Plane Bifurcation of a Fiber-Reinforced Body 
with a Singular a- or n-Curve*

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Plane bifurcations from a finite deformation of an ideal fiber-reinforced body in which a singular a- or n-curve appears are considered. The relation between incremental stresses and incremental strains was obtained from the constitutive equation of the ideal fiber-reinforced composites, and jump conditions across the singular-stress curves for the incremental plane deformations were derived from their equilibrium. Four simple bifurcation problems were solved and compared as application examples.

Key Words: Elasticity, Plane Deformation, Singular Stress, Jump Condition, Fiber-Reinforced Body, Kinematical Constraint, a-Curve, n-Curve

1. Introduction

There are a considerable number of theories proposed to describe various static and dynamic behaviors of a fiber-reinforced body. Among them, the so-called "ideal fiber-reinforced composites theory," i.e., the theory of continua with internal kinematical constraints of inextensibility along the fiber direction and of incompressibility in the volume has been often used, when the tensile modulus of the fibers predominates over the matrix rigidity and when only the gross (macroscopic) behavior of the composite is considered. Resorting to this theory makes the stress and deformation analysis very simple, especially for plane deformations. Two-dimensional solutions in the theory, however, often involve singular fiber lines and normal lines in which the tensile stress component is infinite and across which other components are discontinuous. They are called "singular a- and n-curves," respectively, and "singular-stress curves" collectively.

On the basis of the inextensible (but not incompressible) theory, the present author treated some bifurcation problems of fiber-reinforced elastic slabs and strips. All of them except for Ref. (8) were concerned with cases in which no singular-stress curves appeared. There seems to be no literature available on bifurcation analysis involving singular-stress curves.

In the present paper are considered bifurcations from a finite deformation of an elastic body with internal kinematical constraints of incompressibility and inextensibility. It was assumed that the body was in a plane deformation state prior to and after bifurcation. No parallelism of the fibers was assumed. We derived a relation between incremental stresses and incremental strains and jump conditions across (or equilibrium conditions of) the singular a- and n-curves for the incremental plane deformation. Then, these conditions were applied to four contrasting simple bifurcation problems in which the singular stresses appeared prior to and/or after bifurcation.

2. Relation between Incremental Stresses and Incremental Strains

In this and next section, the two-dimensional Cartesian tensor notation is used. Greek subscripts have a range of (1,2).

The ideal fiber-reinforced material (incompressible and inextensible) has the following constitutive equation:[7]

\[ T_{ab} = P(\delta_{ab} - a_a a_b) + \tau_{ab} \delta + S(\gamma) (a_a n_b - n_a a_b) \] (1)

for the plane deformation, where \( T_{ab} \) denotes the plane Cauchy stress tensor, and \( P \) and \( \tau_{ab} \) reaction stresses associated with the two kinematical constraints. Symbol \( S(\gamma) \) is the Kronecker delta. Shering stress \( S(\gamma) \) is a function of the amount of shear \( \gamma \) and satisfies the following:

\[ S(0) = 0, \quad S(-1) = S(1). \] (2)

In Eq. (1), \( a_a \) is a unit vector tangent to the fibers in a finitely deformed state, and is given by

\[ a_a = \delta_{ab} n_b. \] (3)

If a unit vector tangent to the fibers in the natural state is denoted by \( a_a \), and the plane deformation matrix tensor, by \( F_{ab} \). Vector \( n_a \) is a unit vector normal to \( a_a \). Denote a unit vector normal to \( a_a \) by \( n_a \), and the following relation between \( a_a \) and \( n_a \) is valid:

\[ n_a = (F^{-1})_{ab} a_b. \] (4)

where \( (F^{-1})_{ab} \) is the inverse tensor of \( F_{ab} \). Trajectories of \( n_a \) and \( a_a \) are called...
"a-curves" and "n-curves," respectively.

Derivation of the relation between the incremental stresses and incremental strains from the constitutive equation (1) needs the material derivatives of $a_0$ and $n_0$, with respect to time. Taking the material time derivatives of Eqs. (3) and (4) and using the identities:

$$ F_{ab}^{\text{t}} = (D_{ab} + W_{ab}) F_{\text{pt}}, $$

$$ (p^{-1})_{ab}^{\text{t}} = (p^{-1})_{ab}^* (D_{ab} + W_{ab}), $$

with $D_{ab}$ and $W_{ab}$ denoting the deformation rate and spin tensors, yields

$$ \dot{a}_a = (D_{ab} + W_{ab}) a_b, \quad \dot{n}_a = (D_{ab} + W_{ab}) n_b. $$

Next, the incremental stress, incremental strain and rotation, and incremental reaction stresses for time increment $\Delta t$ are defined as follows:

$$ s_{ab}^{\text{t}} = D_{ab} s_{ab} + W_{ab} s_{ab} \Delta t, $$

$$ e_{ab}^{\text{t}} = D_{ab} e_{ab} + W_{ab} e_{ab} \Delta t, $$

$$ (p^{-1})_{ab}^{\text{t}} = (p^{-1})_{ab}^{*\text{t}} D_{ab} + W_{ab}, $$

Finally, taking the material time derivatives of Eqs. (6)-(9), we obtain

$$ s_{ab}^{\text{tt}} = \dot{D}_{ab} s_{ab} + \dot{W}_{ab} s_{ab} + 2S^{(t)} (\dot{D}_{ab} e_{ab} + \dot{W}_{ab} e_{ab}), $$

$$ e_{ab}^{\text{tt}} = \dot{D}_{ab} e_{ab} + \dot{W}_{ab} e_{ab} + 2S^{(t)} (\dot{D}_{ab} s_{ab} + \dot{W}_{ab} s_{ab}), $$

where use has been made of

$$ \gamma = 2D_{ab} s_{ab} h_b. $$

In a similar manner, the constraint conditions of incompressibility and inextensibility for the incremental deformations are obtained as follows:

$$ c_{ab}^{\text{tt}} a_b = 0, \quad c_{ab}^{\text{tt}} n_b = 0. $$

3. Jump Conditions across Singular-Stress Curves

In the ideal fiber-reinforced material, a singular tensile stress represented by the Dirac delta function often appears along the $a$-$n$-curve. Denote its intensity (resultant force) by $F_a$ along the $a$-curve and by $F_n$ along the $n$-curve. To balance the singular tensile stress, there must appear discontinuities in the normal and shearing stresses across the curve. Let us consider jump conditions of these discontinuities, which are easily obtained from equilibrium of the curves in their normal and tangent directions as follows: Across the $a$-curve,

(a) \[ T_{a0} n_0 + F_a k_a n_0 + (DF_a / d\gamma) s_a = 0; \quad (13a) \]

and across the $n$-curve,

(b) \[ T_{a0} n_0 + F_n k_n a_0 + (DF_n / d\gamma) s_a = 0, \quad (13b) \]

where $k_a$ and $k_n$ are curvatures of the $a$- and $n$-curves given by

$$ k_a = \frac{\gamma}{2\alpha}, \quad k_n = \frac{3\beta}{2}, $$

and $\alpha$ and $\beta$ are arc lengths along the respective curves. The square brackets denote the jump of enclosed quantities across the curves.

Derivation of the jump conditions for the incremental deformations from Eqs. (13a) and (13b) needs the material time derivatives of curvatures $k_a$ and $\kappa_n$. Taking the material time derivatives of Eqs. (14), and using Eq. (6), and the identity

$$ (S_{aa})^{(t)} = \frac{\kappa_a}{2\alpha} - \frac{3\kappa_a}{2} (D_{ab} W_{ab}), $$

we obtain the following after some computations:

$$ \kappa_a = \gamma a, \quad \kappa_n = \gamma a, $$

Next, we define the incremental resultant forces as follows:

$$ f_a = F_a n_0, \quad f_n = F_n n_0. $$

Finally, taking the material time derivatives of Eqs. (13a) and (13b), and using Eqs. (12)-(18), we obtain

$$ \{s_{ab}^{\text{tt}} + s_{ab}^{\text{t}} (\dot{D}_{ab} e_{ab} + \dot{W}_{ab} e_{ab}) + 2S^{(t)} (\dot{D}_{ab} s_{ab} + \dot{W}_{ab} s_{ab}) \} = \{F_a + F_n\} \gamma a, $$

$$ + \{F_a + F_n\} \gamma a, $$

$$ \{s_{ab}^{\text{tt}} + s_{ab}^{\text{t}} (\dot{D}_{ab} e_{ab} + \dot{W}_{ab} e_{ab}) + 2S^{(t)} (\dot{D}_{ab} s_{ab} + \dot{W}_{ab} s_{ab}) \} = \{F_a + F_n\} \gamma a, $$

4. Application Examples (A)

In this section, the jump conditions for cases (a) and (b) are applied to two simple bifurcation problems. Let us consider an infinite fiber-reinforced body with a circular cylindrical cavity with radius $r_0$, and employ a cylindrical polar coordinate system $(r, \theta, z)$ as shown in Fig. 1. Let us assume the body to be reinforced in the $\theta$-direction for the first problem (I) and in the $r$-direction for the second problem (II). The cavity surface is subject to tensile traction $p_0$. A singular stress appears along the $a$-curve at $r = r_0$ for (I) and along the $n$-curve at $r = r_0$ for (II). Jump conditions (13a) and (13b) give its magnitude:

$$ F_a = -n_0 p_0, \quad (r = r_0), $$

$$ F_n = -a_0 p_0, \quad (r = r_0). $$

These are compressive. The interior of the body ($r > r_0$) sustains no deformation and is free from stresses in both of the cases. Therefore,

$$ T_{a0} = 0, \quad (r > r_0), $$

It was examined whether the body whose cavity surface sustains the concentrated compressive hoop stress bifurcates or not, and when the bifurcation begins, supposing it does occur. Let us denote the incremental
displacements associated with the bifurcation by \( u_r \) and \( u_q \). The two constraint conditions (12) yield
\[
\varepsilon_{rr} = \frac{3u_r}{2} r, \quad \varepsilon_{qq} = 0
\]
and
\[
\varepsilon_{rr} = \frac{3u_r}{2} (r/a) + \frac{u_q}{2} (n^2-1) \frac{1}{n} \sin n\theta.
\]
A set of solutions for Eqs. (22) are given by
\[
u = C\sin n\theta, \quad u_q = C\cos n\theta \quad (n=2,3,\ldots),
\]
with \( C \) the integral constant, from which it follows that
\[
\varepsilon_{rr} = \frac{3u_r}{2} (r/a) = \frac{u_q}{2} (n^2-1) \frac{1}{n} \sin n\theta.
\]
Substituting Eqs. (21)–(24) into Eq. (10) into gives
\[
(1) \quad \sigma_{rr} = \frac{r}{r+a} \epsilon_{rr}, \quad \sigma_{qq} = \frac{r}{r+a} \epsilon_{qq},
\]
\[
(2) \quad \sigma_{rr} = \frac{r}{r+a} \epsilon_{rr}, \quad \sigma_{qq} = \epsilon_{qq},
\]
\[
(3) \quad \sigma_{rr} = \epsilon_{rr} (n^2-1) \frac{1}{n} \sin n\theta,
\]
where \( G \) is the shear modulus given by
\[
G = \frac{k}{a}.
\]
Since the interior of the body \( r > r_0 \) is free from initial stresses, the equilibrium equations for the bifurcation are the same as the conventional ones. Substituting Eqs. (25) into them yields
\[
(1) \quad \sigma_{rr} = \frac{r}{r+a} \epsilon_{rr} (n^2-1) \frac{1}{n} \sin n\theta,
\]
\[
(2) \quad \sigma_{qq} = \epsilon_{qq},
\]
\[
(3) \quad \sigma_{rr} = \epsilon_{rr} (n^2-1) \frac{1}{n} \sin n\theta,
\]
Solving the above equations with respect to \( p \) and \( t \), and substituting the results into Eqs. (25), we obtain the same solutions for
\[
\sigma_{rr} = \frac{r}{r+a} \epsilon_{rr} (n^2-1) \frac{1}{n} \sin n\theta,
\]
\[
\sigma_{qq} = \epsilon_{qq},
\]
\[
\sigma_{rr} = \epsilon_{rr} (n^2-1) \frac{1}{n} \sin n\theta.
\]
in the $y$-direction in the third case (III) and in the $x$-direction in the last case (IV). The slab is subjected to compressive axial load $p_0$, at infinity ($y=\infty$), and its surface ($x=\infty$) are free from traction.

It is clear that the state of deformation and stress is given by

\begin{align}
\gamma = 0, \quad S(0) = 0, \quad P_a = 0, \quad F_n = 0, \\
(\text{III}) \quad T_a = p_0, \quad P = 0, \\
(\text{IV}) \quad P = -p_0, \quad T_a = 0.
\end{align}

(31a) (32a) (32b)

We consider the bifurcation of the slab under the uniform compression. Let us denote the incremental displacements associated with the bifurcation by $u$ and $v$. The two constraint conditions (19) yield

\begin{align}
\varepsilon_x = 3u/3x = 0, \quad \varepsilon_y = 3v/3y = 0.
\end{align}

(33)

If we consider only solutions which vary in a sinusoidal way in $y$, the set of solutions for Eqs. (33) are given by

\begin{align}
w &= \sin \gamma/\lambda, \quad v = 0
\end{align}

(34)

with $\lambda$ the half wavelength, from which it follows that

\begin{align}
\varepsilon_{xy} = \frac{1}{2} \frac{du}{dy} \sqrt{2C(y/\lambda) \cos \gamma/\lambda}.
\end{align}

(35)

Substituting Eqs. (31)-(33) into Eq. (10) gives

\begin{align}
(\text{III}) \quad s_{xx} = -p(x,y), \quad s_{yy} = t(x,y), \\
(\text{IV}) \quad s_{xx} = t(x,y), \quad s_{yy} = -p(x,y),
\end{align}

(36a) (36b)

\begin{align}
\sigma_{xy} = \frac{(G-p_0/2)}{C(y/\lambda) \cos \gamma/\lambda}.
\end{align}

(37)

where $G$ is the shear modulus given by Eq. (27). Since the slab is subjected to an axial compression, the equilibrium equations for the incremental stresses are given by

\begin{align}
3\varepsilon_{xx}/3x + 3\varepsilon_{xy}/3y - p_0 &= 0, \\
3\varepsilon_{yy}/3y + 3\varepsilon_{xy}/3x &= 0,
\end{align}

(38)

which are rewritten with the aid of Eqs. (35)-(37) as

\begin{align}
(\text{III}) \quad 3\varepsilon_{xx} = (G-p_0)C(y/\lambda)^2 \sin \gamma/\lambda, \\
3\varepsilon_{yy} = 0,
\end{align}

(39a) (39b)

\begin{align}
(\text{IV}) \quad 3\varepsilon_{xx} = 0, \\
3\varepsilon_{yy} = (G-p_0)C(y/\lambda)^2 \sin \gamma/\lambda.
\end{align}

(39b)

Solving the above equations and substituting the results into Eqs. (38) yield

\begin{align}
\sigma_{xx} = \frac{(G-p_0)}{C(y/\lambda)^2} x \sin \gamma/\lambda, \\
\sigma_{yy} = 0.
\end{align}

(40)

Finally, applying jump conditions (19a) and (19b) to solutions (31a)-(33), (35), (37), and (40) at $x=\infty$ yields

\begin{align}
(\text{III}) \quad P_0 = G, \quad \varepsilon_{xx} = -G \sin \gamma/\lambda, \\
(\text{IV}) \quad P_0 = G, \quad \varepsilon_{xx} = 0.
\end{align}

(41a) (41b)

Thus, the following conclusions are drawn: Neither the axially nor the transversely fiber-reinforced slabs under the axial compression suffer any singular initial stress, but both the slabs begin to bifurcate when the compression reaches $p_0 = G$. The bifurcation involves a singular incremental stress given by the second equation of (41a) in the former case, but not in the latter case.

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References


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