A Torsion-free Axisymmetric Problem of a Cylindrical
Crack in a Transversely Isotropic Body*

By Hideaki KASANO**, Hiroyuki MATSUMOTO***
and Ichiro NAKAHARA***

This paper considers a cylindrical crack in a transversely isotropic elastic body on the macroscopic level, which models a crack-like defect on an interface between fiber and matrix in the fiber reinforced composite materials. Numerical calculations are carried out for E.glass/epoxy and graphite/epoxy composites when they are subjected to a constant internal pressure or shear traction linearly distributing in the axial direction on the crack surfaces. Stress intensity factors are evaluated for different values of the crack-length and the elastic constants.

Key Words: Composite Material, Homogeneous Anisotropic Model, Cylindrical Crack, Transversely Isotropic Body, Elasticity

1. Introduction

The fracture of fiber reinforced composite materials often initiates from localized imperfections located near to or on an interface between matrix and fiber^{(1),(2)}. The most common forms of these imperfections are voids and crackings introduced during fabrications. In studying fracture problems of this kind the imperfections are usually idealized as line cracks in two dimensions and circular or elliptical cracks in three dimensions. On the other hand two approaches are proposed to analyze the crack problems of the fiber reinforced composites⁽³⁾. The first assumes a homogeneous anisotropic elastic model at the global scale level and the second considers a non-homogeneous one since such composite is nonhomogeneous in itself. Many investigations have been done using these models⁽³⁾ and also the theoretical results using these two models are compared with the experimental data⁽⁴⁾. Most of these investigations, however, are concerned with cracks with plane surfaces and a few are with curved surfaces⁽⁵⁾. Particularly when crack-like defects on the interface are distributing in the circumferential direction around a cylindrical fiber as well as in the axial one, we may model the problem with a cylindrical crack. Erdogan et al.⁽⁶⁾ considered a two-phase composite body with a cylindrical interface crack using a singular integral equation method, but a similar problem of a homogeneous anisotropic body has not yet been studied. So we investigate for a solution method to this problem in the framework of the three dimensional anisotropic elasticity theory, using a transversely isotropic elastic model with a cylindrical crack.

We calculate the stress intensity factors and examine the effects of the crack-length and the elastic constants on them.

2. Solution Method

The displacements and stresses in a transversely isotropic body can be expressed in terms of two stress functions for a torsion-free axisymmetric problem. Introducing the cylindrical coordinates r, θ, z where the z axis is taken parallel to the principal material direction and chosen as the axis of the geometric symmetry, we have⁽⁷⁾

$$\left. \begin{aligned} u_r &= \frac{\partial \phi_i}{\partial r}, & w_z &= k_i \frac{\partial \phi_i}{\partial z} \\ \frac{\sigma_r}{c_{44}} &= -(1+k_i) \left(\beta_i \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi_i \\ \frac{\sigma_z}{c_{44}} &= (1+k_i) \mu_i \frac{\partial^2 \phi_i}{\partial z^2} \\ \frac{\sigma_\theta}{c_{44}} &= -(1+k_i) \left(\beta_i \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \right) \phi_i \\ \frac{\tau_{rz}}{c_{44}} &= (1+k_i) \frac{\partial^2 \phi_i}{\partial r \partial z} & i &= 1, 2 \end{aligned} \right\} \dots\dots (1)$$

where the stress functions ϕ_i must satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \mu_i \frac{\partial^2}{\partial z^2} \right) \phi_i = 0 \quad (i=1, 2) \quad \dots\dots\dots (2)$$

and, by using five independent elastic constants c_{mn} of the transversely isotropic elastic body, we have $k_i = (c_{11}\mu_i - c_{44}) / (c_{13} + c_{44})$, $\beta_i = (c_{11} - c_{12}) / c_{44}(1+k_i)$ and further μ_i 's are the roots of the following quadratic equation

$$c_{11}c_{44}\mu^2 + [c_{13}(2c_{44} + c_{12}) - c_{11}c_{33}]\mu + c_{33}c_{44} = 0 \quad \dots\dots\dots (3)$$

Fig. 1 Macromechanical model

* Received 16th August, 1982.

** Research Associate, Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo.

*** Professor, ibid.

Consider a transversely isotropic elastic body of infinite extent containing a cylindrical crack of diameter $2b$ and length $2c$ when it is subjected to the normal and shear tractions $p \cdot p(\zeta)$ and $p \cdot q(\zeta)$ on the crack surfaces as shown in Fig.1. We can then state the boundary conditions as follows.

$$\begin{aligned} (\sigma_r^+)_{\rho=1} &= (\sigma_r^-)_{\rho=1} \dots\dots\dots (4.a) \\ (\tau_{rz}^+)_{\rho=1} &= (\tau_{rz}^-)_{\rho=1} \dots\dots\dots (4.b) \\ (\sigma_r^+)_{\rho=1} &= (\sigma_r^-)_{\rho=1} = -p \cdot p(\zeta) \quad |\zeta| < \beta \dots\dots\dots (4.c) \\ (\tau_{rz}^+)_{\rho=1} &= (\tau_{rz}^-)_{\rho=1} = -p \cdot q(\zeta) \quad |\zeta| < \beta \dots\dots\dots (4.d) \\ (u_r^+)_{\rho=1} - (u_r^-)_{\rho=1} &= 0 \quad |\zeta| \geq \beta \dots\dots\dots (4.e) \\ (w_z^+)_{\rho=1} - (w_z^-)_{\rho=1} &= 0 \quad |\zeta| \geq \beta \dots\dots\dots (4.f) \end{aligned}$$

where $\rho=r/b$, $\zeta=z/b$, $\beta=c/b$ and the superscripts "+", "-" represent the domains of $r>b$ and $r\leq b$, respectively. We also assume a constant p positive $p(\zeta)=p(-\zeta)$, $q(\zeta)=-q(-\zeta)$. In order to seek a solution satisfying the boundary conditions (4.a)~(4.f) we take the stress functions for $r>b$

$$c_{ii}\phi_i^+ = \int_0^\infty g_i(\lambda) K_0(\sqrt{\mu_i}\lambda\rho) \cos \lambda\zeta d\lambda \quad (i=1,2) \dots\dots\dots (5)$$

and substituting them into Eq.(1) we have

$$\left. \begin{aligned} c_{ii}b u_r^+ &= -\Sigma \int_0^\infty g_i(\lambda) \sqrt{\mu_i} \lambda K_1(\sqrt{\mu_i}\lambda\rho) \cos \lambda\zeta d\lambda \\ c_{ii}b w_z^+ &= -\Sigma \int_0^\infty g_i(\lambda) k_i \lambda K_0(\sqrt{\mu_i}\lambda\rho) \sin \lambda\zeta d\lambda \\ b^2 \sigma_r^+ &= \Sigma \int_0^\infty g_i(\lambda) (1+k_i) \lambda \left\{ \lambda K_0(\sqrt{\mu_i}\lambda\rho) + \frac{\beta_i \sqrt{\mu_i}}{\rho} K_1(\sqrt{\mu_i}\lambda\rho) \right\} \cos \lambda\zeta d\lambda \\ b^2 \tau_{rz}^+ &= \Sigma \int_0^\infty g_i(\lambda) \sqrt{\mu_i} \lambda^2 (1+k_i) K_1(\sqrt{\mu_i}\lambda\rho) \sin \lambda\zeta d\lambda \end{aligned} \right\} \dots\dots\dots (6)$$

where Σ represents the sums of $i=1$ and 2 . Similarly for $r\leq b$ we take

$$c_{ii}\phi_i^- = \int_0^\infty f_i(\lambda) I_0(\sqrt{\mu_i}\lambda\rho) \cos \lambda\zeta d\lambda \quad (i=1,2) \dots\dots\dots (7)$$

and then the following displacements and stresses are obtained.

$$\left. \begin{aligned} c_{ii}b u_r^- &= \Sigma \int_0^\infty f_i(\lambda) \sqrt{\mu_i} \lambda I_1(\sqrt{\mu_i}\lambda\rho) \cos \lambda\zeta d\lambda \\ c_{ii}b w_z^- &= -\Sigma \int_0^\infty f_i(\lambda) k_i \lambda I_0(\sqrt{\mu_i}\lambda\rho) \sin \lambda\zeta d\lambda \\ b^2 \sigma_r^- &= \Sigma \int_0^\infty f_i(\lambda) (1+k_i) \lambda \left\{ \lambda I_0(\sqrt{\mu_i}\lambda\rho) - \frac{\beta_i \sqrt{\mu_i}}{\rho} I_1(\sqrt{\mu_i}\lambda\rho) \right\} \cos \lambda\zeta d\lambda \\ b^2 \tau_{rz}^- &= -\Sigma \int_0^\infty f_i(\lambda) \sqrt{\mu_i} \lambda^2 (1+k_i) I_1(\sqrt{\mu_i}\lambda\rho) \sin \lambda\zeta d\lambda \end{aligned} \right\} \dots\dots\dots (8)$$

where $f_i(\lambda)$'s and $g_i(\lambda)$'s are unknown functions which are to be determined by the boundary conditions, and $I_0(\)$, $I_1(\)$ and $K_0(\)$, $K_1(\)$ are modified Bessel functions of the first and the second kinds, respectively. Putting $\rho=1$ in the expressions for σ_r and τ_{rz} of Eqs.(6), (8) and substituting them into the boundary conditions (4.a,b) and further applying the method of Fourier transforms we have $\bar{g}_i(\lambda)$'s in terms of $\bar{f}_i(\lambda)$'s as follows

$$\left. \begin{aligned} \bar{g}_1(\lambda) &= -\frac{\mu_1 \epsilon_1(\lambda) + \mu_2 \tau_1(\lambda)}{\mu_1 \epsilon_2(\lambda) - \mu_2 \epsilon_1(\lambda)} \bar{f}_1(\lambda) - \frac{\tau_2(\lambda) + \epsilon_2(\lambda)}{\mu_1 \epsilon_2(\lambda) - \mu_2 \epsilon_1(\lambda)} \mu_2 \bar{f}_2(\lambda) \\ \bar{g}_2(\lambda) &= \frac{\tau_1(\lambda) + \epsilon_1(\lambda)}{\mu_1 \epsilon_2(\lambda) - \mu_2 \epsilon_1(\lambda)} \mu_1 \bar{f}_1(\lambda) + \frac{\mu_1 \tau_2(\lambda) + \mu_2 \epsilon_1(\lambda)}{\mu_1 \epsilon_2(\lambda) - \mu_2 \epsilon_1(\lambda)} \bar{f}_2(\lambda) \end{aligned} \right\} \dots\dots\dots (9)$$

where $i=1,2$ and

$$\left. \begin{aligned} \bar{f}_i(\lambda) &= f_i(\lambda) (1+k_i) \lambda I_1(\sqrt{\mu_i}\lambda) / \sqrt{\mu_i}, \quad \bar{g}_i(\lambda) = g_i(\lambda) (1+k_i) \lambda K_1(\sqrt{\mu_i}\lambda) / \sqrt{\mu_i} \\ \tau_i(\lambda) &= \sqrt{\mu_i} \lambda I_0(\sqrt{\mu_i}\lambda) / I_1(\sqrt{\mu_i}\lambda) - \beta_i \mu_i, \quad \epsilon_i(\lambda) = \sqrt{\mu_i} \lambda K_0(\sqrt{\mu_i}\lambda) / K_1(\sqrt{\mu_i}\lambda) + \beta_i \mu_i \end{aligned} \right\} \dots\dots\dots (10)$$

Substituting Eq.(9) into Eq.(6) together with Eq.(10), we have the following expressions for $(u_r^+)_{\rho=1}$ and $(w_z^+)_{\rho=1}$.

$$\frac{c_{ii}(u_r^+)_{\rho=1}}{pb} = \int_0^\infty D_r^+(\lambda) \cos \lambda\zeta d\lambda, \quad \frac{c_{ii}(w_z^+)_{\rho=1}}{pb} = \int_0^\infty D_z^+(\lambda) \sin \lambda\zeta d\lambda \dots\dots\dots (11)$$

where

$$\begin{aligned}
 p b^2 D_r^+(\lambda) &= \frac{(1+k_2) \{ \mu_1 \kappa_2(\lambda) + \mu_2 \tau_1(\lambda) \} - (1+k_1) \{ \mu_2 \tau_1(\lambda) + \kappa_1(\lambda) \}}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} \mu_1 \bar{f}_1(\lambda) \\
 &+ \frac{(1+k_2) \mu_1 \{ \tau_2(\lambda) + \kappa_2(\lambda) \} - (1+k_1) \{ \mu_1 \tau_2(\lambda) + \mu_2 \kappa_1(\lambda) \}}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} \mu_2 \bar{f}_2(\lambda) \\
 p b^2 \lambda D_z^+(\lambda) &= \frac{\left[\frac{k_1(1+k_2) \{ \kappa_1(\lambda) - \beta_1 \mu_1 \} \{ \mu_1 \kappa_2(\lambda) + \mu_2 \tau_1(\lambda) \}}{-k_2(1+k_1) \{ \kappa_2(\lambda) - \beta_2 \mu_2 \} \mu_1 \{ \tau_1(\lambda) + \kappa_1(\lambda) \}} \right]}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} \bar{f}_1(\lambda) \\
 &+ \frac{k_1(1+k_2) \{ \kappa_1(\lambda) - \beta_1 \mu_1 \} \mu_2 \{ \tau_2(\lambda) + \kappa_2(\lambda) \} - k_2(1+k_1) \{ \kappa_2(\lambda) - \beta_2 \mu_2 \} \{ \mu_1 \tau_2(\lambda) + \mu_2 \kappa_1(\lambda) \}}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} \bar{f}_2(\lambda)
 \end{aligned} \quad (12)$$

Similar expressions for $(i\bar{t}_r^-)_{p=1}$ and $(w_z^-)_{p=1}$ are obtained from Eq.(8) together with Eq.(10).

$$\frac{c_{44}(u_r^-)_{p=1}}{p b} = \int_0^\infty D_r^-(\lambda) \cos \lambda \zeta d\lambda, \quad \frac{c_{44}(w_z^-)_{p=1}}{p b} = \int_0^\infty D_z^-(\lambda) \sin \lambda \zeta d\lambda \quad (13)$$

where

$$\left. \begin{aligned}
 p b^2 D_r^-(\lambda) &= \frac{\mu_1}{1+k_1} \bar{f}_1(\lambda) + \frac{\mu_2}{1+k_2} \bar{f}_2(\lambda) \\
 p b^2 \lambda D_z^-(\lambda) &= -\frac{k_1}{1+k_1} \{ \tau_1(\lambda) + \beta_1 \mu_1 \} \bar{f}_1(\lambda) - \frac{k_2}{1+k_2} \{ \tau_2(\lambda) + \beta_2 \mu_2 \} \bar{f}_2(\lambda)
 \end{aligned} \right\} \quad (14)$$

From Eqs.(11) and (13) we have

$$\frac{c_{44}}{p b} \{ (u_r^+)_{p=1} - (u_r^-)_{p=1} \} = \int_0^\infty D_r(\lambda) \cos \lambda \zeta d\lambda \quad (15)$$

$$\frac{c_{44}}{p b} \{ (w_z^+)_{p=1} - (w_z^-)_{p=1} \} = \int_0^\infty D_z(\lambda) \sin \lambda \zeta d\lambda \quad (16)$$

where

$$\left. \begin{aligned}
 p b^2 D_r(\lambda) &= \frac{\mu_1 \mu_2 (k_2 - k_1)}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} [\{ \tau_1(\lambda) + \kappa_1(\lambda) \} \bar{f}_1(\lambda) \\
 &+ \{ \tau_2(\lambda) + \kappa_2(\lambda) \} \bar{f}_2(\lambda)] \\
 p b^2 \lambda D_z(\lambda) &= -\frac{k_2 - k_1}{(1+k_1)(1+k_2) \{ \mu_1 \kappa_2(\lambda) - \mu_2 \kappa_1(\lambda) \}} [\mu_1 \kappa_2(\lambda) \{ \tau_1(\lambda) + \kappa_1(\lambda) \} \bar{f}_1(\lambda) \\
 &+ \mu_2 \kappa_1(\lambda) \{ \tau_2(\lambda) + \kappa_2(\lambda) \} \bar{f}_2(\lambda)]
 \end{aligned} \right\} \quad (17)$$

On the other hand the stresses are obtained as follows by using Eq.(10)

$$\frac{(\sigma_r^+)_{p=1}}{p} = \frac{(\sigma_r^-)_{p=1}}{p} = \int_0^\infty S_r(\lambda) \cos \lambda \zeta d\lambda \quad (18)$$

$$\frac{(\tau_{rz}^+)_{p=1}}{p} = \frac{(\tau_{rz}^-)_{p=1}}{p} = \int_0^\infty S_{rz}(\lambda) \sin \lambda \zeta d\lambda \quad (19)$$

where

$$p b^2 S_r(\lambda) = \tau_1(\lambda) \bar{f}_1(\lambda) + \tau_2(\lambda) \bar{f}_2(\lambda), \quad p b^2 \lambda^{-1} S_{rz}(\lambda) = -\mu_1 \bar{f}_1(\lambda) - \mu_2 \bar{f}_2(\lambda) \quad (20)$$

Expressing $\bar{f}_i(\lambda)$'s with $D_r(\lambda)$ and $D_z(\lambda)$ in Eq.(17) and substituting them into Eq.(20), we have

$$S_r(\lambda) = \psi_{11}(\lambda) D_r(\lambda) + \psi_{12}(\lambda) D_z(\lambda), \quad S_{rz}(\lambda) = \psi_{12}(\lambda) D_r(\lambda) + \psi_{22}(\lambda) D_z(\lambda) \quad (21)$$

where

$$\left. \begin{aligned}
 \psi_{11}(\lambda) &= -\frac{(1+k_1)(1+k_2)}{\mu_1 \mu_2 (k_1 - k_2)} \frac{\mu_1 \{ \tau_1(\lambda) + \kappa_1(\lambda) \} \tau_2(\lambda) \kappa_2(\lambda) - \mu_2 \{ \tau_2(\lambda) + \kappa_2(\lambda) \} \tau_1(\lambda) \kappa_1(\lambda)}{\{ \tau_1(\lambda) + \kappa_1(\lambda) \} \{ \tau_2(\lambda) + \kappa_2(\lambda) \}} \\
 \psi_{12}(\lambda) &= \frac{(1+k_1)(1+k_2)}{k_1 - k_2} \frac{\{ \tau_1(\lambda) \kappa_2(\lambda) - \tau_2(\lambda) \kappa_1(\lambda) \}}{\{ \tau_1(\lambda) + \kappa_1(\lambda) \} \{ \tau_2(\lambda) + \kappa_2(\lambda) \}} \lambda \\
 \psi_{22}(\lambda) &= -\frac{(1+k_1)(1+k_2)}{k_1 - k_2} \frac{\mu_1 \{ \tau_2(\lambda) + \kappa_2(\lambda) \} - \mu_2 \{ \tau_1(\lambda) + \kappa_1(\lambda) \}}{\{ \tau_1(\lambda) + \kappa_1(\lambda) \} \{ \tau_2(\lambda) + \kappa_2(\lambda) \}} \frac{\lambda}{\lambda^2}
 \end{aligned} \right\} \quad (22)$$

For a sufficiently large value of λ , Eq.(22) is approximated by

$$\lim_{\lambda \rightarrow \infty} \psi_{11}(\lambda) = \alpha_{11} \lambda, \quad \lim_{\lambda \rightarrow \infty} \psi_{12}(\lambda) = O(1), \quad \lim_{\lambda \rightarrow \infty} \psi_{22}(\lambda) = \alpha_{22} \lambda \quad (23)$$

hence we establish

$$\psi_{11}(\lambda) = \lambda \{ \alpha_{11} + \bar{\psi}_{11}(\lambda) \}, \quad \psi_{12}(\lambda) = \lambda \bar{\psi}_{12}(\lambda), \quad \psi_{22}(\lambda) = \lambda \{ \alpha_{22} + \bar{\psi}_{22}(\lambda) \} \quad (24)$$

and substituting them into Eq. (21), we obtain the following expressions.

$$\left. \begin{aligned} S_r(\lambda) &= \lambda \{ \alpha_{11} D_r(\lambda) + \bar{\psi}_{11}(\lambda) D_r(\lambda) + \bar{\psi}_{12}(\lambda) D_z(\lambda) \} \\ S_z(\lambda) &= \lambda \{ \alpha_{22} D_z(\lambda) + \bar{\psi}_{12}(\lambda) D_r(\lambda) + \bar{\psi}_{22}(\lambda) D_z(\lambda) \} \end{aligned} \right\} \dots\dots\dots (25)$$

where α_{11} and α_{22} are the constants given by

$$\alpha_{11} = -\frac{(1+k_1)(1+k_2)(\sqrt{\mu_1}-\sqrt{\mu_2})}{2(k_1-k_2)\sqrt{\mu_1\mu_2}}, \quad \alpha_{22} = -\frac{(1+k_1)(1+k_2)(\sqrt{\mu_1}-\sqrt{\mu_2})}{2(k_1-k_2)} \dots\dots\dots (26)$$

and $\bar{\psi}_{11}(\lambda) = \bar{\psi}_{22}(\lambda) = O(1/\lambda^2)$, $\bar{\psi}_{12}(\lambda) = O(1/\lambda)$

Substituting Eq. (25) into Eqs. (18), (19) and equating them with the boundary conditions (4.c,d), and also Eqs. (15), (16) with Eqs. (4.e,f) we have the following simultaneous dual integral equations for the unknown functions of $D_r(\lambda)$ and $D_z(\lambda)$.

$$\int_0^\infty \lambda \{ \alpha_{11} D_r(\lambda) + \bar{\psi}_{11}(\lambda) D_r(\lambda) + \bar{\psi}_{12}(\lambda) D_z(\lambda) \} \cos \lambda \zeta d\lambda = -p(\zeta) \quad 0 \leq \zeta < \beta \dots\dots\dots (27-a)$$

$$\int_0^\infty \lambda \{ \alpha_{22} D_z(\lambda) + \bar{\psi}_{12}(\lambda) D_r(\lambda) + \bar{\psi}_{22}(\lambda) D_z(\lambda) \} \sin \lambda \zeta d\lambda = -q(\zeta) \quad 0 \leq \zeta < \beta \dots\dots\dots (27-b)$$

$$\int_0^\infty D_r(\lambda) \cos \lambda \zeta d\lambda = 0 \quad \zeta \geq \beta \dots\dots\dots (28-a)$$

$$\int_0^\infty D_z(\lambda) \sin \lambda \zeta d\lambda = 0 \quad \zeta \geq \beta \dots\dots\dots (28-b)$$

We can solve these integral equations by use of the series expansion method. That is, we assume $D_r(\lambda)$ and $D_z(\lambda)$ in the following infinite series of Bessel functions of the first kind with the unknown constants a_j and b_j .

$$D_r(\lambda) = \sum_{j=0}^\infty a_j \lambda^{-1} J_{2j+1}(\lambda\beta), \quad D_z(\lambda) = \sum_{j=0}^\infty b_j \lambda^{-1} J_{2j+2}(\lambda\beta) \dots\dots\dots (29)$$

Then we find that Eqs. (28) are automatically satisfied⁽⁸⁾. Substituting Eq. (29) into Eq. (27) we get

$$\int_0^\infty \left[\sum_{j=0}^\infty a_j \{ \alpha_{11} + \bar{\psi}_{11}(\lambda) \} J_{2j+1}(\lambda\beta) + \sum_{j=0}^\infty b_j \bar{\psi}_{12}(\lambda) J_{2j+2}(\lambda\beta) \right] \cos \lambda \zeta d\lambda = -p(\zeta) \quad 0 \leq \zeta < \beta \dots\dots\dots (30)$$

$$\int_0^\infty \left[\sum_{j=0}^\infty a_j \bar{\psi}_{12}(\lambda) J_{2j+1}(\lambda\beta) + \sum_{j=0}^\infty b_j \{ \alpha_{22} + \bar{\psi}_{22}(\lambda) \} J_{2j+2}(\lambda\beta) \right] \sin \lambda \zeta d\lambda = -q(\zeta) \quad 0 \leq \zeta < \beta \dots\dots\dots (31)$$

and noting that $\cos \lambda \zeta$, $\sin \lambda \zeta$, $p(\zeta)$ and $q(\zeta)$ are expandable in the form of Fourier series such as

$$\left. \begin{aligned} \cos \lambda \zeta &= \sum_{i=0}^\infty \varepsilon_i J_{2i}(\lambda\beta) \cos 2i\phi, \quad \sin \lambda \zeta = \sum_{i=0}^\infty 2J_{2i+1}(\lambda\beta) \sin (2i+1)\phi \\ p(\zeta) &= \sum_{i=0}^\infty p^i \cos 2i\phi, \quad q(\zeta) = \sum_{i=0}^\infty q^i \sin (2i+1)\phi \end{aligned} \right\} \dots\dots\dots (32)$$

with $\phi = \sin^{-1} \zeta/\beta$, we reduce Eqs. (30) and (31) to

$$\sum_{i=0}^\infty \varepsilon_i \left[\sum_{j=0}^\infty a_j \int_0^\infty \{ \alpha_{11} + \bar{\psi}_{11}(\lambda) \} J_{2j}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda + \sum_{j=0}^\infty b_j \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2i}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda \right] \cos 2i\phi = - \sum_{i=0}^\infty p^i \cos 2i\phi \dots\dots\dots (33)$$

$$\sum_{i=0}^\infty 2 \left[\sum_{j=0}^\infty a_j \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda + \sum_{j=0}^\infty b_j \int_0^\infty \{ \alpha_{22} + \bar{\psi}_{22}(\lambda) \} J_{2i+1}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda \right] \sin (2i+1)\phi = - \sum_{i=0}^\infty q^i \sin (2i+1)\phi \dots\dots\dots (34)$$

where $\varepsilon_0=1$, $\varepsilon_i=2$ ($i \geq 1$) and p^i 's and q^i 's are known constants determined from the load conditions. Consequently equating the respective coefficients of $\cos 2i\phi$ and $\sin (2i+1)\phi$ in Eqs. (33) and (34) we have the following simultaneous algebraic equations of a_j and b_j .

$$\left. \begin{aligned} \sum_{j=0}^\infty a_j (\alpha_{11} P_{ij}^0 + P_{ij}^1) + \sum_{j=0}^\infty b_j Q_{ij}^2 &= -p^i/\varepsilon_i \quad i=0, 1, 2, \dots \\ \sum_{j=0}^\infty a_j P_{ij}^2 + \sum_{j=0}^\infty b_j (\alpha_{22} Q_{ij}^0 + Q_{ij}^1) &= -q^i/2 \quad i=0, 1, 2, \dots \end{aligned} \right\} \dots\dots\dots (35)$$

where

$$\left. \begin{aligned} P_{ij}^1 &= \int_0^\infty \bar{\psi}_{11}(\lambda) J_{2i}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda, \quad P_{ij}^2 = \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda \\ Q_{ij}^1 &= \int_0^\infty \bar{\psi}_{22}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda, \quad Q_{ij}^2 = \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2i}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda \end{aligned} \right\} \dots\dots\dots (36)$$

and also P_{ij}^0 , Q_{ij}^0 reduce to ⁽⁹⁾

$$\left. \begin{aligned} P_{ij}^0 &= \int_0^\infty J_{2i}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda = Q_{ij}^0 = \int_0^\infty J_{2i+1}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda = (-)^{j-i}/2\beta \quad (i \leq j) \\ &= (-)^{i-j-1}/2\beta \quad (i > j) \end{aligned} \right\} \dots\dots\dots(37)$$

From Eq.(35) we can get the unknown coefficients a_j and b_j and then determine all the displacements and the stresses. Substituting Eq.(29) into Eq.(25) and further into Eqs.(18) and (19) we have

$$\frac{(\sigma_r^+)_{s+1}}{p} = \frac{(\sigma_r^-)_{s+1}}{p} = \sum_{j=0}^\infty a_j \alpha_{11} \int_0^\infty J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda + \sum_{j=0}^\infty a_j \int_0^\infty \bar{\psi}_{11}(\lambda) J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda + \sum_{j=0}^\infty b_j \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda \dots\dots\dots(38)$$

$$\frac{(\tau_{rz}^+)_{s+1}}{p} = \frac{(\tau_{rz}^-)_{s+1}}{p} = \sum_{j=0}^\infty b_j \alpha_{22} \int_0^\infty J_{2j+2}(\lambda\beta) \sin \lambda \zeta d\lambda + \sum_{j=0}^\infty b_j \int_0^\infty \bar{\psi}_{22}(\lambda) J_{2j+2}(\lambda\beta) \sin \lambda \zeta d\lambda + \sum_{j=0}^\infty a_j \int_0^\infty \bar{\psi}_{12}(\lambda) J_{2j+1}(\lambda\beta) \sin \lambda \zeta d\lambda \dots\dots\dots(39)$$

Particularly in the neighbourhood of the crack-tips, since the first terms of Eqs.(38) and (39) are dominant, we have ⁽¹⁰⁾

$$\frac{(\sigma_r^+)_{s+1}}{p} = \frac{(\sigma_r^-)_{s+1}}{p} \cong \alpha_{11} \sum_{j=0}^\infty a_j \int_0^\infty J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda = \alpha_{11} \sum_{j=0}^\infty a_j \frac{(-)^{j+1}}{\sqrt{\zeta^2 - \beta^2}} \left(\frac{\beta}{\zeta + \sqrt{\zeta^2 - \beta^2}} \right)^{2j+1} \quad \zeta \geq \beta \dots\dots\dots(40)$$

$$\frac{(\tau_{rz}^+)_{s+1}}{p} = \frac{(\tau_{rz}^-)_{s+1}}{p} \cong \alpha_{22} \sum_{j=0}^\infty b_j \int_0^\infty J_{2j+2}(\lambda\beta) \sin \lambda \zeta d\lambda = \alpha_{22} \sum_{j=0}^\infty b_j \frac{(-)^{j+1}}{\sqrt{\zeta^2 - \beta^2}} \left(\frac{\beta}{\zeta + \sqrt{\zeta^2 - \beta^2}} \right)^{2j+2} \quad \zeta \geq \beta \dots\dots\dots(41)$$

Then we define the stress intensity factors by

$$\left. \begin{aligned} K_I &= \lim_{z \rightarrow +c} \sqrt{2\pi(z-c)} (\sigma_r)_{s+1} = \lim_{\zeta \rightarrow +\beta} \sqrt{2\pi b(\zeta-\beta)} (\sigma_r)_{s+1} \\ K_{II} &= \lim_{z \rightarrow +c} \sqrt{2\pi(z-c)} (\tau_{rz})_{s+1} = \lim_{\zeta \rightarrow +\beta} \sqrt{2\pi b(\zeta-\beta)} (\tau_{rz})_{s+1} \end{aligned} \right\} \dots\dots\dots(42)$$

and substituting Eqs.(40) and (41) into the above equations, we get

$$\frac{K_I}{p\sqrt{\pi c}} = \frac{\alpha_{11}}{\beta} \sum_{j=0}^\infty (-)^{j+1} a_j, \quad \frac{K_{II}}{p\sqrt{\pi c}} = \frac{\alpha_{22}}{\beta} \sum_{j=0}^\infty (-)^{j+1} b_j \dots\dots\dots(43)$$

On the other hand substituting $\bar{f}_z(\lambda)$ expressed with $D_r(\lambda)$ and $D_z(\lambda)$ of Eq.(17) into Eqs.(12) and (14) we have

$$D_r^+(\lambda) = \phi_s^+(\lambda) + \psi_s(\lambda) D_s(\lambda), \quad D_r^-(\lambda) = \phi_s^-(\lambda) D_r(\lambda) + \psi_s(\lambda) D_s(\lambda) \dots\dots\dots(44)$$

where

$$\left. \begin{aligned} \phi_s^+(\lambda) &= \frac{(1+k_2)\tau_1(\lambda)\{\tau_2(\lambda)+\kappa_2(\lambda)\} - (1+k_1)\tau_2(\lambda)\{\tau_1(\lambda)+\kappa_1(\lambda)\}}{(k_2-k_1)\{\tau_1(\lambda)+\kappa_1(\lambda)\}\{\tau_2(\lambda)+\kappa_2(\lambda)\}} \\ \phi_s^-(\lambda) &= -\frac{(1+k_2)\kappa_1(\lambda)\{\tau_2(\lambda)+\kappa_2(\lambda)\} - (1+k_1)\kappa_2(\lambda)\{\tau_1(\lambda)+\kappa_1(\lambda)\}}{(k_2-k_1)\{\tau_1(\lambda)+\kappa_1(\lambda)\}\{\tau_2(\lambda)+\kappa_2(\lambda)\}} \\ \psi_s(\lambda) &= -\frac{(1+k_2)\mu_1\{\tau_2(\lambda)+\kappa_2(\lambda)\} - (1+k_1)\mu_2\{\tau_1(\lambda)+\kappa_1(\lambda)\}}{(k_2-k_1)\{\tau_1(\lambda)+\kappa_1(\lambda)\}\{\tau_2(\lambda)+\kappa_2(\lambda)\}} \lambda \end{aligned} \right\} \dots\dots\dots(45)$$

Noting that Eq.(45) is approximated as follows for a sufficiently large value of λ

$$\lim_{\lambda \rightarrow \infty} \{\phi_s^+(\lambda), \phi_s^-(\lambda), \psi_s(\lambda)\} = \{\delta_{11}, -\delta_{11}, \delta_{22}\} \dots\dots\dots(46)$$

, we establish

$$\phi_s^+(\lambda) = \delta_{11} + \bar{\psi}_s(\lambda), \quad \phi_s^-(\lambda) = -\delta_{11} + \bar{\psi}_s(\lambda), \quad \psi_s(\lambda) = \delta_{22} + \bar{\psi}_s(\lambda) \dots\dots\dots(47)$$

and then substituting them into Eq.(44) we get

$$\left. \begin{aligned} D_r^+(\lambda) &= \{\delta_{11} + \bar{\psi}_s(\lambda)\} D_r(\lambda) + \{\delta_{22} + \bar{\psi}_s(\lambda)\} D_s(\lambda) \\ D_r^-(\lambda) &= \{-\delta_{11} + \bar{\psi}_s(\lambda)\} D_r(\lambda) + \{\delta_{22} + \bar{\psi}_s(\lambda)\} D_s(\lambda) \end{aligned} \right\} \dots\dots\dots(48)$$

where

$$\delta_{11} = \frac{1}{2}, \quad \delta_{22} = \frac{(1+k_1)\sqrt{\mu_2} - (1+k_2)\sqrt{\mu_1}}{2(k_2-k_1)} \dots\dots\dots(49)$$

and $\bar{\psi}_s(\lambda) = O(1/\lambda)$, $\bar{\psi}_s(\lambda) = O(1/\lambda^2)$

Substituting Eq. (25) into Eq. (48) and further into Eqs. (11) and (13) we have

$$\frac{c_{44}(u_r^+)_{s+1}}{pb} = \sum_{j=0}^{\infty} a_j \left\{ \delta_{11} \int_0^{\infty} \lambda^{-1} J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda + \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda \right\} \\ + \sum_{j=0}^{\infty} b_j \left\{ \delta_{22} \int_0^{\infty} \lambda^{-1} J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda + \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda \right\} \dots (50)$$

$$\frac{c_{44}(u_r^-)_{s+1}}{pb} = \sum_{j=0}^{\infty} a_j \left\{ -\delta_{11} \int_0^{\infty} \lambda^{-1} J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda + \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda \right\} \\ + \sum_{j=0}^{\infty} b_j \left\{ \delta_{22} \int_0^{\infty} \lambda^{-1} J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda + \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda \right\} \dots (51)$$

Particularly on the crack surfaces ($|\zeta| \leq \beta$), since we have

$$\left. \begin{aligned} \int_0^{\infty} \lambda^{-1} J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda &= (-)^j U_{2j+1}(\zeta/\beta) / (2j+1) \\ \int_0^{\infty} \lambda^{-1} J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda &= (-)^{j+1} T_{2j+2}(\zeta/\beta) / (2j+2) \end{aligned} \right\} \dots (52)$$

, Eqs. (50) and (51) reduce to

$$\frac{c_{44}(u_r^+)_{s+1}}{pb} = \sum_{j=0}^{\infty} a_j \left\{ \frac{(-)^j U_{2j+1}(\zeta/\beta)}{(2j+1)} + A_j(\zeta) \right\} + \sum_{j=0}^{\infty} b_j \left\{ \frac{(-)^{j+1} T_{2j+2}(\zeta/\beta)}{(2j+2)} + B_j(\zeta) \right\} \dots (53)$$

$$\frac{c_{44}(u_r^-)_{s+1}}{pb} = \sum_{j=0}^{\infty} a_j \left\{ \frac{(-)^{j+1} U_{2j+1}(\zeta/\beta)}{(2j+1)} + A_j(\zeta) \right\} + \sum_{j=0}^{\infty} b_j \left\{ \frac{(-)^j T_{2j+2}(\zeta/\beta)}{(2j+2)} + B_j(\zeta) \right\} \dots (54)$$

where

$$A_j(\zeta) = \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+1}(\lambda\beta) \cos \lambda \zeta d\lambda, \quad B_j(\zeta) = \int_0^{\infty} \lambda^{-1} \bar{\psi}_s(\lambda) J_{2j+2}(\lambda\beta) \cos \lambda \zeta d\lambda \dots (55)$$

and $T_{2j+2}(\cdot)$'s are Tchebysheff polynomials and $U_{2j+1}(\cdot)$'s are Tchebysheff functions of the second kind.

3. Anisotropic Parameters and Engineering Constants

The foregoing solutions contain μ_z , k_z and β_z in the form of anisotropic parameters which can be expressed with five independent elastic constants c_{11} , c_{12} , c_{13} , c_{33} and c_{44} of a transversely isotropic elastic body. On the other hand, when we examine the obtained results in connection with the elastic constants, it is convenient to express these parameters with the engineering constants E , G , ν , E' , G' and ν' (11). That is,

$$\left. \begin{aligned} \mu_1 &= \frac{1}{2} (A \pm \sqrt{A^2 - 4B}), \quad A = \frac{2(1+\nu)(E'/E)(E'/E - \nu'G'/G)}{(G'/G)(E'/E - \nu'^2)} \\ B &= \frac{(E'/E)^2(1-\nu'^2)}{(E'/E - \nu'^2)}, \quad k_i = \frac{(E'/E - \nu'^2)\mu_i - (G'/2G)((1-\nu)(E'/E) - 2\nu'^2)}{(1+\nu)\nu'(E'/E) + (G'/2G)((1-\nu)(E'/E) - 2\nu'^2)} \\ \beta_i &= \frac{2}{(G'/G)(1+k_i)} \quad (i=1, 2) \end{aligned} \right\} \dots (56)$$

where E , G and ν represent the longitudinal and transverse moduli and Poisson's ratio in the plane of isotropy, respectively, while E' , G' and ν' are the corresponding ones in the principal material direction. These elastic constants are subjected to the following constraint conditions.

$$E, E', G, G' > 0, \quad |\nu'| < (E'/E)^{1/2}, \quad -1 < \nu < 1 - 2\nu'^2/(E'/E) \dots (57)$$

4. Numerical Calculations

In solving the simultaneous linear algebraic equations of Eq. (35) we must evaluate the infinite integrals given by Eq. (36). Here we separate a finite and an asymptotic part from these integrals, employing the same procedures as in Ref. (12), the former being evaluated using Simpson's rule and the latter given in closed form. That is,

$$\left. \begin{aligned} P_{ij}^1 &= \int_0^{10} \bar{\psi}_{11}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda + \int_{10}^{\infty} \bar{\psi}_{11}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda \\ P_{ij}^2 &= \int_0^{10} \bar{\psi}_{12}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda + \int_{10}^{\infty} \bar{\psi}_{12}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+1}(\lambda\beta) d\lambda \\ Q_{ij}^1 &= \int_0^{10} \bar{\psi}_{22}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda + \int_{10}^{\infty} \bar{\psi}_{22}(\lambda) J_{2i+1}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda \\ Q_{ij}^2 &= \int_0^{10} \bar{\psi}_{12}(\lambda) J_{2i}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda + \int_{10}^{\infty} \bar{\psi}_{12}(\lambda) J_{2i}(\lambda\beta) J_{2j+2}(\lambda\beta) d\lambda \end{aligned} \right\} \dots (58)$$

where $\tilde{\psi}_{11}(\lambda)$, $\tilde{\psi}_{12}(\lambda)$, and $\tilde{\psi}_{22}(\lambda)$ are asymptotic expressions for $\bar{\psi}_{11}(\lambda)$, $\bar{\psi}_{12}(\lambda)$, and $\bar{\psi}_{22}(\lambda)$ and they are given by

$$\tilde{\psi}_{11}(\lambda) = \bar{\alpha}_{11}/\lambda^2, \quad \tilde{\psi}_{12}(\lambda) = \bar{\alpha}_{12}/\lambda, \quad \tilde{\psi}_{22}(\lambda) = \bar{\alpha}_{22}/\lambda^2 \dots (59)$$

in which

$$\left. \begin{aligned} \bar{\alpha}_{11} &= -\frac{(1+k_1)(1+k_2)}{2(k_1-k_2)} \left(\frac{\beta_1^2 \mu_1^2 - \beta_1 \mu_1 - 1/8}{\mu_1 \sqrt{\mu_1}} - \frac{\beta_2^2 \mu_2^2 - \beta_2 \mu_2 - 1/8}{\mu_2 \sqrt{\mu_2}} \right) \\ \bar{\alpha}_{12} &= -\frac{(1+k_1)(1+k_2)}{2(k_1-k_2)} \left(\frac{\beta_1 \mu_1 - 1/2}{\sqrt{\mu_1}} - \frac{\beta_2 \mu_2 - 1/2}{\sqrt{\mu_2}} \right) \\ \bar{\alpha}_{22} &= -\frac{3(1+k_1)(1+k_2)(\sqrt{\mu_1} - \sqrt{\mu_2})}{16(k_1-k_2)\sqrt{\mu_1 \mu_2}} \end{aligned} \right\} \dots (60)$$

Substituting Eq.(59) into the asymptotic parts of Eq.(58), we have

$$\left. \begin{aligned} P_{1j}^1 &= \int_0^{i_0} \tilde{\psi}_{11}(\lambda) J_{2i}(\lambda \beta) J_{2j+1}(\lambda \beta) d\lambda + \bar{\alpha}_{11} \beta f_2(2j+1, \lambda_0 \beta) \\ P_{1j}^2 &= \int_0^{i_0} \tilde{\psi}_{12}(\lambda) J_{2i+1}(\lambda \beta) J_{2j+1}(\lambda \beta) d\lambda + \bar{\alpha}_{12} f_1(2i+1, 2j+1, \lambda_0 \beta) \\ Q_{1j}^1 &= \int_0^{i_0} \tilde{\psi}_{22}(\lambda) J_{2i+1}(\lambda \beta) J_{2j+2}(\lambda \beta) d\lambda + \bar{\alpha}_{22} \beta f_2(2i+1, 2j+2, \lambda_0 \beta) \\ Q_{1j}^2 &= \int_0^{i_0} \tilde{\psi}_{12}(\lambda) J_{2i}(\lambda \beta) J_{2j+2}(\lambda \beta) d\lambda + \bar{\alpha}_{12} f_1(2i, 2j+2, \lambda_0 \beta) \end{aligned} \right\} \dots (61)$$

where

$$f_k(m, n, x_0) = \int_{x_0}^{\infty} \frac{J_m(x) J_n(x)}{x^k} dx \dots (62)$$

is given in closed form⁽¹²⁾.

5. Numerical Results

Numerical calculations are carried out for the unidirectionally fiber reinforced composite materials such as E.glass/epoxy and graphite/epoxy as shown in Table 1 when they are subjected to a constant internal pressure $p(\zeta)=1$ or a shear traction $q(\zeta)=\zeta/\beta$ linearly distributing in the axial direction on the crack surfaces. The unknown coefficients a_j and b_j are truncated after 5~10 terms in an infinite system of algebraic equations of Eq.(35). Tables 2 and 3 show the nondimensional stress intensity factors $K_I/p\sqrt{\pi c}$ and $K_{II}/p\sqrt{\pi c}$ for the case of the constant internal pressure and shear traction, respectively. In the case of the constant internal pressure they correspond to the plane strain solutions for sufficiently small values of crack-length ($\beta=0$), when K_{II} vanishes and K_I remains all the same for any material. But although Mode II deformation as well as Mode I occurs as the crack-length is increased, the former is considerably smaller than the latter. On the other hand the relation is reversed for the axially distributing shear traction. Figures 2 and 3 show the effects of the crack-length on the stress intensity factors. While $K_I/p\sqrt{\pi c}$ for the constant internal pressure monotonously decreases with an increased value of crack-length, $K_{II}/p\sqrt{\pi c}$ once increases and then it gradually decreases. The nondimensional stress intensity factors $K_I/p\sqrt{\pi c}$ for the fiber reinforced composite materials are larger than those of the isotropic one, but the relation is reversed for $K_{II}/p\sqrt{\pi c}$. For the axially distributing shear fraction the nondimensional stress intensity factors $K_I/p\sqrt{\pi c}$ and $K_{II}/p\sqrt{\pi c}$ grad-

ually increase with an increased value of crack-length and their values of the fiber reinforced composites are smaller than those of the isotropic material. Figures 4(a), (b), and (c) show the crack opening displacements for different crack-lengths of $\beta=0.1, 1$ and 10 , respectively, when the constant internal pressure p is applied on the crack surfaces. The displacements have the largest values at the center of the crack surfaces when the crack-length is small and they decrease gradually towards the crack-tip, whereas they are almost constant when it is large, except near the end

of the crack. The crack opening displacement for E.glass/epoxy composite is about the same as that of the isotropic material, but that of graphite/epoxy composite is larger by 25~50%. Figures 5 and 6 show the effects of the elastic constants on the stress intensity factors $K_I/p\sqrt{\pi c}$ and $K_{II}/p\sqrt{\pi c}$ for $\beta=2$ where E'/E varies from unity to 50 for different values of $G'/G=1, 1.5$, and 2.5 with the fixed Poisson's ratio $\nu=\nu'=0.25$. The stress intensity factors K_I and K_{II} for constant internal pressure

Table 1 Elastic constants

	E'/E	G'/E	ν	ν'
E.glass/Epoxy ($\nu_g=0.63$)	3.44	0.37	0.37	0.24
Graphite/Epoxy ($\nu_g=0.5$)	11.5	0.56	0.32	0.30
Isotropy	1.0	0.38	0.30	0.30

Table 2 Stress intensity factors

$$[p(\zeta)=1, q(\zeta)=0]$$

$\beta=c/b$	Eglass/Epoxy		Graphite/Epoxy		Isotropy	
	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$
0*	1.0	0	1.0	0	1.0	0
0.01	0.9996	0.00004	0.9997	0.00004	0.9996	-0.0005
0.1	0.9748	-0.0004	0.9830	0.0040	0.9754	-0.0058
1	0.5738	-0.0506	0.6291	-0.0330	0.5408	-0.0838
10	0.18	-0.039	0.20	-0.040	0.17	-0.055

(*) Plane Strain Solution

Table 3 Stress intensity factors

$$[p(\zeta)=0, q(\zeta)=\zeta/8]$$

$\beta=c/b$	Eglass/Epoxy		Graphite/Epoxy		Isotropy	
	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$	$K_I/p\sqrt{\pi c}$	$K_{II}/p\sqrt{\pi c}$
0	0	0.5	0	0.5	0	0.5
0.01	-0.2×10^{-6}	0.49999	-0.1×10^{-6}	0.49999	-0.3×10^{-6}	0.49999
0.1	-0.00011	0.49985	-0.00005	0.49992	-0.00018	0.49976
1.0	0.00009	0.49205	-0.00243	0.49430	0.00664	0.48937
10.0	0.060	0.603	0.025	0.555	0.19	0.708

increase or decrease gradually, respectively, when E'/E is small, whereas they are almost constant when $E'/E \geq 30$. Both K_I and K_{II} tend to become a little larger with an increased value of G'/G . In contrast, for the case of the shear traction, the effects of E'/E and G'/G on the stress intensity factors are almost negligible except for sufficiently small values of E'/E .

6. Concluding Remarks

A cylindrical crack in a homogeneous transversely isotropic elastic body is considered in the framework of the three dimensional anisotropic elasticity theory in order to elucidate the macroscopic stress field near the crack-tips in the unidirectionally reinforced composite materials with a cylindrical interface crack. Two harmonic-type stress functions are employed and satisfaction of the boundary conditions through the method of Fourier transforms leads to simultaneous dual integral equations which are reduced to an infinite system of algebraic equations by use of the series expansion method. The following conclusions are drawn.

- (1) Both deformations of Mode I and II occur at the same time except for sufficiently small values of the crack-length. The stress intensity factor K_{II} is much smaller than K_I when a constant internal pressure is applied on the crack surfaces, but the relation is reversed when a shear traction is linearly distributed in the axial direction.

Fig. 2 Effects of crack-length on stress intensity factors

Fig. 3 Effects of crack-length on stress intensity factors

- (2) In the case of the constant internal pressure p , nondimensional stress intensity factor $K_I/p\sqrt{\pi c}$ is independent of the elastic constants for sufficiently small lengths of the crack $\beta=c/b$ and it corresponds to the plane strain solution of $K_I/p\sqrt{\pi c}=1$. But as it monotonously decreases with an increased value of β , some differences appear depending on the materials and stress intensity factor is the largest for the graphite/epoxy composite followed by the E.glass/epoxy composite and the isotropic material.
- (3) On the other hand in the case of the linearly distributed shear traction pz/c in the axial direction, the nondimensional stress intensity factors $K_{II}/p\sqrt{\pi c}$ attain 0.5 for sufficiently small values of β irrespective of the kinds of materials. But it decreases a little with an increased value of β and thereafter it gradually increases. The stress intensity factor of the isotropic material is the largest followed by the E.glass/epoxy and the graphite/epoxy composites.
- (4) When the constant internal pressure is applied on the crack surfaces of $\beta=2$, nondimensional stress intensity factor $K_I/p\sqrt{\pi c}$ becomes larger with an increased value of E'/E and/or G'/G , but, when the shear traction is distributed in the axial direction, $K_{II}/p\sqrt{\pi c}$ is almost constant for all values of E'/E and/or G'/G .

(a) $\beta=0.1$ (b) $\beta=1$ (c) $\beta=10$

Fig. 4 Crack opening displacements

Fig. 5 Effects of elastic constants on stress intensity factors

Fig. 6 Effects of elastic constants on stress intensity factors

References

- [1] Owen, M.J., and Dukes, R., J. Strain Anal., 2-4(1967), 272.
 - [2] Erdogan, F., Engng. Frac. Mech., 4 (1972), 811.
 - [3] Sih, G.C. and Chen, E.P., Mechanics of Fracture, Vol.6 Cracks in Composite Materials, (1981), Nijhoff Pub.
 - [4] Sih, G.C. et al., J. Comp. Mat., 9(1975), 167.
 - [5] Panasyuk, V.V. et al., Engng. Fract. Mech., 14-2(1981), 245.
 - [6] Erdogan, F. and Ozbek, T., Trans ASME, Ser. E, 36-4(1969), 865.
 - [7] Elliot, H.A., Proc. Camb. Philo. Soc., 44-4(1948), 522.
 - [8] Moriguchi, S. et al., Tables of Mathematical Formulae III, (1971), Iwanami Pub., 203.
 - [9] Watson, G.N., A Treatise on the Theory of Bessel Functions (2nd Ed.), (1966), 404, Camb. Univ. Press.
 - [10] *ibid*(8).
 - [11] Kasano, H. et al., Bull JSME, 25-200 (1982), 143.
 - [12] Kasano, H. et al., Bull JSME, 24-195 (1981), 1521.
 - [13] Chen, C.H., J. Comp. Mat., 1(1967), 30.
 - [14] Chen, C.H. and Cheng, S., Trans ASME, Ser. E., 37-1(1970), 186.
-