Lateral Vibrations of a Rotating Shaft Driven by a Universal Joint*

(1st Report, Generation of Even Multiple Vibrations by Secondary Moment)

By Hiroshi ŌTA** and Masayoshi KATO***

The causes and characteristics of lateral vibrations occurring when a rotating shaft is driven by a universal joint (so-called Hooke's joint) are the subject of this study. Dynamical considerations are made in relation to the mechanism of a moment transmitted by the universal joint. The analytical results regarding a secondary moment generated by the universal joint show that a driven shaft has a number of forced vibrations owing to the secondary moment. Each of these vibrations makes a resonance when an angular velocity of the drive shaft coincides with one of the even integer submultiples of natural angular frequencies in the driven shaft system. The experimental results show good coincidence with the analytical results, so the validity of the analyses is confirmed.

Key Words: Vibration of Rotating Body, Universal Joint, Transmission Mechanism of Moment, Natural Angular Frequency, Resonance, Lateral Vibration of shaft

1. Introduction

As is well known, a shaft driven by a universal joint makes a resonance at an angular velocity ω which is equal to one of the integer submultiples of natural angular frequency p ($p = N\omega$, $N = 2, 3, 4, \dots$), and has a violently whirling lateral vibration. Each of these vibrations has a frequency of integer multiples of the shaft angular velocity. Vibrations with only the frequencies which are equal to odd integer multiples of the angular velocity ($N = 2k + 1$, $k = 1, 2, 3, \dots$) have been treated analytically^{(1)~(4)} Fujii et al.⁽⁵⁾ assumed the vibrations of even multiples of the angular velocity ($N = 2k$) caused by frictions between a cross-pin and yokes of a universal joint. There have been many investigations regarding 2ω vibration,^{(6)~(8)} in which analyses are performed under the assumption that a drive shaft has both a constant angular velocity and a constant drive torque. However, if the angular velocity of the drive shaft is assumed to be constant, then the drive torque can, in fact, no longer be constant, and it should be determined only by a load resisting the rotation of the driven shaft.

In the present study, the authors suppose that the drive shaft rotates with a constant angular velocity ω and that the secondary moment, as well as the angular velocity of the driven shaft, is not influenced by the deflection in the driven shaft. Analytical and experimental results show that an even multiple vibration is caused by the secondary moment when the driven shaft is subjected to a moment

resisting its rotation. This secondary moment is valid for the even multiple vibration regardless of friction between a cross-pin and yokes of a universal joint.

2. Moment Transmitted to Driven Shaft through Universal Joint

A universal joint considered here is assumed to have no play, no friction, and no misalignment in its center. As shown in Fig. 1, let the center lines of a drive shaft and a driven shaft be the z_{a1} and z axes, respectively. These two axes cross at the joint center A and make the joint angle α_a . In two stationary rectangular co-ordinate systems $A-x_{a1}y_{a1}z_{a1}$ and $A-x_ay_az$, the y_a axis is perpendicular to the $z_{a1}Az$ plane. Consider the following two pins forming the cross-pin of the universal joint: one is called the Ψ pin, which always exists in the $x_{a1}y_a$ plane, and the other is the Θ pin, which always exists in the x_ay_a plane. A revolutionary angle of the drive shaft is represented by Ψ , and that of the driven shaft by Θ . These angles Ψ and Θ are zero when the Ψ pin and Θ pin coincide with the x_{a1} and y_a axes, respectively. Let one end of the Ψ pin be P, and that of the Θ pin be Q.

We introduce here the unit vectors $i, j, k, k_1, p(\Psi)$, and $q(\Theta)$ as shown in Fig. 2. The unit vectors i, j, k , and k_1 have the same directions as the x_a, y_a, z , and z_{a1} axes; $p(\Psi)$ and $q(\Theta)$ have the same ones as $\overline{AP}$ and $\overline{AQ}$, respectively, and they are given by the following equation (1) as can be seen from Fig. 2.

$$\begin{aligned} p(\Psi) &= \cos \alpha_a \cos \Psi i + \sin \Psi j + \sin \alpha_a \cos \Psi k \\ q(\Theta) &= -\sin \Theta i + \cos \Theta j \end{aligned} \quad \dots\dots\dots (1)$$

Scalar product of $p(\Psi)$ and $q(\Theta)$ must be zero because of their orthogonal condition

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** Professor, Faculty of Engineering, Nagoya University, Chikusa-ku, Nagoya.

*** Assistant, Faculty of Engineering, Nagoya University.

Fig. 1 Moment transmitted to driven shaft through universal joint (when there is no friction between a cross-pin and yokes, i.e., $M_{\dot{\Theta}} = 0$)

as follows:

$$p(\Psi) \cdot q(\Theta) = 0 \quad (2)$$

Substitution of Eq. (1) into Eq. (2) yields the following relation⁽⁹⁾

$$\tan \Theta = \frac{\tan \Psi}{\cos \alpha_a} \quad (3)$$

that is,

$$\sin \Theta = \frac{\sin \Psi}{\sqrt{H(\Psi)}}, \quad \cos \Theta = \frac{\cos \alpha_a \cos \Psi}{\sqrt{H(\Psi)}} \quad \left(-\frac{\pi}{2} < \alpha_a < \frac{\pi}{2}\right) \quad (4)$$

where

$$H(\Psi) = 1 - \sin^2 \alpha_a \cos^2 \Psi \quad (5)$$

The Ψ pin moment which the drive yoke exerts on the Ψ pin is equal to the Θ pin moment which the Ψ pin exerts on the driven yoke through the Θ pin by the law of action and reaction. It is assumed

that there exists no friction between the cross-pin and the yokes in this study. Accordingly, the Ψ pin moment given by the drive shaft has no component in the direction of Ψ pin (AP); and the Θ pin moment acting on the driven yoke either has no component in the direction of Θ pin (AQ). Thus the universal joint can transmit only a moment perpendicular to the plane PAQ containing the cross-pin. This moment vector M_{a0} is expressed by the vector product as shown in Fig. 2:

$$M_{a0} = M_{a0} p(\Psi) \times q(\Theta) \quad (6)$$

Generally, since the direction of the moment vector M_{a0} coincides neither with the center line of the drive shaft (z_{a1} axis) nor with that of the driven shaft (z axis), the vector M_{a0} has a component perpendicular both to the drive shaft and to the driven shaft.

As shown in Fig. 1, the moment M_{a0} is resolved into the following three components: M_{Θ} in the z direction, and $M_{\dot{\Theta}}$ and $M_{\dot{\Theta}}$ in the AQ and AT directions perpendicular to the z direction, respectively. The moment M_{Θ} has influence on the rotation of the driven shaft, and the moment $M_{\dot{\Theta}}$ and $M_{\dot{\Theta}}$ have influence on the lateral vibration of the driven shaft. Reference to Fig. 2, and use of Eqs. (1), (4), and (6) yield the following equations:

$$\left. \begin{aligned} M_{\Theta} &= M_{a0} \cdot k = M_{a0} \sqrt{H(\Psi)} \\ M_{\dot{\Theta}} &= M_{a0} \cdot q(\Theta) = 0 \\ M_{\dot{\Theta}} &= M_{a0} \cdot q\left(\Theta + \frac{\pi}{2}\right) = M_{a0} \sin \alpha_a \cos \Psi \end{aligned} \right\} \quad (7)$$

The components M_{Ψ} , $M_{\dot{\Psi}}$, and $M_{\dot{\Psi}}$ of the moment M_{a0} in the z_{a1} , $p(\Psi)$, and $p(\Psi + \pi/2)$ directions are derived as follows:

$$\left. \begin{aligned} M_{\Psi} &= M_{a0} \cdot k_1 = M_{a0} \cdot (-\sin \alpha_a i + \cos \alpha_a k) \\ &= M_{a0} \frac{\cos \alpha_a}{\sqrt{H(\Psi)}} \\ M_{\dot{\Psi}} &= M_{a0} \cdot p(\Psi) = 0 \\ M_{\dot{\Psi}} &= M_{a0} \cdot p\left(\Psi + \frac{\pi}{2}\right) = -M_{a0} \frac{\sin \alpha_a \sin \Psi}{\sqrt{H(\Psi)}} \end{aligned} \right\} \quad (8)$$

Fig. 2 Relation between stationary rectangular co-ordinate systems A - $x_{a1}y_{a1}z_{a1}$ and A - $x_a y_a z$ ($\angle PAQ = \angle QAT = \pi/2$)

Differentiating Eq. (2) with respect to time t , and using Eqs. (1), (4), (7), and (8), we have

$$\frac{d}{dt}\{p(\Psi) \cdot q(\Theta)\} = \dot{\Psi} \frac{\cos \alpha_a}{\sqrt{H(\Psi)}} - \dot{\Theta} \sqrt{H(\Psi)}$$

$$= \frac{1}{M_{ao}} (\dot{\Psi} M_{\Psi} - \dot{\Theta} M_{\Theta}) \equiv 0 \quad (9)$$

where the dot over the variables Ψ and Θ means differentiation with respect to t . From Eq. (9) the following two equations are obtained:

$$\dot{\Theta} = \dot{\Psi} \frac{\cos \alpha_a}{H(\Psi)} \quad (10)$$

$$M_{\Psi} \dot{\Psi} = M_{\Theta} \dot{\Theta} \quad (11)$$

Equation (10) shows a familiar relation⁽⁹⁾ between the angular velocity of the drive shaft and that of the driven shaft. Equation (11) shows clearly that the time rates of work done by the drive shaft (the left-hand side of Eq. (11)) and the driven yoke (the right-hand side) are equal to each other; this shows the truism that there is no energy loss in a frictionless universal joint.

Figure 1 shows all the moments and forces acting on the shaft driven by a universal joint which has no friction between the cross-pin and the yokes. In this figure the vectors R and $-R$ are the reactions given by the bearing B and the drive shaft, respectively; they are proportional to the moment M_{Θ}'' in magnitude, and coincide with Θ pin (AQ) in direction.

In the following, the drive shaft is

assumed to rotate with a constant angular velocity ω , and Ψ pin is assumed to exist in a plane containing both the z_{a1} and z axes when $t=0$, namely,

$$\Psi = \omega t \quad (12)$$

Here, a revolutionary angle Θ is given as a known function of time; if a dynamical property is given to the driven shaft system, then M_{Θ} is obtained from Eq. (7), and a secondary moment M_{Θ}'' is determined by

$$M_{\Theta}'' = M_{\Theta} \sin \alpha_a \frac{\cos \Psi}{\sqrt{H(\Psi)}} \quad (13)$$

Consider a driven shaft system such as shown in Fig. 3. A flexible shaft S_h has a uniform circular cross section of length l , and has a balanced symmetrical rotor D. A point A supported by a rigid drive shaft is a center of the universal joint. One end of the flexible shaft S_h is simply supported at the point A. The rotor D is mounted at a position S which is at distances of a and b from the shaft ends A and B, respectively. The secondary moment and also the angular velocity of the driven shaft are assumed not to be influenced by deflection in the driven shaft.

For convenience, an equivalent force F and an equivalent moment M at the point S are considered instead of a moment M_j which acts at the point A in the direction perpendicular to the driven shaft. Simultaneous action of F and M causes the same deflection and angle of deflection in the driven shaft as caused only by the moment M_j . Use of the theory of elasticity gives

$$F = \frac{3M_j}{2a}, \quad M = -\frac{M_j}{2} \quad (14)$$

whether the support condition at the shaft end B is simple or fixed. The positive directions of the vectors F , M , and M_j are shown in Fig. 3.

Let a torsional angle at the point S with respect to the point A be $\Theta_s - \Theta$, the positive direction of which is the same as the shaft rotation. The torsional angle $\Theta_s - \Theta$ and the rotating angle Θ_s are given by the following equations:

$$M_{\Theta} = -\delta_s(\Theta_s - \Theta) \quad (15)$$

Fig. 3 Driven shaft and rotor system

Fig. 4 Variation of coefficient ϵ by joint angle α_a

Fig. 5 Force F and moment M equivalent to secondary moment M_{Θ}''

$$I_p \ddot{\theta}_s = -\delta_t (\theta_s - \theta) - M_b \dots (16)$$

where, I_p denotes a polar moment of inertia of the rotor D, δ_t a spring constant for shaft torsion between A and S, and M_b a constant resisting moment acting on the driven shaft end B.

Equation (10) can be transformed into the following form*:

$$\dot{\theta} = \omega(1 + 2 \sum_{N=2,4,\dots} \epsilon^{N/2} \cos N\omega t) \dots (17)$$

Here the coefficient ϵ is given by the following equation:

$$\epsilon = \cos \alpha_a \sum_{k=1}^{\infty} 2^k C_{k-1} \left(\frac{\sin \alpha_a}{2} \right)^{2k} \dots (18)$$

which is a function only of the joint angle α_a as shown in Fig. 4. Equation (17) shows clearly that the component fluctuating with the angular velocity $N\omega$ is $2\omega\epsilon^{N/2}$ in magnitude. Since the coefficient ϵ increases quadratically with respect to the joint angle α_a , the quadratic equation

$$\epsilon = 0.262 \alpha_a^2 \dots (18)'$$

serves as an approximation of the infinite polynomial (18). The broken line in Fig. 4 indicates the value ϵ derived from Eq. (18)', and it shows a fairly good approximation to Eq. (18) when α_a is smaller than $\pi/6$. Substitution of Eq. (17) into Eq. (16) gives the following differential equation with respect to the torsional angle $\theta_s - \theta$:

$$I_p \frac{d^2}{dt^2} (\theta_s - \theta) + \delta_t (\theta_s - \theta) = 2I_p \omega^2 \sum_{N=2,4,\dots} \epsilon^{N/2} N \sin N\omega t - M_b \dots (19)$$

Substitution of Eq. (19) into Eq. (15) yields

$$\left. \begin{aligned} M_\theta &= M_b - \delta_t \sum_{N=2,4,\dots} \epsilon^{N/2} d_N \sin N\omega t \\ d_N &= \frac{2NI_p \omega^2}{\delta_t - I_p (N\omega)^2} \end{aligned} \right\} \dots (20)$$

As pointed out in the introduction, a drive torque M_ψ acting on the drive yoke should not be presumed in advance, and it is determined necessarily by Eqs. (7) and (8), namely,

$$\begin{aligned} M_\psi &= M_\theta \frac{\cos \alpha_a}{H(\psi)} \\ &= (M_b - \delta_t \sum_{N=2,4,\dots} \epsilon^{N/2} d_N \sin N\omega t) \frac{\cos \alpha_a}{H(\psi)} \end{aligned} \dots (21)$$

Here one may set up a stationary rectangular co-ordinate system $O-xyz$ parallel to $A-x_a y_a z_a$ in which point S coincides with the origin O when there is no deflection in the driven shaft. Substituting M_ψ in

Eq. (13) into M_j in Eq. (14), we obtain the equivalent force F and moment M expressed on the $O-xy$ plane (Fig. 5). Comparison of Eq. (10) with Eq. (17) yields

$$\frac{\cos \alpha_a}{H(\psi)} = 1 + 2 \sum_{N=2,4,\dots} \epsilon^{N/2} \cos N\psi \dots (22)$$

Use of Eqs. (4), (13), (20), (22), and Fig. 5 gives the following x components F_x and M_x , and y components F_y and M_y :

$$\left. \begin{aligned} F_x &= -\frac{3M_\theta''}{2a} \sin \theta \\ &= -\frac{3}{4a} \tan \alpha_a \sin 2\omega t \\ &\quad \times (1 + 2 \sum_{N=2,4,\dots} \epsilon^{N/2} \cos N\omega t) \\ &\quad \times (M_b - \delta_t \sum_{N=2,4,\dots} \epsilon^{N/2} d_N \sin N\omega t) \\ F_y &= \frac{3M_\theta''}{2a} \cos \theta \\ &= \frac{3}{4a} \sin \alpha_a (1 + \cos 2\omega t) \\ &\quad \times (1 + 2 \sum_{N=2,4,\dots} \epsilon^{N/2} \cos N\omega t) \\ &\quad \times (M_b - \delta_t \sum_{N=2,4,\dots} \epsilon^{N/2} d_N \sin N\omega t) \\ M_x &= \frac{aF_y}{3}, \quad M_y = -\frac{aF_x}{3} \end{aligned} \right\} \dots (23)$$

Apparently, the equivalent force and moment (23) have the components of whirling velocities $\pm N\omega$ ($N=2,4,6,\dots$). Consequently, when any one among $\pm N\omega$ is nearly equal to the natural angular frequency p in relation to a whirling motion of the shaft (i.e., $\omega \approx \pm p/N$ holds well), the shaft undergoes a violent lateral vibration whirling with any one angular velocity among $\pm N\omega$. Also, when the drive shaft rotates with an angular velocity nearly equal to one of the even integer submultiples of the natural frequency regarding torsional vibration of the driven shaft, the shaft may have a violent lateral vibration accompanied by a violent torsional vibration.

3. Experiments

Experiments are performed for the purpose of confirming the analytical results obtained in Chap. 2. Lateral deflections and angles of deflection at the mounting point S of a rotor D are measured for the joint angle α_a and the axial moment M_θ .

A universal joint used in the experiments is shown in Fig. 6. This joint is not one available on the market but one produced especially for the present investigation with the following structure and characteristics: Both single-row radial ball bearings (#6002) and thrust ball bearings (#51104) are used between the cross-pin and yokes. Consequently, play and frictions between the cross-pin and yokes will be smaller than those of the conventional universal joint obtained on the market. The coefficient of the viscous friction to the relative swinging motion between the cross-pin and the yokes

* Although the direct derivation of Eq. (17) is very difficult, the equivalence of Eq. (17) to Eq. (10) may be proved rather easily.

is about 0.3 sNm/rad , and the moment caused by Coulomb friction is $8.1 \times 10^{-4} \text{ Nm}$. The magnitude of play is about 0.01 mm . The driven shaft S_h is made of mild steel with a uniform circular section, and one end B of it is supported in fixed condition (Fig. 3). The dimensions of the driven shaft are $l = 625.4 \text{ mm}$ in length ($a = 237.5 \text{ mm}$, $b = 387.9 \text{ mm}$) and 17.97 mm in diameter.

The experiments are performed as follows: A moment M_b is externally given at the shaft end B, and its direction is counterclockwise as viewed from the drive shaft. Under the action of the moment M_b , the drive shaft is fixed in relation to rotation. Therefore $\omega = 0$ and $\dot{d}_N = 0$ hold, and from Eq. (20) the moment M_θ exerted at the point A is

$$M_\theta = M_b \quad \dots\dots\dots (20)'$$

Thereupon static deflections and angles of deflection at the point S are measured. The following equations

$$\left. \begin{aligned} \frac{aF_x}{M_\theta} &= \frac{a}{M_\theta} (ax + \gamma\theta_x) \\ \frac{aF_y}{M_\theta} &= \frac{a}{M_\theta} (ay + \gamma\theta_y) \end{aligned} \right\} \dots\dots\dots (24)$$

are used for non-dimensional quantities. They are shown as functions of Ψ in Fig. 7, where α and γ in Eq. (24) are the spring constants of the shaft. The solid line and the broken line in Fig. 7 represent the calculated values from the following equations

$$\left. \begin{aligned} \frac{aF_x}{M_\theta} &= -\frac{3 \sin \alpha_a \sin 2\Psi}{4H(\Psi)} \\ \frac{aF_y}{M_\theta} &= \frac{3 \sin \alpha_a \cos \alpha_a (1 + \cos 2\Psi)}{4H(\Psi)} \end{aligned} \right\} \dots\dots\dots (25)$$

which are given by substitution of Eqs. (4), (13), and (20) into Eq. (23). Figure 8 shows the measured results for the purpose of confirmation of the proportional relation (25) between the equivalent forces F_x , F_y and the moment M_θ when $\Psi = \pi/6 \text{ rad}$ and $M_\theta = 0 \sim 16.8 \text{ Nm}$. Since these measured values coincide well with the calculated values, it is clear that the lateral deflection of the shaft is generated by the secondary moment M_θ , and the

Fig. 6 Universal joint

analytical results derived in Chap. 2 are valid.

4. Conclusions

In this study, consideration has been given to the mechanism of a moment transmitted by a universal joint. Also, a dynamically clear interpretation has been proposed for generation of a secondary moment. Thus causes and characteristics of a lateral vibration generated in the

Fig. 7 Variations of equivalent forces F_x , F_y by revolutionary angle Ψ of drive shaft

Fig. 8 Variations of equivalent forces F_x , F_y by drive moment M_θ acting on driven shaft

shaft driven by a universal joint have been clarified. The results obtained may be summarized as follows:

(1) Owing to the secondary moment generated by the universal joint, the driven shaft makes a resonance and has a number of forced lateral vibrations. These vibrations increase when the drive shaft rotates with an angular velocity nearly equal to one of the even integer submultiples of natural angular frequencies in the driven shaft system. The whirling angular velocities of these vibrations are equal to even integer multiples of the angular velocity of the drive shaft.

(2) An equivalent force and moment acting on a rotor mounted on the shaft may be used instead of the secondary moment.

(3) When the drive shaft rotates with a constant angular velocity, the shaft driven by a universal joint rotates with the angular velocity represented by simple Fourier series (Eq. (17)). Use of this equation can facilitate vibration analyses for the driven shaft system.

(4) The experimental results show validity of the present analyses.

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