A Numerical Method for Free Vibration Analysis of Structures with Small Design Changes*

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Small design changes are usually repeated at design phase of structures, and it appears to be of practical significance to develop effective numerical methods for analysis of free vibration problems of structures with small perturbations. This paper presents a numerical method for perturbation analysis of matrix eigenvalue problems, where the eigenpairs necessary to be considered are assumed to be non-degenerate in unperturbed states. The proposed method is based on an iteration scheme combined with the use of generalized inverses of singular matrices. The scheme is designed to be easily implemented on large-scale finite element programs. To see effectiveness and fundamental properties of the proposed method, some numerical results are given for lateral vibrations of cantilever beams and circular plates.

Key Words: Free Vibration, FEM, Eigenvalue Problem, Perturbation, Generalized Inverse, Iteration, Design Change

1. Introduction

At the design phase of structures, we usually repeat static and dynamic analyses to determine the optimum stiffness and mass distributions of structures. The finite element method is considered a numerical tool convenient for such purposes. Since eigenvalue analysis for vibration problems is much more costly than the static one, efforts have been made to develop efficient algorithms. In standard algorithms, eigenpairs of structures with design changes are usually calculated independently of the original structures.(1),(2) However, it is often necessary to evaluate the variations of eigenpairs caused by small design changes. Therefore, it appears to be of practical significance to develop numerical methods well-suited for such objectives.

Theoretically, small design changes in structures may be regarded as perturbations to the governing equations of the original systems. In this paper, we will propose a numerical method for perturbation analysis of matrix eigenvalue problems arising in finite element calculations of free vibrations of structures. When the applied perturbations are small, the variations of eigendata are expected to be also small, and in such cases we can adopt the standard perturbation method based on a formal power series of small design parameters.(3) In fact, there have been proposed various approaches of this kind in conjunction with the sensitivity analysis of structural eigenvalue problems.(4),(5),(6) However,

the perturbation schemes for higher order terms of power series are in general very complicated and have some difficulties in implementation on computers. Furthermore, some of such schemes need the whole eigenpairs of unperturbed systems, and hence the band properties of matrices are not available in finite element calculations. Besides the conventional perturbation schemes, there has been developed another type of approaches based on the substructure concept.(7),(8),(9) Some of such methods appear well-suited for a special type of design changes, but may be difficult to use for general purposes.

Our proposed approach is based on an iteration scheme combined with the use of generalized inverse of singular matrices. Due to the iteration nature, it is well adapted to automatic computations, and gives exact solutions to the matrix eigenvalue problems when the iteration process converges. The number of iterations required for convergence is small for small applied perturbations, and the process needs only information on small number of eigenpairs in the unperturbed systems. Furthermore, band properties of matrices are as effectively utilized in our method as in the approach of Ref. 6, and we can take full advantage of the finite element solvers.

In this paper, the fundamental theory and the iteration scheme of our method are presented for non-degenerate eigenvalue problems. Further, we discuss efficiency of our method by roughly comparing the number of necessary operations with that of the inverse iteration method. In order to show validity of the proposed method, we will take for numerical examples free vibration problems of cantilever beams with various types of perturbations, and a circular plate with a rigid ring. These examples will show that our method is applicable to a variety of structural problems with design changes.

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2. Basic Theory

In the finite element analysis of free vibration problems of structures, we usually have algebraic eigenvalue problems of the form (1), (2)

$$K\varphi = \lambda M\varphi, \quad (1)$$

where K and M are the stiffness and mass matrices of the system, respectively, φ is the column vector for nodal displacements, and λ the eigenvalue representing the square of the angular frequency ω . The orders of the matrices and vectors are all equal to a positive integer n . Usually, both K and M are symmetric, and, moreover, K and M are positive semi-definite and positive definite, respectively. Then problem (1) has n eigenvalues, which can be arranged as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n, \quad (2)$$

where the equal signs are prepared for possible multiple eigenvalues. Furthermore, we can choose the associated eigenvectors φ_i ($1 \leq i \leq n$) to be M -orthonormal in the sense that

$$\varphi_i^t M \varphi_j = \delta_{ij} \quad (1 \leq i, j \leq n), \quad (3)$$

where the superscript t indicates the transpose of vectors and matrices, and δ_{ij} is Kronecker's delta. We call each $\{\lambda_i, \varphi_i\}$ an eigenpair of Eq. (1).

Consider the case when certain small design changes are introduced in the structures. Then the vibration equation (1) becomes

$$(K + \delta K)\varphi = \lambda(M + \delta M)\varphi, \quad (4)$$

where δK and δM , respectively, are the variations or perturbations of K and M due to design changes. We assume that δK and δM are symmetric and have the same orders as K and M . Henceforth, δ will be used as symbols for variations of various quantities.

Let us consider a specific eigenpair $\{\lambda_i, \varphi_i\}$ ($1 \leq i \leq n$) of Eq. (1), which we assume to be non-degenerate. That is, any eigenvector associated with λ_i is a multiple of φ_i . Due to δK and δM , the eigenpair $\{\lambda_i, \varphi_i\}$ varies as

$$\{\lambda_i, \varphi_i\} \rightarrow \{\lambda_i + \delta\lambda_i, \varphi_i + \delta\varphi_i\}. \quad (5)$$

The variation $\delta\varphi_i$ of φ_i is generally expressed as

$$\delta\varphi_i = \alpha\varphi_i + \psi_i \quad \text{with} \quad \varphi_i^t M \psi_i = 0, \quad (6)$$

where $\alpha\varphi_i$ with scalar α denotes the component of $\delta\varphi_i$ proportional to φ_i , and ψ_i the component M -orthogonal to φ_i . Then the eigenvector of the perturbation problem (4) is written as

$$\varphi_i + \delta\varphi_i = (1 + \alpha)\varphi_i + \psi_i. \quad (7)$$

For relatively small perturbations, the absolute value of α can be expected to be

also small. Then $1 + \alpha \neq 0$, and we find that $(\varphi_i + \delta\varphi_i)/(1 + \alpha) = \varphi_i + \psi_i/(1 + \alpha)$ is also an eigenvector of (4). Thus we can assume a priori that $\delta\varphi_i$ is M -orthogonal to φ_i without loss of generality:

$$\varphi_i^t M \delta\varphi_i = 0. \quad (8)$$

Substituting $\{\lambda_i + \delta\lambda_i, \varphi_i + \delta\varphi_i\}$ into Eq. (4) with Eq. (1) taken into account, we have

$$(K - \lambda_i M)\delta\varphi_i = g(\delta\lambda_i, \delta\varphi_i) \quad (9)$$

with

$$g(\delta\lambda_i, \delta\varphi_i) = \delta\lambda_i(M + \delta M)(\varphi_i + \delta\varphi_i) + (\lambda_i \delta M - \delta K)(\varphi_i + \delta\varphi_i). \quad (10)$$

Once δK and δM are given, the right-hand side of Eq. (9) is determined by the vector function of $\delta\lambda_i$ and $\delta\varphi_i$ given by Eq. (10).

To deal with Eq. (9), let us consider the matrix equation

$$Au = f \quad \text{with} \quad A = K - \lambda_i M. \quad (11)$$

Since $A\varphi_i = (K - \lambda_i M)\varphi_i = 0$ for $\varphi_i \neq 0$, the coefficient matrix A in Eq. (11) is singular, and the solution vector u may not exist for a general given vector f . The rank of A is $n-1$ because λ_i is assumed to be non-degenerate. We will clarify the condition under which Eq. (11) has solutions, and also obtain an explicit form of the possible solutions.

Denote the unit column vector e_j ($1 \leq j \leq n$) whose j -th component is unity and others are zero. Then we can express φ_i as

$$\varphi_i = \sum_{j=1}^n \varphi_{ij} e_j, \quad (12)$$

where φ_{ij} ($1 \leq j \leq n$) is nothing but the j -th component of φ_i . Since $\varphi_i \neq 0$, some components of φ_{ij} 's are not zero. If $\varphi_{ii} \neq 0$, then the vectors $\varphi_i, e_2, e_3, \dots, e_n$ are linearly independent, and hence can be regarded as a basis of the n -dimensional space. Consequently, the matrix T defined by

$$T = [\varphi_i \ e_2 \ e_3 \ \dots \ e_n] \quad (13)$$

is nonsingular, and we can introduce the transformation

$$u = Tv, \quad (14)$$

where v is a column vector of order n . Multiplying T^{-1} to both sides of Eq. (11) with Eq. (14) substituted into, we have

$$T^{-1}ATv = T^{-1}f. \quad (15)$$

Noting that $A\varphi_i = 0$, $A^{-1} = A$, and that $e_j^t A e_k$ ($2 \leq j, k \leq n$) coincides with the (j, k) -component of A , we find

$$\begin{bmatrix} 0 & (0')^t \\ 0' & A' \end{bmatrix} \begin{bmatrix} v_1 \\ v' \end{bmatrix} = \begin{bmatrix} \varphi_i^t f \\ f' \end{bmatrix}, \quad (16)$$

where $0'$ is the zero column vector of order

$n-1$, A' the square matrix of order $n-1$ obtained from A by excluding the first row and column, v_1 the first component of v , and v' and f' are column vectors of order $n-1$ respectively obtained from v and f by excluding the first components. From Eq. (16), we have

$$\varphi_1 f = 0, \quad A' v' = f'. \quad (17, 18)$$

Eq. (17) is the solvability condition for Eq. (15), and is also that for Eq. (11) due to the nonsingularity of T . That is, Eq. (15) or Eq. (11) is solvable if and only if f satisfies Eq. (17). Notice here that A' is nonsingular, since the rank of $T'AT$ with the nonsingular matrix T remains the same as the rank of A ($=n-1$). (10) Now the solution of Eq. (15) under the condition (17) is expressed as

$$v = \begin{Bmatrix} v_1 \\ (A')^{-1} f' \end{Bmatrix}, \quad (19)$$

where v_1 is an arbitrary number. From Eqs. (14) and (19), we can obtain the solution of Eq. (11) as

$$u = Tv = v_1 \varphi_1 + \bar{u} \quad \text{with} \quad \bar{u} = \begin{Bmatrix} 0 \\ (A')^{-1} f' \end{Bmatrix}. \quad (20)$$

As may be seen from the above expression, the factor $v_1 \varphi_1$ is indefinite, and hence an additional condition can be introduced to determine u uniquely. Corresponding to Eq. (8), we will impose the following on u :

$$\varphi_1 Mu = 0. \quad (21)$$

With Eqs. (21) and (3) taken into account, u may be uniquely specified as

$$u = \bar{u} - (\varphi_1 M \bar{u}) \varphi_1. \quad (22)$$

The operator, relating the above u to each f with Eq. (17), is a kind of generalized inverse of A and is denoted by A^* (11):

$$u = A^* f. \quad (23)$$

When $\varphi_{11} \neq 0$, an explicit expression of A^* is given by

$$A^* = (I - \varphi_1 \varphi_1' M) \begin{bmatrix} 0 & (0')' \\ 0' & (A')^{-1} \end{bmatrix}, \quad (24)$$

where I is the identity matrix of order n . In the derivation of Eq. (24), we have assumed $\varphi_{11} \neq 0$: when $\varphi_{1j} \neq 0$ for $j \neq 1$, we can obtain a similar explicit expression. To assure numerical stability, the choice of j is very important.

Now we return to Eq. (9). Corresponding to Eq. (17), we have the following solvability condition for Eq. (9):

$$\begin{aligned} \delta \lambda_i (\varphi_i + \delta \varphi_i)' (M + \delta M) \varphi_i \\ + (\varphi_i + \delta \varphi_i)' (\lambda_i \delta M - \delta K) \varphi_i = 0, \end{aligned} \quad (25)$$

or, if the coefficient of $\delta \lambda_i$ is not zero,

$$\delta \lambda_i = \Lambda(\delta \varphi_i), \quad (26)$$

with

$$\Lambda(\delta \varphi_i) = \frac{(\varphi_i + \delta \varphi_i)' (\delta K - \lambda_i \delta M) \varphi_i}{(\varphi_i + \delta \varphi_i)' (M + \delta M) \varphi_i}. \quad (27)$$

The denominator of Eq. (26) is expected to be close to $\varphi_i' M \varphi_i = 1$ and different from zero, since $\delta \varphi_i$ is small for small perturbations. Eq. (26) implies that $\delta \lambda_i$ is uniquely determined once $\delta \varphi_i$ is given. Substituting Eq. (26) into Eq. (9) and using the generalized inverse $(K - \lambda_i M)^*$, we find that

$$\begin{aligned} \delta \varphi_i &= (K - \lambda_i M)^* g(\delta \lambda_i, \delta \varphi_i) \\ &= (K - \lambda_i M)^* g(\Lambda(\delta \varphi_i), \delta \varphi_i). \end{aligned} \quad (28)$$

We must solve the above for $\delta \varphi_i$. However, this is an implicit relation in $\delta \varphi_i$, and in general needs some iteration processes to solve.

Notice here that we can obtain some valuable formulas from Eq. (26). Neglecting the term $\delta \varphi_i$ in the right-hand side of this equation, we have

$$\delta \lambda_i^{(0)} = \varphi_i' (\delta K - \lambda_i \delta M) \varphi_i / \varphi_i' (M + \delta M) \varphi_i, \quad (29)$$

which coincides with the formula derived from the one term Rayleigh-Ritz approximation based on φ_i . (4) Further neglecting the term δM in the denominator of Eq. (29) yields

$$\delta \lambda_i^{(0)} = \varphi_i' (\delta K - \lambda_i \delta M) \varphi_i, \quad (30)$$

which is a wellknown formula of the first order approximation in perturbation analysis. (4)

3. Computational Techniques

We need some iteration processes to solve Eqs. (26) and (28). First we assume a starting approximation $\delta \varphi_i^{(0)}$ of $\delta \varphi_i$, say $\delta \varphi_i^{(0)} = 0$, and then use the following recursive formula ($k = 1, 2, \dots$):

$$\delta \lambda_i^{(k)} = \Lambda(\delta \varphi_i^{(k-1)}), \quad (31)$$

$$\delta \varphi_i^{(k)} = (K - \lambda_i M)^* g(\delta \lambda_i^{(k)}, \delta \varphi_i^{(k-1)}). \quad (32)$$

This iteration process is stopped for a certain k by some termination conditions, and $\{\delta \lambda_i^{(k)}, \delta \varphi_i^{(k)}\}$ is employed as an approximation of $\{\delta \lambda_i, \delta \varphi_i\}$. As may be proved mathematically, $\delta \varphi_i^{(k)}$ ($k \geq 1$) are all M -orthogonal to φ_i and the approximate eigenpair converges to an exact solution if δK and δM are sufficiently small, that is, if we consider sufficiently small design changes. (12)

In practical calculation, we need no explicit expression of the generalized inverse matrix $A^* = (K - \lambda_i M)^*$. As may be seen from Eq. (20), the most essential step in our scheme is the calculation of vectors of the form $(A')^{-1} f'$, that is, the solution of simultaneous equations $A' v' = f'$. The matrix A' derived from a band matrix $A = K - \lambda_i M$ is also of band type, and hence

we can make use of standard finite element solvers for band matrices.(1)

We can consider another iteration scheme without use of generalized inverse operations. Using a parameter λ^* , which is close to but a little different from λ_i , we have the following equation, instead of Eq. (9):

$$(K - \lambda^* M) \delta \varphi_i = g^*(\delta \lambda_i, \delta \varphi_i) \\ \equiv g(\delta \lambda_i, \delta \varphi_i) + (\lambda_i - \lambda^*) M \delta \varphi_i. \quad (33)$$

In this case, the inverse matrix $(K - \lambda^* M)^{-1}$ exists due to the nonsingularity of $K - \lambda^* M$, and hence we have

$$\delta \varphi_i = (K - \lambda^* M)^{-1} g^*(\delta \lambda_i, \delta \varphi_i). \quad (34)$$

In order to derive the condition corresponding to Eq. (17), φ_i is multiplied to both sides of Eq. (33). Noting the K- and M-orthogonality of $\delta \varphi_i$ to φ_i , we have again

$$\delta \lambda_i = \Lambda(\delta \varphi_i). \quad (35)$$

Then we have another iteration scheme based on Eqs. (34) and (35) instead of Eqs. (31) and (32). Theoretically, the k-th approximation $\delta \varphi_i^{(k)}$ of $\delta \varphi_i$ is M-orthogonal to φ_i if the starting approximation $\delta \varphi_i^{(0)}$ is so. In the actual calculation, however, the M-orthogonality condition is likely to be contaminated by round-off errors, and hence it is recommended to perform a reorthogonalization process like Eq. (22) in every step.

Let us evaluate the efficiency of the present method and compare it with that of some standard method in which a perturbed problem is solved independently of the unperturbed one. As such a reference method, we take the inverse iteration method in which each necessary eigenpair is obtained separately as in our method.(1) Table 1 summarizes the kinds and numbers of main operations for these two methods. Dominant numbers of multiplications and

divisions in each operation are also included. For simplicity, we assume that the matrices $K, M, \delta K$ and δM are all of band type with half-bandwidth m_K , and that the number ℓ of iterations required for convergence is common to these two methods. As a method of solution for linear simultaneous equations, we consider the upper-triangularization scheme based on the Gauss elimination: the symmetric and band properties of matrices are taken into account.(1) The exchange of rows and columns is not employed in this scheme. We neglect in the table the number of inner products between vectors, and, furthermore, we assume that the vectors $K \varphi_i, M \varphi_i, \delta K \varphi_i$ and $\delta M \varphi_i$ are calculated beforehand and stored for use in iterations. In the proposed method we can use two different iteration schemes: one is based on Eqs. (31) and (32), and the other on Eqs. (34) and (35). However, they are essentially identical to each other in the number of main operations.

As shown in Table 1, the inverse iteration method needs one triangulation for every design change, and further triangulations are necessary if the so-called shift parameter is updated during iteration.(1) On the other hand, our method needs one and only one triangulation process for the same unperturbed problem. In the present method, the number of products between matrix and vector is twice as large as that in the inverse iteration method.

From the above observation, our method is considered superior to the inverse iteration method when m_K is large and ℓ is small. Assume that the inverse iteration method employs only one triangulation for each perturbation. Then rough estimation of the dominant numbers of multiplications and divisions in these methods shows that the present method becomes more efficient than the inverse iteration method when $m_K^2 n / 2 \geq 2 m_K n \ell$, i.e., $m_K \geq 4 \ell$. It is important to note that the main operations of matrices and vectors are common to these

Table 1 Comparison of main operations between the present method and the inverse iteration method

Operation	Dominant Term of Numbers of Products and Divisions in Each Operation	Number of Operations	
		Present Method	Inverse Iteration Method
Upper triangulation of $K - \lambda_i M$	$m_K^2 n / 2$	1	≥ 1 per each perturbation ≤ 1 per each iteration step
Solution of linear simultaneous equations by backward substitution	$2 m_K n$	ℓ	ℓ
Product between a matrix and a vector	$2 m_K n$	2ℓ	ℓ

two methods. That is, the present method can make full use of techniques developed for the inverse iteration method. It is even possible to implement both methods by a single program.

4. Numerical Examples

To show the effectiveness of our method, we will provide some typical numerical results for vibration problems.

The necessary eigenpairs of the unperturbed system must be obtained with sufficient accuracy. To this end, we use the bisection method combined with the inverse iteration method.(1),(2) For the iteration, the following starting and stopping conditions are used:

$$\delta \varphi_i^{(0)} = 0, \quad \delta \lambda_i^{(0)} = 0, \quad (36)$$

$$|(\delta \lambda_i^{(n)} - \delta \lambda_i^{(n-1)}) / \delta \lambda_i^{(n)}| \leq \epsilon, \quad (37)$$

where ϵ is the tolerance of error. The present calculations are performed with $\epsilon = 10^{-8}$ and in double precision. The iteration process is stopped at ℓ -th iteration where Eq. (37) is first satisfied: it is also stopped if Eq. (37) is not satisfied within 100 iterations. As already mentioned, it is necessary to exclude a certain row and a certain column from $K - \lambda_i M$ to obtain its generalized inverse. In our calculation, such row or column number is chosen the component number of φ_i where the absolute values of nodal displacements attain the maximum. We will use finite elements for lateral vibration of beams and axisymmetric circular plates. As for the interpolation functions for the lateral displacements, we employ the piecewise cubic polynomials of Hermite type. The employed element subdivisions are uniform, and the number of elements is denoted by N .

The first example is on lateral vibration of a uniform cantilever beam with a concentrated mass or a spring at the free end as shown in Fig. 1. In this case, we

Fig. 1 A cantilever beam with a concentrated mass or a spring at the free edge

take as the unperturbed system a cantilever beam without a concentrated mass and a spring. The length of the beam is denoted by L , the mass density of the beam by ρ , the cross-sectional area by A , and the bending rigidity by EI . The present method based on Eqs. (31) and (32) is adopted for various values of N , m_1/m_0 and k_1/k_0 , where m_1 is the concentrated mass, m_0 the mass of the beam ($m_0 = \rho AL$), k_1 the spring constant and k_0 is the equivalent spring constant of the beam defined by $k_0 = EI/L^3$. In this problem, the element stiffness and mass matrices can be exactly evaluated.

Table 2 shows the number of iterations and the calculated non-dimensional frequency parameter $\alpha = L(\rho A \lambda / EI)^{1/4}$ for the first mode with $m_1/m_0 = 1$ and $k_1 = 0$. The results are obtained for various values of N , and the exact solution of α is given by the minimum root of

$$1 + \cosh \alpha \cos \alpha + \frac{m_1}{m_0} \alpha [-\cosh \alpha \sin \alpha + \cos \alpha \sinh \alpha] = 0. \quad (38)$$

It is confirmed that the present method gives an exact finite element solution when the iteration process converges. The accuracy of the finite element solutions appears to be sufficient for $N \geq 10$. But more important to note is that the number of iterations is insensitive to N , that is, insensitive to degree of discretization. This property appears to be useful to

Table 2 Effect of N on the number of iterations: cantilever beam (first mode, $m_1/m_0 = 1$, $k_1 = 0$)

N	α	Number of Iterations
2	1.247 928	6
4	1.247 918	6
8	1.247 917	6
10	1.247 917	6
20	1.247 917	6
40	1.247 917	6
Exact	1.247 917	—

Table 3 Number of iterations for the first five modes: cantilever beam ($k_1 = 0$, $N = 10$)

m_1/m_0	i				
	1	2	3	4	5
0.02	4	7	8	8	10
0.04	4	8	10	12	14
0.08	5	10	13	23	62
0.20	5	12	52	—	—
0.40	5	18	—	—	—
0.80	6	—	—	—	—
1.00	6	—	—	—	—

estimate the necessary amount of calculations by refined finite element meshes.

For various values of m_1/m_0 with $k_1 = 0$ and $N = 10$, we observe convergence characters of the iteration scheme applied to the calculation of the first five modes. Table 3 summarizes the number of iterations required for convergence, and Fig. 2 illustrates the relation between α and m_1/m_0 for the first three modes. The number of iterations is not included in the table when convergence is not achieved within 100 iterations. The dotted lines in the figure denote the solutions obtained by the inverse iteration method for the region where the present method fails in convergence. We can see that, even for fairly large values of m_1/m_0 , the first mode can be obtained by the present method with a very small number of iterations. However, convergence behaviors are worse for higher modes.

Here we discuss the validity of the other iteration scheme based on Eqs. (34) and (35) under the same conditions as those of Table 3. Table 4 shows the number of iterations for $\lambda^* = 0.9\lambda_1$ and $\lambda^* = 0.99\lambda_1$. As λ^* approaches λ_1 , the number of iterations reduces and becomes nearly equal to that in Table 3. Hereafter, we will only show the results by a scheme based on Eqs. (31) and (32).

Fig. 2 Variation of α with m_1/m_0 : cantilever beam

Tables 5 and 6 respectively show the number of iterations and the non-dimensional frequency parameter α of the first five modes for various values of k_1/k_0 with $m_1 = 0$ and $N = 10$. Convergence is achieved within five iterations in all cases considered here. In this problem, the concentrated stiffness seems to have little influence on the change of higher mode shapes and the frequencies are relatively insensitive to this type of perturbations. From the results in Fig. 2 and Table 6, we can observe the general property that frequency is reduced either by increase of mass or by decrease of stiffness.

As a second example, we consider the axisymmetric lateral vibration of a circular plate with a rigid uniform ring along the edge, as shown in Fig. 3. The plate is assumed to be uniform and clamped at the center, and the edge rotation is constrained to satisfy the displacement compatibility between the plate and the ring. We denote the thickness of the plate

Table 4 Effect of λ^* on the number of iterations: cantilever beam ($k_1 = 0$, $N = 10$)

λ^*	$\lambda^* = 0.9\lambda_1$					$\lambda^* = 0.99\lambda_1$				
	i					i				
m_1/m_0	1	2	3	4	5	1	2	3	4	5
0.02	5	10	13	16	21	4	7	7	9	11
0.04	5	10	17	28	47	4	8	10	12	18
0.08	5	10	24	79	—	5	10	13	26	77
0.20	5	14	—	—	—	5	13	60	—	—
0.40	5	37	—	—	—	5	20	—	—	—
0.80	6	—	—	—	—	6	—	—	—	—
1.00	6	—	—	—	—	6	—	—	—	—

Table 5 Number of iterations for the first five modes: cantilever beam ($m_1 = 0$, $N = 10$)

k_1/k_0	i				
	1	2	3	4	5
0.02	4	4	3	3	3
0.20	4	4	3	3	3
0.40	4	4	4	4	3
0.80	5	5	4	4	4
1.00	5	5	4	4	4

Table 6 Variation of α with k_1/k_0 : cantilever beam ($m_1 = 0$, $N = 10$)

k_1/k_0	i				
	1	2	3	4	5
0.02	1.8781	4.6944	7.8558	11.0008	14.1550
0.20	1.9047	4.6961	7.8562	11.0009	14.1550
0.40	1.9328	4.6980	7.8566	11.0011	14.1551
0.80	1.9854	4.7019	7.8574	11.0014	14.1553
1.00	2.0100	4.7039	7.8578	11.0015	14.1553

by h , the mass density of the plate by ρ , Young's modulus by E , Poisson's ratio by ν , and the radius of the plate by R . In this case, a circular plate without a rigid ring is adopted as the unperturbed system. We use the 3-point Gauss quadrature for the calculations of element stiffness and mass matrices, and N is taken as 20. In Fig. 4, the first four non-dimensional frequency parameters $\beta = R(\rho h \lambda / D)^{1/4}$ are depicted against m_1/m_0 , where D is the bending stiffness defined by $D = Eh^3 / \{12(1-\nu^2)\}$, m_0 the mass of the plate ($m_0 = \pi \rho R^2 h$), m_1 the mass of the ring, and ν is taken as 0.3. The number of iterations required in the present method is given in Table 7. We can observe that the proposed method is valid for the first two modes when $m_1/m_0 \leq 1$. The general behavior of iteration for the present problem appears to be similar to that for the problem of a beam with a concentrated mass.

As the final example, we apply our method to the lateral vibration problem of a cantilever beam with variable rectangular cross-section. This example appears to be suitable to see the applicability of the present method to distributed perturbations. We assume that the thickness of the cantilever beam is uniform but the width of the beam varies linearly in the longitudinal direction: the widths of the beam at the fixed and the free ends are denoted by b_0 and b_1 , respectively. In the analysis, b_0 is kept constant, say $b_0 = 1$, while b_1 changes parametrically. A uniform beam

Fig. 3 A circular plate with a rigid ring along the edge

Table 7 Number of iterations for the first four modes: circular plate ($N = 20$)

m_1/m_0	i			
	1	2	3	4
0.02	4	6	6	7
0.04	4	6	7	8
0.06	4	6	8	9
0.08	4	6	8	10
0.10	4	8	9	11
0.20	5	10	12	17
0.40	5	12	24	—
0.60	5	15	61	—
0.80	5	22	—	—
1.00	5	37	—	—

Fig. 4 Variation of β with m_1/m_0 : circular plate

Fig. 5 Variation of α with b_1/b_0 : cantilever beam with variable width

with $b_1/b_0 = 1$ is adopted as the unperturbed system, and N is taken as 10. The 4-point Gauss quadrature is used to obtain the exact element stiffness and mass matrices.

In Fig. 5, the non-dimensional frequency parameters α ($= L(\rho A \lambda / EI)^{1/4}$) are shown against b_1/b_0 for the first three modes. Notations are essentially the same as in the first example, but the quantities A and EI are taken as those for the uniform beam ($b_1 = b_0$). Table 8 gives the number of iterations required for convergence for $0 \leq b_1/b_0 \leq 2$. Since both mass and stiffness of beam increase with the parameter b_1/b_0 , it is rather difficult to predict the change of frequencies with b_1/b_0 . The calculated results in Fig. 5 show that frequencies are more sensitive to the change of mass and decrease monotonically with an increase of b_1/b_0 . We can also see that the present method is available for this type of perturbations.

5. Conclusions

We have proposed an iteration scheme for perturbation analysis of matrix eigenvalue problems arising from finite element vibration analysis of structures with small design changes. Our method utilizes the generalized inverse operations for band matrices, which can be easily implemented on computers. We have shown some numerical results for lateral vibrations of beams and circular plates with various kinds of perturbations. Our method has turned out to be effective for analysis of non-degenerate eigenpairs with small perturbations. These numerical examples are relatively simple and may be insufficient to discuss the validity of the present method applied to more complicated practical problems. However, we can consider the examples appropriate to clarify the fundamental properties of the present method. We have also compared the number of dominant operations in the present method with that

Table 8 Number of iterations for the first five modes: cantilever beam with variable width ($N = 10$)

b_1/b_0	i				
	1	2	3	4	5
0.00	38	71	—	—	—
0.20	27	33	45	53	58
0.40	17	18	26	28	30
0.60	12	13	17	18	18
0.80	8	9	11	11	11
1.20	8	9	11	11	11
1.40	12	13	16	17	18
1.60	18	21	24	28	29
1.80	29	37	36	50	56
1.90	44	59	67	73	89
2.00	—	—	—	—	—

in the inverse iteration method.

We have treated free vibration problems only, but similar eigenvalue problems arise in various fields of engineering. A typical example is the buckling problem where we are usually interested in the smallest eigenvalue and the associated mode. Thus the present method is expected to be efficient for such a problem, since it gives generally good convergence behaviors for the first eigenpair.

It is possible to apply our method to a degenerate eigenvalue problem, and such analyses will be reported elsewhere.

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