Vibrations of a Combined System of Circular Plates and a Shell of Revolution*

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In this paper, we analyze the asymmetric vibrations of a combined system of a barrel-like shell of revolution and circular plate lids. The Lagrangian of the combined system is expressed in quadratic forms of boundary values, and the frequency equations can be obtained from the minimum conditions of that Lagrangian with respect to unknown boundary values.

Effects of the circular plate lids, the ratio of thickness of the circular plate to that of the shell of revolution, circumferential wave numbers upon natural frequencies of a combined system are clarified in a discussion of numerical results. Transfer phenomena of the mode shapes are also investigated.

Key Words: Vibration, Combined System, Shells of Revolution, Circular Plate lids, Natural Frequency, Mode Shapes

1. Introduction

Vibration problems of a combined system which consists of a circular cylindrical shell, a conical shell, a spherical shell etc. have been studied by the following researchers. Vibration of a combination of circular plates and a cylindrical shell^{(1),(2)}, and vibrations of a system connecting two cylindrical shells⁽³⁾ have been studied by Takahashi et al, vibrations of a combined system of a circular cylindrical shell with variable thickness and circular plates with variable thickness by using improved thick plate theory by Suzuki et al^{(4),(5)}, axisymmetric vibrations of thin drums by Hirano⁽⁶⁾, a combined system of a spherical and a conical shell by Saunders and Paslay⁽⁷⁾, and one of a cylindrical and a conical shell by Hu and Raney⁽⁸⁾. The circular cylindrical shells, conical shells and spherical shells belong to the category of shells of revolution in a broad sense. However, few papers are published on the vibrations of a combined system of shells of revolution of the general type, except for the shells mentioned above, and other elastic bodies. Hence the characteristics of that vibration have not yet been clarified sufficiently.

In view of the circumstances, the authors analyze in this paper free vibrations of a combined system of a barrel-like shell of revolution⁽⁹⁾ which is formerly analyzed by them and circular plate lids without hole. As for methods of analysis, the Lagrangian of vibration of the circular plate and that of the shell of revolution are expressed with their boundary values (displacements and slopes at boundary). Then the Lagrangian of the combined system is obtained in quadratic forms of the unknown boundary values by using the

conditions of continuity. The frequency equation and the relations among the boundary values are obtained from minimum conditions of the Lagrangians with respect to the unknown boundary values.

Effects of the circular plate lids, thickness of the plates, ratio of thickness of the circular plate to that of the shell of revolution, slenderness of the shell of revolution and circumferential wave numbers upon natural frequencies of the combined system are clarified in discussions of numerical results. Also obtained are the mode shapes of the circular plate and the shell of revolution and the transfer phenomena in them are investigated.

2. Theory

2.1 Lagrangian of a Shell of Revolution in terms of Boundary Values

The Cartesian coordinates are assumed as (X, Y) and shells of revolution of which the middle surface is developed by rotating a circular arc lying on the X - Y plane with respect to Y axis are considered. Let the center of a shell of revolution be O . Take angular coordinates θ (between the normal to the arbitrary point A on the middle surface and X axis), ϕ as shown in Fig.1 and z axis in the direction of the normal to the point A with positive inward. Denoting the principal

Fig.1 Middle surface of a shell of revolution and coordinate system

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radius of curvature of the meridian as R_1 (=const.), we put k as a ratio of the principal radius of curvature of the parallel circle at $\theta=0$ to that of the meridian R_1 . We denote the displacements in the θ , ϕ and z directions by $\{u(\theta, \phi), v(\theta, \phi), w(\theta, \phi)\} \sin p't$, where p' and t are the circular frequency and the time. Putting the thickness of the shell as k , we obtain the Lagrangian for a thin shell of revolution with $h/R_1 \ll 1$ in a period of vibration from Ref.(9),

$$\begin{aligned} -L/\left(\frac{D}{R_1^2}\right) = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} \left\{ -\alpha'(u^2 + v^2 + w^2) + \beta \left[\left(\frac{\partial u}{\partial \theta} - \phi' u + \phi'' \frac{\partial v}{\partial \phi} - (1+\phi)w \right)^2 \right. \right. \\ + 2\zeta \left(\frac{\partial u}{\partial \theta} - w \right) \left(\phi' u - \phi'' \frac{\partial v}{\partial \phi} + \phi w \right) + \frac{\zeta}{2} \left(\phi'' \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \theta} + \frac{1}{\phi''} \frac{\partial \phi''}{\partial \theta} v \right)^2 \Big] \\ + \left(\frac{\partial u}{\partial \theta} - \phi' u + \phi'' \frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \theta^2} - \phi' \frac{\partial w}{\partial \theta} + (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} \right)^2 \\ + 2\zeta \left(\frac{\partial u}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \left(\phi' u - \phi'' \frac{\partial v}{\partial \phi} + \phi' \frac{\partial w}{\partial \theta} - (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} \right) \\ \left. + 2\zeta \left(\phi'' \frac{\partial u}{\partial \phi} + \phi \left(\phi' v + \frac{\partial v}{\partial \theta} \right) + \phi'' \left(\phi' \frac{\partial w}{\partial \phi} + \frac{\partial^2 w}{\partial \theta \partial \phi} \right) \right)^2 \right\} \frac{1}{\phi''} d\theta d\phi \quad (1) \end{aligned}$$

where

$$\phi = \cos \theta / (\cos \theta + k - 1), \quad \phi' = \phi \tan \theta, \quad \phi'' = \phi / \cos \theta \quad (2)$$

and

$$\alpha' = \gamma h (p')^2 R_1^2 / g D, \quad \beta = 12 R_1^2 / k^2, \quad D = E h^3 / 12 (1 - \nu^2), \quad \zeta = 1 - \nu \quad (3)$$

The symbols α , β , D , ζ denote the nondimensional frequency, the parameter of thickness, the bending rigidity of the plate, the parameter including Poisson's ratio and E , ν , γ , g denote Young's modulus, Poisson's ratio, the specific weight, the acceleration due to gravity, respectively. We put the values of θ at both ends as θ_1 and θ_2 and assume the shell is symmetric about the $\theta=0$ -plane and put $\theta_2 = -\theta_1 = \theta_0$. We obtain the following equations of motion by the first variation of Eq.(1).

$$E_1 = E_2 = E_3 = 0 \quad (4)$$

where,

$$\begin{aligned} E_1 = \phi'' \left(\frac{\partial T_1}{\partial \theta} + \frac{\partial T_2}{\partial \phi} + T_3 \sin \theta \right) + \alpha' u, \quad E_2 = \phi'' \left(\frac{\partial T_2}{\partial \theta} + \frac{\partial T_3}{\partial \phi} - T_1 \phi'' \sin \theta \right) + \alpha' v, \\ E_3 = \phi'' \left(\frac{\partial T_3}{\partial \theta} - \frac{\partial^2 T_1}{\partial \phi^2} + \phi'' \frac{\partial T_2}{\partial \phi} - T_2 \right) + \alpha' w, \\ \phi'' T_1 = \frac{\partial^2 w}{\partial \theta^2} - (1-\zeta) \phi' \frac{\partial w}{\partial \theta} + \beta ((\zeta-1)\phi-1)w + (1-\zeta)(\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} \\ + (1-\zeta)\phi''(\phi+\beta) \frac{\partial v}{\partial \phi} + (\beta+1) \left\{ \frac{\partial u}{\partial \theta} + (\zeta-1)\phi' u \right\}, \\ \phi'' T_2 = 2\zeta \phi'' \left(\frac{\partial^2 w}{\partial \theta \partial \phi} + \phi' \frac{\partial w}{\partial \phi} \right) + \frac{\beta \zeta}{2} \left\{ \left(1 + \frac{4}{\beta} \phi^2 \right) \left(\frac{\partial v}{\partial \theta} + \phi' v \right) + \phi'' \left(1 + \frac{4}{\beta} \phi \right) \frac{\partial u}{\partial \phi} \right\}, \\ - \phi'' T_3 = \frac{\partial^3 w}{\partial \theta^3} - \phi'' \frac{\partial^2 w}{\partial \theta^2} - ((1-\zeta)\phi + (\phi'')^2) \frac{\partial^2 v}{\partial \theta} + (1+\zeta)(\phi'')^2 \frac{\partial^2 w}{\partial \theta \partial \phi^2} \\ + (2+\zeta)\phi''(\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} + (1+\zeta)\phi \phi'' \frac{\partial^2 v}{\partial \theta \partial \phi} + \phi'' \left\{ (1+2\zeta)\phi \phi' + (1-\zeta) \frac{d\phi}{d\theta} \right\} \frac{\partial v}{\partial \phi} \\ + \frac{\partial^3 u}{\partial \theta^3} - \phi'' \frac{\partial^2 u}{\partial \theta^2} - ((1-\zeta)\phi + (\phi'')^2) u + 2\zeta(\phi'')^2 \frac{\partial^2 u}{\partial \phi^2}, \\ \phi'' T_4 = \frac{\partial^2 w}{\partial \theta^2} - (1-\zeta) \left\{ \phi' \frac{\partial w}{\partial \theta} - (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} - \phi \phi'' \frac{\partial v}{\partial \phi} + \phi' u \right\} + \frac{\partial u}{\partial \theta}, \\ T_5 = \frac{\beta \zeta}{2} \left\{ \frac{4}{\beta} \phi'' \left(\frac{\partial^2 w}{\partial \theta \partial \phi} + \phi' \frac{\partial w}{\partial \phi} \right) + \left(1 + \frac{4}{\beta} \phi \right) \left(\frac{\partial v}{\partial \theta} + \phi' v \right) + \left(1 + \frac{4}{\beta} \right) \phi'' \frac{\partial u}{\partial \phi} \right\}, \\ T_6 = (1-\zeta) \phi \frac{\partial^2 w}{\partial \theta^2} - \phi \phi'' \frac{\partial w}{\partial \theta} - \beta (1-\zeta + \phi) w + \phi (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} + \phi'' (\beta + \phi^2) \frac{\partial v}{\partial \phi} \\ + (\beta + \phi) \left\{ (1-\zeta) \frac{\partial u}{\partial \theta} - \phi' u \right\}, \\ T_7 = \phi'' \left\{ (1-\zeta) \frac{\partial^2 w}{\partial \theta^2} - \phi' \frac{\partial w}{\partial \theta} + (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} + \phi \phi'' \frac{\partial v}{\partial \phi} + (1-\zeta) \frac{\partial u}{\partial \theta} - \phi' u \right\}, \\ T_8 = 2\zeta \left\{ \phi'' \left(\frac{\partial^2 w}{\partial \theta \partial \phi} + \phi' \frac{\partial w}{\partial \phi} + \frac{\partial u}{\partial \phi} \right) + \phi \left(\frac{\partial v}{\partial \theta} + \phi' v \right) \right\}, \\ T_9 = (1-\zeta) \frac{\partial^2 w}{\partial \theta^2} - \phi' \frac{\partial w}{\partial \theta} + \beta (\zeta-1-\phi) w + (\phi'')^2 \frac{\partial^2 w}{\partial \phi^2} + \phi'' (\beta + \phi) \frac{\partial v}{\partial \phi} + (\beta+1) \left\{ (1-\zeta) \frac{\partial u}{\partial \theta} - \phi' u \right\}, \\ \phi'' T_{10} = \beta \left\{ 1 + 2(1-\zeta)\phi + \phi^2 \right\} w + (\zeta-1-\phi) \phi'' \frac{\partial v}{\partial \phi} + ((\zeta-1)\phi-1) \frac{\partial u}{\partial \theta} + \phi' (1-\zeta + \phi) u \quad (5) \end{aligned}$$

Assuming the solutions of equations of motion (4) as u , v and w , we integrate the Lagrangian by parts with help of equations E_1 , E_2 , E_3 , and for complete shells of revolution we obtain,

$$-L/\left(\frac{D}{R_1^2}\right) = \int_0^{2\pi} \left(T_1 u + T_2 v + T_3 w + T_4 \frac{\partial w}{\partial \theta} \right) \Big|_{\theta=-\theta_0}^{\theta=\theta_0} d\phi \quad (6)$$

Putting the integration constants as C_{nq} , C_{nq}^* ($q=1, 2, 3, 4$), according to Ref.(9), the general solutions of equations of motion are expressed as follows;

$$\left. \begin{aligned} u &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (C_{nq} u_{nq} + C_{nq}^* u_{nq}^*) \cos n\phi, \\ v &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (C_{nq} v_{nq} + C_{nq}^* v_{nq}^*) \sin n\phi, \\ w &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (C_{nq} w_{nq} + C_{nq}^* w_{nq}^*) \cos n\phi \end{aligned} \right\} \quad (7)$$

where, $(u_{nq}, v_{nq}, w_{nq})_{q=1\sim 4}$ and $(u_{nq}^*, v_{nq}^*, w_{nq}^*)_{q=1\sim 4}$ are the series solutions which have been obtained in Ref.(9) and they are the solutions for the symmetric vibration and the antisymmetric one respectively. We expand displacements and slope at boundary in Fourier series as follows, with double signs taken in the same order.

$$\left. \begin{aligned} u|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} (\pm a_{n1} + a_{n1}^*) \cos n\phi, \\ v|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} (a_{n2} \pm a_{n2}^*) \sin n\phi, \\ w|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} (a_{n3} \pm a_{n3}^*) \cos n\phi, \\ \frac{\partial w}{\partial \theta}|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} (\pm a_{n4} + a_{n4}^*) \cos n\phi \end{aligned} \right\} \dots\dots (8)$$

We call $a_{nq}, a_{nq}^* (q=1\sim 4)$ boundary values hereafter. Substituting Eq.(7) into Eq.(8), we obtain the integration constants in terms of $u_{n1}(\theta_0), u_{n2}(\theta_0), \partial w_{n1}/\partial \theta|_{\theta=\theta_0}$, etc. which are abbreviated as u_{n1}, u_{n2}, w'_{n1} etc.

$$c_{nq} = \frac{1}{\Delta_0} \sum_{p=1}^4 (-1)^{p+q} a_{np} \Delta_{pq} \quad (q=1, 2, 3, 4) \dots\dots\dots (9)$$

where

$$\Delta_0 = \begin{vmatrix} u_{n1} & u_{n2} & u_{n3} & u_{n4} \\ v_{n1} & v_{n2} & v_{n3} & v_{n4} \\ w_{n1} & w_{n2} & w_{n3} & w_{n4} \\ w'_{n1} & w'_{n2} & w'_{n3} & w'_{n4} \end{vmatrix} \dots\dots\dots (10)$$

The symbol Δ_{pq} denotes the 3×3 minor determinant obtained by eliminating p -th row and q -th column from Δ_0 . By replacing $(u_{nq}, v_{nq}, w_{nq})_{q=1\sim 4}$ with $(u_{nq}^*, v_{nq}^*, w_{nq}^*)_{q=1\sim 4}$ in Δ_0 and $(\Delta_{pq})_{p,q=1\sim 4}$ we obtain Δ^*_{0} and $(\Delta^*_{pq})_{p,q=1\sim 4}$ and furthermore by replacing $a_{np}, \Delta_0, \Delta_{pq}$ with $a_{np}^*, \Delta^*_{0}, \Delta^*_{pq}$ in c_{nq} we obtain the expression c_{nq}^* . Substituting Eq.(7) into Eq.(5) and putting $\theta=\pm\theta_0$, the following equations are obtained with double signs taken in the same order.

$$\left. \begin{aligned} T_1|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (C_{nq} T_{1q} \pm C_{nq}^* T_{1q}^*) \cos n\phi, \\ T_2|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (\pm C_{nq} T_{2q} + C_{nq}^* T_{2q}^*) \sin n\phi, \\ T_3|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (\pm C_{nq} T_{3q} + C_{nq}^* T_{3q}^*) \cos n\phi, \\ T_4|_{\theta=\pm\theta_0} &= \sum_{n=1}^{\infty} \sum_{q=1}^4 (C_{nq} T_{4q} \pm C_{nq}^* T_{4q}^*) \cos n\phi \end{aligned} \right\} \dots\dots\dots (11)$$

Substituting Eqs.(8), (9) and (11) into Eq.(6), the Lagrangian is expressed in quadratic forms of the boundary values as follows;

$$\left. \begin{aligned} L &= L_{RS} + L_{RA}, \quad -L_{RS} / \left(\frac{2\pi D}{R^2} \right) \\ &= \left[\sum_{p=1}^4 \sum_{q=1}^4 S_{pq} a_{np}^* a_{nq} + \sum_{i,j=1}^4 S_{ij} a_{ni} a_{nj} \right] / \Delta_0, \\ S_{pp} &= \sum_{n=1}^{\infty} (-1)^{p+q} \Delta_{pq} T_{pq}, \\ S_{ij} &= \sum_{n=1}^{\infty} [(-1)^{q+i} \Delta_{iq} T_{jq} + (-1)^{j+i} \Delta_{jq} T_{iq}] \end{aligned} \right\} \dots\dots\dots (12)$$

We can obtain L_{RA} by replacing $a_{np}, \Delta_0, \Delta_{pq}$ and T_{pq} with $a_{np}^*, \Delta^*_{0}, \Delta^*_{pq}$ and T^*_{pq} in L_{RS} .

Fig.2 Coordinate system of a circular plate

2.2 Lagrangian of a Circular Plate Without a Hole in Terms of Boundary Values

The Lagrangian for inplane vibrations and for flexural ones of a circular plate without a hole in terms of boundary values has already been obtained in Ref.(1) and so we show here their essentials.

Let the radius, the bending rigidity and Poisson's ratio of the circular plate be $R_0, \bar{D}$ and $\bar{\nu} (=1-\bar{\zeta})$. As shown in Fig.2, we take γ, ϕ as the polar coordinates on the circular plate and take z' -axis perpendicular to this surface. Nondimensional radius is denoted by $\bar{r} = \alpha r / R_0$ and displacements along r, ϕ and z' axes are denoted by $(U, V, W) \sin p't$. We introduce relations such as $\bar{\alpha}' = \bar{\gamma} h (p')^2 R_0^4 / g \bar{D}$, $\bar{\beta} = 12 R_0^2 / h^2$ where $\bar{\gamma}/g$ and $\bar{h}$ are the density and the thickness of the circular plate and expand the displacements and slope at outer edge of the circular plate in Fourier series as follows

$$\left. \begin{aligned} U|_{\bar{r}=\bar{\alpha}} &= \sum_{n=1}^{\infty} b_{n1} \cos n\phi, \\ V|_{\bar{r}=\bar{\alpha}} &= \sum_{n=1}^{\infty} b_{n2} \sin n\phi, \\ W|_{\bar{r}=\bar{\alpha}} &= \sum_{n=1}^{\infty} b_{n3} \cos n\phi, \\ \frac{\partial W}{\partial \bar{r}}|_{\bar{r}=\bar{\alpha}} &= \sum_{n=1}^{\infty} b_{n4} \cos n\phi \end{aligned} \right\} \dots\dots\dots (13)$$

We call $b_{nq} (q=1\sim 4)$ boundary values of the circular plate. Now we obtain, from Ref.(1), L_{PI} the Lagrangian for inplane vibrations and L_{PB} the one for flexural vibrations, both in terms of boundary values.

$$\left. \begin{aligned} -L_{PI} / \left(\frac{\pi \bar{D}}{2 R_0^2} \right) &= P_{11} b_{n1}^2 + P_{22} b_{n2}^2 + P_{12} b_{n1} b_{n2}, \\ P_{11} &= \bar{\alpha}' [\omega' J_n(\omega) J_n'(\omega') / (\eta - 2/(\omega')^2)], \\ P_{22} &= \bar{\alpha}' [\omega' J_n'(\omega) J_n(\omega') / (\eta - 2/(\omega')^2)], \\ P_{12} &= 2 \bar{\alpha}' n [J_n(\omega) J_n'(\omega') / (\eta - 2/(\omega')^2)] \end{aligned} \right\} \dots\dots\dots (14)$$

$$\left. \begin{aligned} -L_{PB} / \left(\frac{\pi \bar{D}}{2 R_0^2} \right) &= Q_{11} b_{n1}^2 + Q_{22} b_{n2}^2 + Q_{12} b_{n1} b_{n2}, \\ Q_{11} &= 2 \bar{\alpha}'^3 J_n'(\bar{\alpha}) I_n'(\bar{\alpha}') / \eta' - \xi n^2, \\ Q_{22} &= 2 \bar{\alpha}'^3 J_n(\bar{\alpha}) I_n(\bar{\alpha}') / \eta' - \xi \bar{\alpha}'^2, \\ Q_{12} &= 2 \xi n^2 \bar{\alpha}' - 2 \bar{\alpha}'^3 [J_n(\bar{\alpha}) I_n'(\bar{\alpha}') + J_n'(\bar{\alpha}) I_n(\bar{\alpha}') / \eta'] \end{aligned} \right\} \dots\dots\dots (15)$$

where, $\kappa^2 = \bar{\alpha}'^2 / \bar{\beta}$, $(\kappa')^2 = 2 \bar{\alpha}'^2 / \bar{\zeta} \bar{\beta}$, $\omega = \bar{\alpha} \kappa$, $\omega' = \bar{\alpha} \kappa'$, $\eta = n^2 J_n(\omega) J_n'(\omega') - \omega \omega' J_n'(\omega) J_n(\omega')$, $\eta' = J_n(\bar{\alpha}) I_n'(\bar{\alpha}') - J_n'(\bar{\alpha}) I_n(\bar{\alpha}')$, and J_n and I_n are Bessel and modified Bessel functions of the first kind and the differentiation is denoted by dash.

2.3 Vibrations of a Combined System of a Shell of Revolution and Circular Plate Without a Hole

We consider the vibrations of a shell of revolution with circular plates lids at both ends. For simplicity the circular plate

and the shell of revolution are assumed to be homogeneous and the ratio of thicknesses is represented by $\rho = \bar{h}/h$. Thus the relations $\bar{D} = \rho^3 D$, $\bar{\zeta} = \zeta$ and $\bar{\gamma} = \gamma$ hold among the quantities. From Fig.3, the conditions of continuity of the plate and the shell of revolution are the same at the upper end and at the lower one such that;

$$\left. \begin{aligned} U &= -u \sin \theta - w \cos \theta, \quad V = v, \\ W &= u \cos \theta - w \sin \theta, \\ \frac{\partial W}{\partial r} &= \frac{1}{R_1} \frac{\partial w}{\partial \theta} \\ \left(\text{or, } \frac{\partial W}{\partial z} &= \frac{\cos \theta + k - 1}{\bar{a}} \frac{\partial w}{\partial \theta} \right) \end{aligned} \right\} \quad (16)$$

Rewriting Eq.(16) with help of Eqs.(8) and (13), the following relations between the boundary values are obtained.

$$\left. \begin{aligned} b_{n1} &= (-a_{n1} \sin \theta - a_{n3} \cos \theta) \\ &\pm (-a_{n1} \sin \theta - a_{n3} \cos \theta), \\ b_{n2} &= a_{n2} \pm a_{n2}, \\ b_{n3} &= \pm (a_{n1} \cos \theta - a_{n3} \sin \theta) \\ &+ (a_{n1} \cos \theta - a_{n3} \sin \theta), \\ b_{n4} &= \pm \frac{\cos \theta + k - 1}{\bar{a}} a_{n4} + \frac{\cos \theta + k - 1}{\bar{a}} a_{n4} \end{aligned} \right\} \quad (17)$$

Now we replace the boundary values of the circular plate as follows.

$$\left. \begin{aligned} b_{n1} &= b'_{n1} \pm b''_{n1}, \quad b_{n2} = b'_{n2} \pm b''_{n2}, \\ b_{n3} &= \pm b'_{n3} + b''_{n3}, \quad b_{n4} = \pm b'_{n4} + b''_{n4} \end{aligned} \right\} \quad (18)$$

Fig.3 Conditions of continuity of a circular plate and a shell of revolution

where, the sign "+" refers to the upper circular plate and "-" to the lower one. Substituting Eq.(18) into Eq.(17) we have,

$$\left. \begin{aligned} b'_{n1} &= -a_{n1} \sin \theta - a_{n3} \cos \theta, \quad b'_{n2} = a_{n2}, \\ b'_{n3} &= a_{n1} \cos \theta - a_{n3} \sin \theta, \\ b'_{n4} &= \frac{\cos \theta + k - 1}{\bar{a}} a_{n4} \end{aligned} \right\} \quad (19)$$

We can obtain $(b^*_{np})_{p=1 \sim 4}$ by replacing a_{np} with a^*_{np} in $(b_{np})_{p=1 \sim 4}$.

The Lagrangian L of the combined system is given by the sum of the Lagrangians of the shell of revolution and the circular plates. Utilizing Eqs.(12), (14), (15) and (19), we obtain

$$L = L_{\text{Symm.}} + L_{\text{Antisymm.}} \quad (20)$$

where,

$$\begin{aligned} -L_{\text{Symm.}} / \left(\frac{2\pi D}{R_1^2} \right) &= -(L_{R3} + 2L_{P1} + 2L_{P2}) / \left(\frac{2\pi D}{R_1^2} \right) \\ &= [S_{11}/\Delta_0 + H_1(P_{11} \sin^2 \theta + Q_{11} \cos^2 \theta)] a_{n1}^2 \\ &+ [S_{22}/\Delta_0 + H_1 P_{22}] a_{n2}^2 + [S_{33}/\Delta_0 \\ &+ H_1(P_{11} \cos^2 \theta + Q_{11} \sin^2 \theta)] a_{n3}^2 \\ &+ [S_{44}/\Delta_0 + (\rho^3/2\bar{a}^2) Q_{22}] a_{n4}^2 \\ &+ [S_{13}/\Delta_0 - H_1 P_{12} \sin \theta] a_{n1} a_{n3} \\ &+ [S_{13}/\Delta_0 + H_1(P_{11} - Q_{11}) \sin 2\theta] a_{n1} a_{n3} \\ &+ [S_{14}/\Delta_0 + H_1 Q_{12} \cos \theta] a_{n1} a_{n4} \\ &+ [S_{23}/\Delta_0 - H_1 P_{12} \cos \theta] a_{n2} a_{n3} \\ &+ S_{24} a_{n2} a_{n4} / \Delta_0 \\ &+ [S_{34}/\Delta_0 - H_1 Q_{12} \sin \theta] a_{n3} a_{n4}, \\ H_1 &= \rho^3 / 2(\cos \theta + k - 1)^2, \\ H_2 &= \rho^3 / 2\bar{a}(\cos \theta + k - 1) \end{aligned} \quad (21)$$

The expression $-L_{\text{Antisymm.}} / \left(\frac{2\pi D}{R_1^2} \right)$ can be obtained by replacing a_{np} ($p=1 \sim 4$), S_{pq} ($p, q=1 \sim 4$), Δ_0 with a^*_{np} ($p=1 \sim 4$), S^*_{pq} ($p, q=1 \sim 4$), Δ^*_0 in Eq.(21).

The frequency equations of the combined system are obtained from the minimum conditions of the Lagrangian with respect to unknown boundary values:

$$\partial L / \partial a_{np} = 0 \quad (a^*_{np} = 0), p=1 \sim 4 \quad (22)$$

Symbol S.O.R.C. denotes a single shell of revolution with both ends clamped. This is the same in Figs.7 and 10. The symbols F_1 and F_2 denote the first and the second modes of vibration for a circular plate with outer edge free and C_1 and C_2 denote those with outer edge clamped. The numbers ① ~ ⑮ and ①' ~ ⑪' denote the points at which the mode shapes are calculated.

Fig.4 Frequency curve

Symmetric, $n=4$, $k=0.7$, $\beta=10^4$, $\rho=1.0$

①~③; 1st mode, ④~⑩; 2nd mode, ⑪~⑮; 3rd mode

Fig.5 Mode shapes

Antisymmetric, $n=4$, $k=0.7$,
 $\beta=10^4$, $\rho=1.0$

①'~⑤'; 1st mode,
⑥'~⑪'; 2nd mode

Fig.6 Mode shapes

Fig.7 Effects of ρ , the thickness ratio, upon the frequency

The values of α^4/β at $\theta_0=0$ are the natural frequencies of flexural vibrations of the circular plate with outer edge free. This is the same in Figs.8~9.

From the above, we obtain four simultaneous equations for the symmetric vibration with respect to $\theta=0$ -plane. And from the minimum condition

$$\partial L / \partial a_{np} = 0 \quad (a_{np}=0), p=1 \sim 4 \quad \dots \dots \dots (23)$$

those for the antisymmetric one can be obtained. By setting the 4×4 determinant of coefficients of these equations at zero, we obtain the frequency equations.

3. Numerical Calculations

In numerical calculations, the value of Poisson's ratio is assigned as $\nu=0.3$. We first fix the values of α , β , k , n , ρ , etc. and calculate the terms included in the Lagrangian of the circular plate and then, we seek for the values of θ which satisfy the frequency equation. We denote these values of θ as θ_0 and obtain the relation between α^4/β and θ_0 . The nature of convergency of solution for the shell of revolution is explained in Ref.(9). As for the solution of the circular plate, we confirm that it coincides with known results⁽¹⁰⁾. The following relations hold among the nondimensional quantities.

$$\begin{aligned} \bar{\alpha} &= \frac{\cos \theta + k - 1}{\sqrt{\rho}} \alpha, \\ \bar{\beta} &= \left(\frac{\cos \theta + k - 1}{\rho} \right)^2 \beta \end{aligned} \quad \dots \dots \dots (24)$$

Figure 4 shows the $\alpha^4/\beta-\theta_0$ curves of a shell of revolution with circular plates lids. For comparison purpose, the curves of a shell of revolution with both ends clamped are also shown in the same figure. They are the first, second, third modes of symmetric vibrations and the first and second modes of antisymmetric ones from below. The values of α^4/β at $\theta_0=0$ are the natural frequencies of flexural vibrations of the circular plate. The symbols F_1 and F_2 denote the first and the second modes of vibration for a circular plate with outer edge free and C_1 and C_2 do those with outer edge clamped. As θ_0 tends to zero, the frequencies for symmetric vibration approach those of the flexural vibrations of a circular plate with outer edge clamped, while the frequencies for antisymmetric vibration approach those of a circular plate with outer

edge free. On the contrary, as θ_0 becomes larger the frequency curves have a local maximum and a local minimum indicating the transfer phenomena, and when $\theta_0 > 0.6$ they come closer to the curves of a shell of revolution with both ends clamped so far as illustrated there-in. The numbers ①~⑮ and ①~⑪ on the curves denote the points at which the mode shapes are calculated and the results are shown in Figs.5 and 6.

Figure 5 shows the mode shapes of the symmetric vibration of a shell of revolution with circular plates lids where w denotes the displacements towards the center of curvature of the shell of revolution and W denotes those of the circular plate in its thickness direction. The figures show the first mode of vibration with ①~③, the second mode with ④~⑩ and the third with ⑪~⑮. The scales of displacements here are magnified or reduced appropriately. This is the same in Fig.6. In the second mode ④~⑩, for example, the mode of ⑥ is similar to that of ① (first mode), then the second mode changes as the frequency curve transfers and the mode of ⑨ resembles that of ⑬ (third mode). Thus even in the same order of the vibration, the transfer phenomena of the mode shapes can be observed as the transfer phenomena occur in the frequency curves. The mode shapes of the shell of revolution are simple and those of the circular plate are complicated when the length of the shell of revolution θ_0 is short; on the contrary, inverted features are observed when θ_0 is large.

Fig.8 Effects of β , the thickness parameter, upon the frequency.

Fig.9 Effects of k , the slenderness ratio of the shell or revolution, upon the frequency.

Figure 6 shows the mode shapes of the antisymmetric vibration and ①~⑤ corresponds to the first mode and ⑥~⑪ does to the second one. The tendencies are almost the same as those of symmetric vibrations.

Effects of various parameters upon natural frequencies are discussed in the following Figs.7 to 10. Since the tendencies of the symmetric vibration and the antisymmetric one are almost the same, we only show the antisymmetric one to avoid complexity.

Figure 7 shows effects of ρ , the ratio of thickness of the circular plate to that of the shell of revolution, upon the frequency. The curves are the first and the second modes of vibration from below. The values of α^4/β at $\theta_0 = 0$ are the natural frequencies of flexural vibrations of the circular plate with outer edge free. This is the same in Figs.8 and 9. The frequency becomes higher as ρ becomes larger. As θ_0 becomes large, the effects of ρ become small and the frequencies of the combined system come close to those of a single shell of revolution with both ends clamped, so far as illustrated herein.

Figure 8 shows effects of β , the thickness parameter of plates, upon the frequency. It gives the curves for $\beta = 3 \times 10^3, 5 \times 10^3, 10^4 (R/h = 15.8, 20.4, 28.9)$. The frequency increases as β becomes smaller, which means increase of thickness.

Figure 9 shows effects of k , the slenderness ratio of the shell, upon the frequency. As the value of k decreases like 1.3, 1.0, 0.7, i.e. the shell becomes slender, the frequency increases. Incidentally, $k=1.0$ means a spherical shell of which the upper and the lower ends are cut off and combined with two circular plate lids there.

Figure 10 shows the relations between the frequency and n , the circumferential wave number. They are the first and the second modes of vibration from below. The frequency increases as n increases. The shell of revolution with circular plate lids gives a lower frequency than that of the single shell of revolution with both ends clamped, but they come close to each other as n increases. Hence, effects of the circular plate lids upon the vibration of a combined system decrease as n increases in the case of combination of parameters illustrated in the figure.

4. Conclusions

In this paper, the vibrations of a combined system of a barrel-like shell of revolution and circular plate lids without a hole are analyzed. The results can be summarized as follows:

- (1) The frequency increases as the thickness becomes large, as the ratio of thickness of the shell of revolution becomes large, as the circumferential wave number increases and as the shell of revolution becomes slender.
- (2) As θ_0 tends to zero, the frequency of a combined system for symmetric vibration approaches that of the flexural vibration of a circular plate with outer edge clamped, while the frequency for antisymmetric vibration approaches that of a circular plate with outer edge free. On the contrary, as θ_0 becomes large the frequency comes closer to that of a

single shell of revolution with both ends clamped.

(3) Depending on the length of the shell of revolution, transfer phenomena can be observed in frequency curves and in mode shapes. The mode shapes of the shell of revolution are simple and those of the circular plate are complicated when the length of the shell of revolution is shorter; on the contrary, inversed features are observed when this length is longer.

Fig.10 Effects of n , the circumferential wave number, upon the frequency.

The numerical calculations in this paper were carried out by using ACOS system 1000 computer of Tohoku University.

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