 can be found from $[F]_1^{(6)}$; the path R_3 as 1-2-3-4-5-16, the path R_6 as 1-6-7-8-9-5-16 and the path R_7 as 1-6-10-11-13-14-16. Following the above mentioned procedure, P_k is:

$$\begin{aligned} P_1 &= [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0] \\ P_2 &= [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0] \\ P_3 &= [1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \\ P_4 &= [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \\ P_5 &= [1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \\ P_6 &= [1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \\ P_7 &= [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1] \end{aligned} \quad (42)$$

Secondly, form the Boolean matrix from Fig. 3, find the reachability matrix R , calculate R^* by equation (13) and then choose the vertex composing the strongly connected portion. The submatrix of Boolean matrix composing the strongly connected portion becomes equations (43) and (44).

$$M_B(1) = \begin{matrix} & \begin{matrix} 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix} \quad (43)$$

$$M_B(2) = \begin{matrix} & \begin{matrix} 10 & 11 & 12 \end{matrix} \\ \begin{matrix} 10 \\ 11 \\ 12 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix} \quad (44)$$

Choose the arc constituting the entrance from the Boolean matrix using the strongly connected component. Remove the arcs of $u_1 = (3, 2)$ and $u_2 = (4, 2)$ and obtain $M_B^0(1)$, to which and u_1 and then from $M_B^2(1)$ find the circuit 2-3-2. Likewise, add the arc u_2 to $M_B^0(1)$ and calculate $M_B^3(1)$, and obtain 2-3-4-2. Secondly, remove the arc $u_3 = (12, 10)$ and obtain $M_B^0(2)$, to which add arc u_3 and calculate $M_B^2(2)$. Thus circuit 10-11-12-10 is obtained. As for the loop, by 1 which appears in the diagonal element vertex 8 is obtained. Find the vector for 3 circuits and one loop $C_i; i=1, 2, \dots, 4$.

$$\begin{aligned} C_1 &= [0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ C_2 &= [0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ C_3 &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0] \\ C_4 &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \end{aligned} \quad (45)$$

Find the contact circuit and loop by equation (24). On this account, define H like equation (46) and obtain equation (47).

$$H = [C_1 \ C_2 \ C_3 \ C_4]^T \quad (46)$$

$$L = H \cdot H^T = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \quad (47)$$

From equation (47), it is found that circuit C_1 is a contact circuit C_2 . Then, find the contact among the path circuit and loop by equation (25).

$$\begin{aligned} HP_1^T &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & HP_2^T &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & HP_3^T &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ HP_4^T &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} & HP_5^T &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} & HP_6^T &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ HP_7^T &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned} \quad (48)$$

In order to calculate the equivalent W_E function, find $\Delta(s)$. Obtain $[D]^{(2)}$ by using equation (47), then $[D]^{(3)}$ can be obtained like equation (52) by using $[\tilde{D}]^{(1)}$. $[D]^{(4)}$ becomes 0, and so by using equations (50) and (51). $\Delta(s)$ can be reduced to equation (49).

$$\begin{aligned} \Delta(s) &= 1 - C_1 - C_2 - C_3 - C_4 - \\ &\quad C_1C_3 - C_1C_4 + C_2C_3 + \\ &\quad C_2C_4 + C_3C_4 - C_1C_3C_4 - \\ &\quad C_2C_3C_4 \end{aligned} \quad (49)$$

By using γ_{ji} of equation (48), $\Delta_k(s)$ can be represented in the form of equation (53).

$$[D]^{(2)} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \\ C \end{matrix} & \begin{bmatrix} 0 & 0 & C_1C_3 & C_1C_4 \\ 0 & 0 & C_2C_3 & C_2C_4 \\ 0 & 0 & 0 & C_3C_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (50)$$

$$[\tilde{D}]^{(1)} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix} & \begin{bmatrix} 0 & 0 & C_3 & C_4 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & C_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (51)$$

$$[D]^{(3)} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \\ \begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix} & \begin{bmatrix} 0 & 0 & C_1C_3C_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (52)$$

$$\begin{aligned} \Delta_k(s) &= 1 - (1-\gamma_{k1})C_1 \\ &\quad - (1-\gamma_{k2})C_2 - (1-\gamma_{k3})C_3 \\ &\quad - (1-\gamma_{k4})C_4 + \\ &\quad (1-\gamma_{k1})C_1(1-\gamma_{k3})C_3 + \\ &\quad (1-\gamma_{k1})C_1(1-\gamma_{k4})C_4 + \\ &\quad (1-\gamma_{k2})C_2(1-\gamma_{k3})C_3 + \\ &\quad (1-\gamma_{k2})C_2(1-\gamma_{k4})C_4 + \\ &\quad (1-\gamma_{k3})C_3(1-\gamma_{k4})C_4 - \\ &\quad (1-\gamma_{k1})C_1(1-\gamma_{k3})C_3(1-\gamma_{k4})C_4 - \\ &\quad (1-\gamma_{k2})C_2(1-\gamma_{k3})C_3(1-\gamma_{k4})C_4 \end{aligned} \quad (53)$$

Substituting γ_{ki} of equation (48) in to equation (48), we get,

$$\begin{aligned} \Delta_1 &= 1 - C_1 - C_2 - C_3 + C_1C_3 + C_2C_3 \\ \Delta_2 &= 1 - C_1 - C_2 - C_4 + C_1C_4 + C_2C_4 \\ \Delta_3 &= 1 - C_3 - C_4 + C_3C_4 \\ \Delta_4 &= \Delta_5 = \Delta_3, \Delta_6 = \Delta_1, \Delta_7 = \Delta_2 \end{aligned} \quad (54)$$

The equivalent W_E functions for the vertexes 1 to 15 and for the vertexes 1 to 16 become equations (55) and (56).

$$W_{E15}(s) = \frac{\sum_{i=1}^2 R_i(s) \Delta_i}{\Delta(s)} \quad (55)$$

$$W_{E16}(s) = \frac{\sum_{i=3}^7 R_i(s) \cdot \Delta_i(s)}{\Delta(s)} \quad (56)$$

In order to make calculation of $\partial W_{E15}/\partial s$, $\partial W_{E16}/\partial s$ by equation (31), calculate Δ , Δ' , Δ_k , R_i and R_i' and then $s = 0$ shall be substituted. In this case, the W function of C_i and R_i becomes equation (57).

$$\begin{aligned} C_1(s) &= W_3(s) \cdot W_5(s) \\ C_2(s) &= W_3(s) \cdot W_4(s) \cdot W_5(s) \\ C_3(s) &= W_{19}(s) \cdot W_{20}(s) \cdot W_{21}(s) \\ C_4(s) &= W_{15}(s) \\ R_1(s) &= W_2(s) W_{10}(s) W_{13}(s) W_{14}(s) W_{17}(s) \\ R_2(s) &= W_2(s) W_{11}(s) W_{18}(s) W_{19}(s) W_{22}(s) \\ R_3(s) &= W_1(s) W_3(s) W_4(s) W_6(s) W_8(s) \\ R_4(s) &= W_2(s) W_9(s) W_3(s) W_4(s) W_6(s) \\ &\quad W_8(s) \\ R_5(s) &= W_2(s) W_{10}(s) W_{12}(s) W_4(s) W_6(s) \\ &\quad W_8(s) \\ R_6(s) &= W_2(s) W_{10}(s) W_{13}(s) W_{14}(s) W_{16}(s) \\ &\quad W_8(s) \\ R_7(s) &= W_2(s) W_{11}(s) W_{18}(s) W_{19}(s) W_{23}(s) \\ &\quad W_{24}(s) \end{aligned} \quad (57)$$

Substitute equation (57) in to equations (52) and (54), and then obtain the calculation of equation (31) and $W_E(0)$ as per $W_{E15}(s)$ and $W_{E16}(s)$.

$$\left. \frac{\partial W_{E15}(s)}{\partial s} \right|_{s=0} = 73.44, W_{E15}(0) = 0.445 \quad (58)$$

$$\text{hence, } \mu_1 = \left. \frac{\partial}{\partial s} \left[\frac{W_{E15}(s)}{W_{E15}(0)} \right] \right|_{s=0} = 164.95 \quad (59)$$

$$\left. \frac{\partial W_{E16}(s)}{\partial s} \right|_{s=0} = 243.90, W_{E16}(0) = 0.555 \quad (60)$$

$$\text{hence, } \mu_2 = \left. \frac{\partial}{\partial s} \left[\frac{W_{E16}(s)}{W_{E16}(0)} \right] \right|_{s=0} = 439.46 \quad (61)$$

12. Conclusions

This report describes a procedure to analyze the calculation of a large scale GERT network by the method of graph theory, the steps in the procedure being as follows:

1) Regarding the detection of circuit in GERT network a method is shown, by which the Boolean matrix, reachability matrix, and strongly connected matrix are calculated, and then the result is obtained from the Boolean matrix by removing the arc composing the strongly connected portion.

2) Regarding the contact between circuit and circuit, loop and loop, loop and circuit, or among path, circuit and loop, a method is shown to detect the contact by using vector and matrix.

3) A method is shown in which the detection of path is made by applying latin square multiplication, and further the detection of circuit is made by applying the strongly connected portion and the entrance of the graph and by jointly applying latin square multiplication.

4) A method is shown to obtain the determinant of network $\Delta(s)$ by the matrix L of

contact circuit and loop as well as latin multiplication, and further to obtain $\Delta_k(s)$ by the calculated HP_j of the matrix of contact path, circuit and loop by using the coefficient of $(1-\gamma_{ji})$.

5) A calculation equation is introduced to obtain an equivalent W_E function as simple as possible, and thereby it has become possible to process a problem in a computer only using parameter matrix.

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