Green's Functions for Axisymmetric Body Force Problems
of an Elastic Half Space and Their Application
(An Elastic Half Space with a Hemispherical Pit)*

By Hisao HASEGAWA**

The principal object of this paper is to show Green's functions for axisymmetric body force problems of an elastic half space. Green's functions are defined as solutions to the problems of an elastic half space subjected to axisymmetric body forces acting on a circle in (i) radial, (ii) circumferential and (iii) axial directions, respectively. It is assumed that the boundary surface of the half space is stress free from applied surface forces.

As an example of application of Green's functions shown in this paper, a method of solution is proposed for stress concentration problems of a half space with a hemispherical pit. Numerical results are compared with the results known. The method of solution proposed here may be called an application of the so-called body force method.

Key Words: Elasticity, Green's Function, Body Force, Elastic Half Space, Hemispherical Pit, Body Force Method, Numerical Analysis

1. Introduction

The principal object of this paper is to show Green's functions for axisymmetric body force problems of an elastic half space and to investigate a method of solution for stress concentration problems of an elastic half space with a hemispherical pit applying Green's functions shown. Green's functions are defined as solutions to the problems of an elastic half space subjected to axisymmetric body forces acting on a circle in the interior of the half space and satisfying the boundary conditions of the stress free from applied surface forces. The axisymmetric body forces acting on a circle are classified into (i) a radial force, (ii) a circumferential force (torsional force) and (iii) an axial force in cylindrical coordinates. Green's functions for the torsional body force (ii) mentioned above and an example of their application have been shown in previous papers(1) - (2). Recently, Green's functions for the body forces (i) and (iii) mentioned above and their application were investigated by Murakami et al. (4), using the so-called Mindlin's solution (5) for point force problems. The investigation by Murakami may be called a valuable work for an extension of Green's functions for the body forces (ii) to Green's functions (i) and (iii) mentioned above, but Green's functions proposed by Murakami are not a closed form solution but an integral expression in terms of Mindlin's solution.

Green's functions shown in this paper are derived by applying a general solution (1) for axisymmetric body force problems of an elastic half space and are given in a closed form. Since Green's functions shown are obtained in both expressions of displacements and stresses, they can be applied to mixed boundary value problems. To confirm the validity of Green's functions shown, these are applied to problems of an elastic half space with a hemispherical pit and the results are compared with known results. (6) (7)

2. Definition of Green's Functions

Let us introduce cylindrical coordinates (r, θ, z) to represent a point of an elastic half space (0 ≤ r < η, 0 ≤ z) as shown in Fig.1. Definition of Green's functions shown in this paper is the same as that of Murakami's one. (6) That is, these Green's functions are defined as solutions to the problems of an elastic half space subjected to axisymmetric body forces

\[ q_i = \frac{1}{2\pi} \delta(r-a)\delta(z-h), \quad (j=1, 2, 3) \]

acting on a circle of a radius a as shown in Fig.1 and satisfying the boundary conditions

\[ z=0, 0 \leq r < \infty; \quad q_r = q_\theta = 0 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2) \]

of no external surface forces at the boundary (z = 0), where \( q_j, \quad (j=1, 2, 3) \), are the

![Fig.1 A half space subjected to body forces on a circle (r = a, z = h)](image-url)
components of an axisymmetric body force vector acting on a circle in $r$- (Fig. 2), $\theta$- (Fig. 3), and $z$- directions respectively; $\delta(\cdot)$ is a Dirac delta function.

3. Fundamental Equations

In this paper we shall prepare fundamental equations to obtain Green's functions defined above. If the displacement components $u_j$ of an elastic solid are independent of the angle $\theta$ of cylindrical coordinates $(r, \theta, z)$, $u_j(r, z)$ can be expressed by

$$2\mu u_j = (1 - \nu) \left[ \rho + \frac{1}{r} \frac{\partial}{\partial r} \right] \delta_j - \text{grad} \cdot \text{div} \phi \tag{3}$$

provided that the stress functions $\phi_j(r, z)$ satisfy the differential equations

$$\left( \rho + \frac{1}{r} \frac{\partial}{\partial r} \right) \phi_j = -\frac{1}{1 - \nu} F_j, \quad \rho = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{4}$$

where $j = 1, 2, 3$ denote the $j$-components of the vector quantities in $r$-, $\theta$-, and $z$-directions respectively; $\delta_j$ is a Kronecker delta function; $\mu$ and $\nu$ are the shear modulus of elasticity and Poisson's ratio respectively; $F_j$ is a body force. From these expressions, we can show that when there are body forces $F_1$ in an elastic half-space, the displacements $u_j(r, z)$ satisfying the boundary conditions (2) are expressed as follows: (here we shall represent expressions of $u_j$ separately for each component of the body forces $F_j$ because simple superpositions of these expressions yield $u_j$ for the case of $F_j$ acting at the same time).

(i) $u_j$ due to only the body force $F_1$, $(F_2 = F_3 = 0)$;

$$2\mu u_j = \frac{2}{\pi(1 - \nu)} \int_0^\infty r F_1(\alpha) \cos \beta z \, dr \, dz$$

where $\bar{F}_1(\alpha)$ is

$$\bar{F}_1 = \frac{2}{\pi(1 - \nu)} \int_0^\infty r F_1(\alpha) \cos \beta z \, dr \, dz$$

and $F_1(\alpha, \beta)$ is an integral transform of $F_1(r, z)$ defined by

$$F_1 = \int_0^\infty F_1(\alpha, \beta) \cos \beta z \, dr \, dz$$

where $J_1(\alpha)$ is the Bessel function of the first kind of order $m$ and we must take $m = 1$ for $j = 1, 2$ and $m = 0$ for $j = 3$.

(ii) $u_j$ due to only the body force $F_2$, $(F_1 = F_3 = 0)$;

$$2\mu u_j = \frac{2}{\pi(1 - \nu)} \int_0^\infty r F_2(\alpha) \cos \beta z \, dr \, dz, \quad u_1 = 0$$

where $\bar{F}_2(\alpha, \beta)$ is expressed by putting $j = 2$ in Eq. (7).

(iii) $u_j$ due to only the body force $F_3$, $(F_1 = F_2 = 0)$;

$$2\mu u_j = \frac{2}{\pi(1 - \nu)} \int_0^\infty r F_3(\alpha) \sin \beta z \, dr \, dz - \int_0^\infty [2(1 - \nu) - \alpha^2] J_1(\alpha) \cos \beta z \, dr \, dz$$

$$+ \int_0^\infty (1 - 2\nu + \alpha^2) J_1(\alpha) \cos \beta z \, dr \, dz$$

$$u_1 = 0,$$

$$2\mu u_j = \frac{2}{\pi(1 - \nu)} \int_0^\infty r F_3(\alpha) \sin \beta z \, dr \, dz$$

$$+ \int_0^\infty (1 - 2\nu + \alpha^2) J_1(\alpha) \cos \beta z \, dr \, dz$$

$$u_1 = 0.$$
where $F_3(a, b)$ is expressed by putting $j = 3$ in Eq. (7) and $F_1(a)$ is expressed by

$$F_1 = \frac{2}{\pi (1 - \nu)} \int_0^\infty \frac{1 - \nu}{a^2 + b^2} \left(\frac{1 - \nu}{a^2 + b^2} \right) dB$$

It is easy to show the expressions of stresses due to the body forces $F_1$ as was investigated in the reference (1), but these expressions are omitted here.

4. Green's Functions

Here we shall show Green's functions defined in chapter 2 by applying the general solutions for axisymmetric body force problems of an elastic half space mentioned in the previous chapters. For the calculations of integrals necessary to derive the following expressions, we used the references (9), (10) and so on.

(i) Green's functions for a radial body force $F_1$: We shall show here the solutions (Green's functions) to the problem of an elastic half space subjected to a radial body force $q_1$ acting on a circle ($r = a$, $z = h$) as shown in Fig. 2 and satisfying the boundary conditions (2) at the surface ($z = 0$). Putting $j = 1$ in $q_j$ of Eq. (1) and substituting it into $F_1$ of Eq. (7), we get

$$F_1 = \frac{1}{2\pi} j_1(\alpha a) \cos \beta h$$

Substituting $F_1$ of Eq. (11) into Eq. (6) yields

$$F_1 = 4(1 - \nu) \left[ j_1(\alpha a) \cos \beta h \right]$$

Substituting $F_1$ and $F_3$ above into Eq. (5), we can obtain

$$2 \mu u_1 = \frac{1}{2D} \left\{ (3 - 4\nu) Q_1(x) + \frac{z^2}{2z} B_1 + (1 - 2\nu) Q_0(x) \right\} + \frac{(1 - 2\nu) z B_1 + \frac{z h}{2z} B_1}{\alpha}$$

$$2 \mu u_1 = -\frac{1}{2D} \left\{ \frac{1}{2x} B_1 + (1 - 2\nu) B_1 + \frac{(1 - 2\nu) z - 2(1 - \nu) h B_1 + \frac{z h}{2z} B_1}{\alpha} \right\}$$

where $Q_m(x_1)$ is a Legendre function of the second kind of order $m$. $x_1 = (x^2 + y^2 + z^2)/2\alpha$, $z_1 = z - h$. $x_2 = x + h$. $B = 8\pi (1 - \nu) (ar)Q_4$, and $B$ and $B'$ will be shown in Eq. (22).

Likewise, we can derive expressions for stresses due to a radial body force $q_1$ as follows:

$$\sigma_r = \frac{1}{2\pi} \left\{ \frac{1}{2\alpha} \left[ (3 - 4\nu) B_1 - (3 - 4\nu) Q_0(x) \right] + \frac{z^2}{2z} B_1 + \frac{z h}{2z} B_1 \right\}$$

$$-\frac{2\nu + (1 - 2\nu) z_1 B_1 - (1 - 2\nu) B_1 + \frac{z h}{2z} B_1}{\alpha}$$

$$\sigma_z = \frac{1}{2\pi} \left\{ \frac{1}{2\alpha} \left[ (3 - 4\nu) Q_1(x) + \frac{z^2}{2z} B_1 + 2(1 - 2\nu) B_1 + 2\nu (1 - 2\nu) B_1 + (1 - 2\nu) Q_0(x) \right] \right\}$$

$$-\frac{h z h}{2z} B_1 + \frac{z h}{2z} B_1$$

$$\tau_r = \frac{1}{2\pi} \left\{ \frac{1}{2\alpha} \left[ B_1 + 2(1 - \nu) B_1 + \frac{h z h}{2z} B_1 \right] \right\}$$

$$\tau_z = \tau_r = 0$$

(ii) Green's functions for the torsional body force $F_2$: Here we shall show the solutions (Green's functions) to the problem of an elastic half space subjected to a torsional body force $q_2$ acting on a circle ($r = a$, $z = h$) as shown in Fig. 3 and satisfying the boundary conditions (2) at the surface ($z = 0$). Putting $j = 2$ in $q_j$ of Eq. (1) and substituting it into $F_2$ of Eq. (7), we get

$$F_2 = \frac{1}{2\pi} j_2(\alpha a) \cos \beta h$$

Substituting $F_2$ shown above into Eq. (6) yields

$$2 \mu u_2 = \frac{1}{2\pi \sqrt{ar}} \frac{1}{\alpha} Q_2(x), u_i = u_2 = 0$$

These expressions have been shown in the previous paper (1). Similarly, stresses are expressed as follows:

$$\sigma_r = \frac{1}{8\pi^2 r^2} \frac{1}{ar} \left\{ 3Q_0(x) + \frac{z}{3z} G_1(x) \right\}$$

$$\tau_r = \frac{1}{8\pi^2 r^2} \frac{1}{ar} \left\{ \frac{z}{3z} G_1(x) \right\}$$

$$\sigma_r = \sigma_z = \tau_r = 0$$

where $G_1(x_1) = x_1 Q_1(x_1) - Q_1(x_1)$.
(iii) Green's functions for the axial body force $F_3$: We shall show the solutions (Green's functions) to the problem of an elastic half space subjected to an axial body force $q_3$ acting on a circle $(r=a, z=h)$ and satisfying the boundary conditions (2). Similar considerations as above (i) and (ii) yield the expressions for $F_3$ and $F_1$ as follows:

\[
F_3 = \frac{1}{2\pi(1-\nu)}(1-2\nu+\alpha z)l(2a)\
\]

Substituting these expressions into Eq. (9), we can derive expressions for displacements and stresses as follows:

\[
2\mu_1 = \frac{1}{2\pi a^2} \left[ 2(3-4\nu)Q_{ax}(a) + \frac{z}{2\pi} B_2 \right] + 2(1-\nu)Q_{ax}(a) + \frac{1-2\nu}{2\pi a^2} B_1 - \frac{z h}{2\pi a^2} B_1
\]

\[
2\mu_2 = \frac{1}{2\pi a^2} \left[ 2(3-4\nu)Q_{ax}(a) + \frac{z}{2\pi} B_2 \right] + 2(1-\nu)Q_{ax}(a) + \frac{1-2\nu}{2\pi a^2} B_1 - \frac{z h}{2\pi a^2} B_1
\]

\[
\sigma_r = \frac{1}{2\pi a^2} \left[ 2(3-4\nu)Q_{ax}(a) + \frac{z}{2\pi} B_2 \right] + 2(1-\nu)Q_{ax}(a) + \frac{1-2\nu}{2\pi a^2} B_1 - \frac{z h}{2\pi a^2} B_1
\]

\[
\sigma_\theta = \frac{1}{2\pi a^2} \left[ 2(3-4\nu)Q_{ax}(a) + \frac{z}{2\pi} B_2 \right] + 2(1-\nu)Q_{ax}(a) + \frac{1-2\nu}{2\pi a^2} B_1 - \frac{z h}{2\pi a^2} B_1
\]

In the above expressions, $B$ and $B'$ represent

\[
B_1 = \frac{1}{x^2-1} G_a(x, z) + \frac{1}{x^2-1} A_1, B_2 = Q_{ax}(x) - \frac{x^2 - 1}{x^2 - 1} G_a(x, z) + \frac{2\pi \sqrt{\rho}}{a} F(k) + \frac{\pi - r}{a + \rho} \Pi(p, k) (a > r)
\]

\[
B_1 = \frac{2\pi \sqrt{\rho}}{a} F(k) + \frac{\pi - r}{a + \rho} \Pi(p, k) (a = r)
\]

\[
B_1 = \frac{2\pi \sqrt{\rho}}{a} F(k) + \frac{\pi - r}{a + \rho} \Pi(p, k) (a < r)
\]

\[
B_1 = \frac{1}{x^2-1} A_1, B_2 = \frac{1}{x^2-1} \left[ \left( 1 - 2x, \frac{z}{2\pi a} \right) A_1 + \frac{1}{x^2-1} \frac{z}{2\pi a} A_1 \right]
\]

\[
B_1 = \frac{1}{x^2-1} \left[ 2Q_{ax}(x) - \frac{2\pi}{x^2-1} A_1 + \frac{1}{x^2-1} \frac{z}{2\pi a} Q_{ax}(x) + \left( 1 - x \right) \frac{z}{2\pi a} G_a(x, z) - \frac{z}{2\pi a} G_a(x, z) \right]
\]

\[
B_1 = \frac{1}{x^2-1} G_a(x, z), B_2 = \frac{1}{x^2-1} A_1, B_3 = \frac{1}{x^2-1} A_1, B_4 = B_5 + B_6 - B_7 - B_8
\]

where $G_2(x) = 4 \frac{Q_{ax}(x)}{x^2-1}, B_m = B_m(x), m = 1, 2, 3,$ and $F(k)$ and $\Pi(p, k)$ are complete elliptic integrals of the first and third kinds respectively, defined by

\[
F(k) = \int_0^\pi \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} d\phi, \Pi(p, k) = \int_0^\pi \frac{1}{(1-p \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}} d\phi
\]

where

\[
k^2 = \left( \frac{a + r}{a + r} \right)^2, p = \frac{4ar}{(a + r)^2}
\]

In Eq. (22), $A_m$ represents

\[
A_1 = \frac{1}{x^2-1} Q_{ax}(x) + \frac{z}{2\pi a} Q_{ax}(x),
\]

\[
A_2 = \frac{2\pi}{a} A_1 + \frac{1}{x^2-1} \left[ 4x - \frac{2\pi}{a} \left( 1 + \frac{2}{x^2-1} \right) + \frac{z}{2\pi a} \right] G_a(x, z) + \frac{4\pi}{a} Q_{ax}(x)
\]

\[
A_3 = G_a(x, z) + \frac{2}{x^2-1} Q_{ax}(x) + \frac{z}{2\pi a} G_a(x, z)
\]

\[
A_4 = Q_{ax}(x) + \frac{z}{a} Q_{ax}(x) - \frac{4}{x^2-1} G_a(x, z)
\]

\[
A_5 = \frac{16\pi}{a} G_a(x, z) + \frac{16\pi}{a} G_a(x, z) - 12 Q_{ax}(x) - \frac{4r}{a} Q_{ax}(x)
\]

Let us consider displacements $u(r, z)$ and stresses $q_{ij}(r, z)$ due to axisymmetric body forces $F_{j}(z)$ acting in a region $\Omega$ of an elastic half space. They can be obtained by superposition of Green's functions shown above, and are expressed by

$$[u_{i}]=\int_{\partial \Omega}[F_{i}(a, h)\frac{d^{2}}{a_{i}h}]+F_{i}(a, h)\frac{d^{2}}{a_{i}h}$$

$$+F_{i}(a, h)\frac{d^{2}}{a_{i}h}]dah$$

...(26)

where the terms with a superscript 1 means the displacements and the stresses of Eqs. (13) and (14), the terms with a superscript 2 are those of Eqs. (16) and (17), and the terms with a superscript 3 are those of Eqs. (20) and (21). Expressions (26) satisfy the boundary conditions (2) of the stress free from applied surface forces at the surface ($z=0$), regardless of the values of $F_{j}(a, h)$. Therefore, Eq. (26) can be applied to boundary value problems of elasticity besides body force problems if we determine the intensities of $F_{j}(a, h)$ such as to satisfy boundary conditions of a problem and take up the imaginary region of acting body forces from a region of consideration of a problem.

Assumption of $F_{j}(a, h)=0$ in Eq. (26) yields a method of solution for torsion problem. Applying the method of solution, the torsion of a rigid cylinder embedded in an elastic half space was investigated in previous papers. (7,8) Therefore, we shall show an example of application of Eq. (26) in the case of $F_{j}(a, h)=0$, $F_{1}(a, h)\neq 0$ and $F_{2}(a, h)\neq 0$, which is a problem other than torsions, in the next chapter.

6. An Example of Application (an elastic half space with a hemispherical pit)

Here we shall propose a method of solution for an elastic half space with a hemispherical pit subjected to (4) pressures at the pit surface and (1) an all-around tension at infinity. The principle of the method of solution is to distribute the body force $F_{1}$ and $F_{2}$ in the interior of a half space with no pits and to determine the intensities of the body forces distributed such as to satisfy boundary conditions of a semihemispherical plane which is to become a pit surface. There are no body forces in the half space with a pit, because we distribute body forces such that the region of acting body forces is regarded as a pit.

6.1 An elastic half space with a pit subjected to pressures

We use polar coordinates $(r, \theta, \psi)$ as shown in Fig. 4. The two dot-chain line of a radius $c$ in Fig. 4 represents a semihemispherical plane which is to become a pit surface, and the dotted line of a radius $e$ represents a plane where body forces are distributed. The boundary conditions of the problem are expressed by

Fig. 4 A half space with a pit ($R=0$) and the plane of distributions of body forces ($R=0$)

(i) $R=c, 0 \leq \theta \leq \pi/2$: $\gamma_{e}=\gamma(\psi)$,

$\tau_{e}=\tau_{\theta}=0$

(ii) $R=c, \pi/2 < \theta < \pi$: $\gamma_{e}=\tau_{\theta}=\tau_{e}=0$

...(27)

where $\gamma(\psi)$ is an axisymmetric pressure acting on the pit surface.

In order to obtain a solution to the problem, we distribute the body forces $q_{1}$ and $q_{2}$ defined by Eq. (1) on the plane shown by the radius $e$ in Fig. 4 and express the intensities of the body forces distributed by $\xi(\psi)$ and $\zeta(\psi)$. In this case, Eq. (26) becomes

$$[u_{i}]=\int_{0}^{\pi} \left[ \xi(\psi)\frac{d^{2}}{a_{i}h} + \zeta(\psi)\frac{d^{2}}{a_{i}h} \right] \psi \sin \psi d\psi$$

...(28)

Note that there are the relations $a=e \sin \psi$ and $h=e \cos \psi$ between the angle $\psi$ and the point $(a, h)$ of acting body forces.

According to the definition of Green's functions, Eq. (28) always satisfies the boundary conditions (11) of Eq. (27) regardless of the values of $\xi(\psi)$ and $\zeta(\psi)$. Therefore, it is sufficient to consider only the boundary conditions (1) of Eq. (27). The stress components $(\sigma_{x}, \tau_{xy}, \tau_{xz})$ in polar coordinates $(R, \theta, \psi)$ are obtained from Eq. (28) as follows:

$$\begin{align*}
\sigma_{x} &= \int_{0}^{\pi} \left[ \xi(\psi) \left( \frac{d^{2}}{a_{x}h} \right) + \zeta(\psi) \left( \frac{d^{2}}{a_{x}h} \right) \right] \psi \sin \psi d\psi \\
\tau_{xy} &= \int_{0}^{\pi} \left[ \xi(\psi) \left( \frac{d^{2}}{a_{y}h} \right) + \zeta(\psi) \left( \frac{d^{2}}{a_{y}h} \right) \right] \psi \sin \psi d\psi \\
\tau_{xz} &= \int_{0}^{\pi} \left[ \xi(\psi) \left( \frac{d^{2}}{a_{z}h} \right) + \zeta(\psi) \left( \frac{d^{2}}{a_{z}h} \right) \right] \psi \sin \psi d\psi
\end{align*}$$

...(29)

where $\sigma_{x}, \tau_{xy}, \tau_{xz}$ represent the stress components.

Application of the boundary conditions (1) of Eq. (27) to Eq. (29) yields

$$\begin{align*}
-\gamma(\psi) &= \int_{0}^{\pi} \left[ \xi(\psi) \frac{d^{2}}{a_{x}h} + \zeta(\psi) \frac{d^{2}}{a_{x}h} \right] \psi \sin \psi d\psi \\
\tau_{xy} &= \int_{0}^{\pi} \left[ \xi(\psi) \frac{d^{2}}{a_{y}h} + \zeta(\psi) \frac{d^{2}}{a_{y}h} \right] \psi \sin \psi d\psi \\
\tau_{xz} &= \int_{0}^{\pi} \left[ \xi(\psi) \frac{d^{2}}{a_{z}h} + \zeta(\psi) \frac{d^{2}}{a_{z}h} \right] \psi \sin \psi d\psi
\end{align*}$$

...(31)
If we have chosen the unknown functions (intensities of body forces distributed) \( \xi(\psi) \) and \( \zeta(\psi) \) which are solutions to the dual integral equations (31), then the stresses and displacements of a point \((r,\psi)\) of a half space with a pit are obtained from Eq. (28). To obtain a solution to the integral equations, we shall reduce them to a set of linear algebraic equations. For this purpose, we shall assume that
\[
\begin{align*}
\xi(\psi) &= \sum_{n=1}^{N} \xi_n \sigma_n(r_n, \psi_n), \\
\zeta(\psi) &= \sum_{n=1}^{N} \zeta_n \sigma_n(r_n, \psi_n), \\
\end{align*}
\]
(0 < \(\psi_n < \pi/2\))

(32)

where \(\xi_n\) and \(\zeta_n\) are unknown constants corresponding to the intensities of the body forces distributed. Assumption of Eq. (36) means that the body forces are distributed on \(N\) isolated points (circles).

Substitution of Eq. (32) into Eq. (31) yields
\[
\begin{align*}
-\mu \frac{d\psi}{dr} &= \sum_{n=1}^{N} \left( \xi_n \frac{d\psi_n}{dr_n} + \zeta_n \frac{d\psi_n}{dr_n} \right), \\
+\frac{1}{r} \frac{d\theta}{dr} &= \sum_{n=1}^{N} \left( \frac{\zeta_n}{r_n} \frac{d\psi_n}{dr_n} \right), \\
\end{align*}
\]

(33)

where \(a_n = \rho \sin \psi_n\) and \(b_n = \rho \cos \psi_n\). Since there are \(2N\) unknown arbitrary constants \(\xi_n\) and \(\zeta_n\) in Eq. (33), we shall consider \(N\) points on the semispherical plane expressed by the two dot-line chain of a radius \(c\), which is to become a pit surface, and choose the values \(\xi_n\) and \(\zeta_n\) such as to satisfy the boundary conditions (1) of Eq. (27) at these selected points (circles). It follows from the above that we can construct a set of simultaneous linear algebraic equations with \(2N\) unknown constants. Solving the equations and substituting their solutions obtained into the expressions
\[
\begin{align*}
\frac{u}{\sigma} &= \sum_{n=1}^{N} \left( \frac{\xi_n}{\sigma_n} \frac{d\sigma_n}{dr_n} + \frac{\zeta_n}{\sigma_n} \frac{d\sigma_n}{dr_n} \right), \\
\end{align*}
\]

(34)

the displacements and stresses at any point of the half space with a pit are obtained.

6.2 An elastic half space with a pit under an all-round tension

By a little alteration of the method of solution described in the previous section, we can consider also the problem of an elastic half space with a pit under an all-round tension \(p\) at infinity. The boundary conditions of the problem are expressed by

(i) \(r = c, 0 \leq \psi \leq \pi/2\): \(a_1 = r_{01} = r_{02} = 0\),
(ii) \(\psi = 0, c < r < \infty\): \(a_2 = r_{02} = 0\),
(iii) \(r = \infty\): \(a_3 = r_{02} = \rho, \sigma_{02} = \rho, \sigma_{01} = \rho\),
other stresses \(= 0\)

(35)

Now, the displacements \(u_3^2\) and stresses \(\sigma_{11}^2\) of an elastic half space with no pits subjected to an all-round tension \(p\) are expressed by
\[
\begin{align*}
u' &= \frac{\rho(1-\nu)}{2\mu(1+\nu)} r, \\
u' &= 0, \quad u' = -\frac{\rho v}{\mu(1+\nu)} \\
\sigma_{11} &= \rho, \quad \sigma_2 = r_{11} = r_{02} = 0
\end{align*}
\]

(36)

For the present problem, equations corresponding to Eq. (28) in the previous section are represented by superposition of Eq. (36) on Eq. (28) as follows:
\[
\begin{align*}
\left\{ \frac{u}{\sigma} \right\} &= \int_{c}^{0} \left[ \xi(\phi') \frac{d\phi'}{dr'} \right] \frac{d\phi}{dr} + \zeta(\phi') \frac{d\phi}{dr} \left( \frac{d\phi}{dr} - \frac{d\phi}{dr} \right), \\
+\frac{1}{r} \frac{d\theta}{dr} &= \int_{c}^{0} \left( \frac{\zeta(\phi')}{r'} \frac{d\phi'}{dr'} \right), \\
\end{align*}
\]

(37)

Expressions (37) always satisfy the boundary conditions (ii) and (iii) of Eq. (35). Therefore, the solution to the problem is obtained by choosing the values \(\xi(\psi)\) and \(\zeta(\psi)\) such as to satisfy the boundary conditions (1) of Eq. (35). Further discussions are similar to those in the previous section and are omitted here.

6.3 Numerical results

Based on the method of solution described above, numerical calculations were performed under the assumption of \(N = 40\) and \(\nu/C = 0.92\), and the assumption of Poisson's ratio \(\nu = 0.25\) for comparison with the results shown by Tsuchida et al. (5) (7). The results are shown in Figs. 5 to 8. Figures 5 and 6 show the distributions of stresses \(\sigma_0\) and displacements \(u_3\) at the surface \(z = 0\) of the half space with a pit subjected to uniform pressure \(p\) or a cosine-form pressure \(p \cos \psi\).
6. Concluding Remarks

Green’s functions for axisymmetric body force problems of an elastic half space were shown in this paper. As an example of application of Green’s functions shown, a method of solution was proposed for stress concentration problems of an elastic half space with a hemispherical pit subjected to pressures at the pit surface or an all-round tension at infinity. Numerical results were compared with the results shown by Tsuchida, Fujita and so on. The principle of the method of solution proposed here is to distribute body forces and to determine the intensities of the body forces distributed such as to satisfy boundary conditions of a problem. As was mentioned by Murakami, the method of solution has wide applications in boundary value problems of elasticity (for example the so-called body force method).

Acknowledgement

The author is indebted to Professors E. Tsuchida (Saitama University) and T. Fujita (National Defence Academy) for kindly making their results available. Thanks are also due to Mr. J. Takebe (Takada Kogyocho) for help with the numerical calculations.

References

(5) Mindlin, R.D., Physics, 2-6 (1936), 195.
(12) Fujita, T., private communication.