Flexural Wave Propagation in Beam with Dispersion *

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The wave distortions caused by the dispersion to a rectangular wave are calculated by a method based on an approximation by a Fourier series expansion and the distinctive features of their dispersive waves are shown. Next, for the case that the rectangular wave is given to the free end of a cantilever, both the results of the dispersive waves according to the finite element model and the experimental results coincide with the ones approximated by Fourier expansion. From these results, it is clear that the wave distortions arise actually. A study of the forming process of the natural modes lead to the conclusion that the modes of higher frequencies are early formed.

Key Words : Vibration of Continuous System, Beam, Dispersion, Flexural Wave, Wave Distortion, Traveling Wave, Standing Wave, Wave Motion

1. Introduction

An audio cartridge is a device that transforms a mechanical vibration into an electrical signal. In it, the input wave as a flexural wave must propagate without a wave distortion. It is well known that the propagation velocity of the flexural wave, namely the phase velocity varies depending upon a wave length or a frequency when this flexural wave propagates in a beam. These phenomena are called a dispersion in general. It has been known since long ago that the traveling waves are distorted by a dispersion when the wave consisting of the components with various frequencies propagates in a dispersive medium. For example, according to a book of Morse P.M., the analytical solution is obtained when an infinitely long beam has an initial displacement with a normal distribution. And it is definitively stated in comparison with a string without dispersion that the traveling wave propagates with the initial shape distorted. A study of a propagation velocity in consideration of a rotatory inertia and a shearing force is also reported. And a flexural wave propagation in a plate is investigated by M-sequence correlation method. However, no study on a wave distortion has been carried out up to date. Particularly, in the case that some shape wave enters a finite length beam, we think it important industrially and technologically to clarify a difference between the input wave and the output wave of such a flexural wave in a dispersive medium.

From this point of view, this paper verifies first of all by an impulse response of a cantilever that a dispersion arises in fact. Next, dispersive waves in the case of rectangular wave input are investigated theoretically and experimentally. Then, it is difficult in general to obtain an analytical solution of a traveling wave for the partial differential equation of fourth order governing the flexural wave. Accordingly, assuming an ideal semi-infinitely long beam and inputting a rectangular wave approximated by Fourier series expansion of 300th order, a method of recomposition at each position is tried in consideration of phase velocities of sinusoidal waves. And then, after discretizing the cantilever by using the finite element method, the numerical solutions are obtained by iterative integration of the fourth transition matrix. It is verified by the comparison between numerical solutions and experimental values for the impulse response that the accuracy of the computation is very high. Though numerical solutions of the infinitely long cantilever have influences of a reflection, dispersive waves of recompositions correspond to output waves of numerical solutions when there are no reflections. The experimental results also agree with dispersive waves of recompositions qualitatively. Moreover, from experimental results, the standing wave, namely the natural modes of flexural vibration are formed by the interaction of all waves including reflected waves of traveling waves at the boundary points, but it is clear that standing waves of different frequencies are not formed at the same time.

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2. Dispersive Wave in the Case of Rectangular Wave Input to Semi-infinitely Long Beam

2.1 Phase velocity of flexural wave

The differential equation governing a flexural wave propagating in a beam is written as follows:

\[ \frac{\partial^2 y(x,t)}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad a = \frac{EI}{\rho A} \quad (1) \]

where the cross section is uniform and rotatory inertia effects and shearing forces are neglected. An external force is not applied. Then \( y \) is a longitudinal deflection in the beam, \( A \) is a sectional area and \( x \) is a spatial co-ordinate. Equation (1) differs apparently from the wave equation of a string and the constant \( a \) does not possess dimension of velocity. Let us assume that the solution of Eq. (1) is a simple harmonic wave of the wavelength \( \lambda \) traveling with the velocity \( v_0 \) in the positive \( x \) direction, so that its form \( y \) is

\[ y(x,t) = A \cos \left( \frac{2\pi}{\lambda} x - vt \right) \quad (2) \]

By substituting Eq. (2) into Eq. (1), the following equation as to the phase velocity is obtained.

\[ v_0 = \left( \frac{EI}{\rho A} \right)^{1/2} \quad (3) \]

where \( f \) is the frequency. It is found that the propagating velocity or the phase velocity of the harmonic wave varies in proportion to the square root of the frequency.

2.2 Output wave for rectangular wave input

Let us consider the dispersive phenomena for the rectangular wave input.

\[ y(t) = \begin{cases} Y & (0 \leq t \leq T) \\ 0 & (T < t < 2T) \end{cases} \quad (4) \]

When the rectangular wave indicated by Eq. (4) is repeated with the period \( T \), the following equation is obtained by Fourier series expansion:

\[ y(t) = \sum_{n=1}^{\infty} \frac{2Y}{n\pi} \left( \cos \frac{2\pi x}{T} \cos \frac{2\pi n t}{T} \right) + \frac{Y}{\sqrt{2}} \left( 1 - \cos \frac{2\pi x}{T} \sin \frac{2\pi t}{T} \right) \quad (5) \]

Now, we consider the approximate rectangular wave \( y(t) \) by way of \( n = 300 \) and \( \nu = 0.5 \). The approximate rectangular wave is composed of harmonic waves of 300 terms with period \( T/\nu \). By getting phase velocities of these harmonic waves by Eq. (3), we compute the time taken for each harmonic wave to move the distance \( x \) mm, and then, the output wave at \( x \) mm is obtained by a recomposition in consideration of necessary times to move the distance \( x \) mm. Thus, this is an ideal propagation system of the semi-infinitely long beam with no reflection and no damping.

The input and output waves are shown in Fig. 1 and Fig. 2. These results are computed for an aluminium rod of 10 mm in diameter used in the experiments. Figure 1 shows a rectangular wave whose period is 20 ms and whose pulse width \( \nu T \) is equal to 10 ms. Gibbs phenomena arise slightly in the input wave in spite of higher approximations of 300 terms. From its output wave at the position of 240 mm, it is found that the wave distortion in the output wave caused

Fig. 1 Rectangular input wave approximated by 300 terms of Fourier expansion and dispersive wave (input wave \( T = 20 \) ms)

Fig. 2 Dispersive wave as viewed from varied observed position (input wave \( T = 0.1 \) s)
by the dispersion cannot be disregarded because of the difference between the input wave and the output wave. Figure 2 shows the case of the detected dispersive waves every 240 mm for the period with 0.1 s. It is found that the output wave is largely distorted as the moving distance increases. Comparing dispersive waves at the same position x=240mm the one in Fig.2 whose period T is longer is clearly close to the input wave in shape. Therefore, the rectangular wave whose period is shorter receives the influence of the dispersion and its wave distortion becomes larger.

From Figs.1 and 2, the distinctive features of the dispersive wave for the rectangular wave input are as follows: (1) First, its deflection arises in the negative side. This seems to be the reason why the single rectangular wave is assumed to be a part of the periodic function. (2) Next, the wave shape goes up to the right and has one sharp peak. (3) After this, it assumes an opposite pattern to the process in (2), namely goes down to the right and comes back to (1). (4) When the moving distance increases or the rectangular wave period becomes shorter, the sharp peak of (2) changes into a slow and large peak and it has a vibrational displacement immediately before this.

2.3 Distortion rate of rectangular wave caused by dispersion

The distortion rate of the rectangular wave caused by the dispersion is defined as follows:

$$D = \frac{\sum_{i=1}^{N} \frac{f(t_i) - g(t_i)}{f(t_i)}}{\Delta T}$$

$$\Delta T = \frac{T}{N}$$

where $f(t_i)$ is the input wave and $g(t_i)$ is the output wave. That is, by turning back for only delay time caused by the propagation for the output wave, it is compared with its input wave. Then the products of absolute value of their difference and the step width $\Delta T$ are calculated and total summations are divided by the total area of the input wave. Figure 3 shows the distortion rate calculated as $N=1290$ in Eq.(6). The distortion

![Distortion rate of rectangular wave](image1)

Fig.3 Distortion rate of rectangular wave

![Finite element model of cantilever](image2)

Fig.4 Finite element model of cantilever

(a) Response from just input time to 340 ms (b) Response from just input time to 7 ms

![Numerical solution of impulse response using finite element method](image3)

Fig.5 Numerical solution of impulse response using finite element method
rate decreases as the period becomes longer. It is found that the decreasing rate is about linear if the chart is shown in double logarithm scales. Moreover, such charts look just like parallel shifts of charts of shorter distances as the moving distance increases.

3. Finite Element Model of Cantilever and Numerical Solution

3.1 Cantilever model

No reflection models are studied in this chapter. The actual system is analyzed in this chapter to verify the validity of discussions in chapter 2. For this reason, we consider a finite element model of the cantilever as shown in Fig.4. Discretizing the aluminum circular rod with a span length 1200 mm into five divisions every 240 mm in length, the equation of motion of the total beam is written as follows:

\[ M\ddot{Y} + CY + KY = F \]  \hspace{1cm} (7)

where \( M \) is the mass matrix, \( C \) is the damping matrix, \( K \) is the stiffness matrix, \( Y \) is the state vector including the displacement and the inclination and \( F \) is the external force vector. With the transition matrix obtained from Eq.(7), the numerical solutions are given by their iterative integration.

3.2 Response for impulse input

Figure 5 shows the displacement response at each position every 240 mm in length from the input position when the initial velocity 0.2 mm/s is added to the position \( Y \). (a) shows responses from the input time to 340 ms and (b) shows responses from just the input time to 7 ms. The frequency of a fundamental wave in (a) is 4.4 Hz and it coincides with the result of the eigen value analysis of the model as shown in Fig.4. The second natural frequency is 28.1 Hz and the third is 79.2 Hz. The different responses at each position are caused by the natural modes. In Fig.5 (b), no response times become longer at positions farther from the input position. These results come from traveling waves which arise instantaneously at the impact of the impulse input. The longer the moving distance, the later becomes the arrival. They can be also explained by the phase velocity of Eq.(3).

3.3 Response for rectangular wave input

The rectangular wave is given to the position of \( Y \) of the cantilever model as shown in Fig.4. The term \( F \) in Eq.(7) is given in this case. For the purpose of constructing the forced displacement, the stiffness of 10Ks and the damping of \( C = 1.97 \times 10^6 \) N/s/m are added to the position of \( Y \). Further, putting suitable values in elements of the damping matrix of Eq.(7) in order to reduce the influences of the reflection, the numerical solutions are carried out. Figure 6 shows responses at the position in 240 mm when the rectangular wave having a pulse width 50 ms is given to the position of \( Y \). The input wave is regarded approxi-
mately as a rectangular wave though the seventh natural vibration of 403 Hz remains slightly. Though the input wave is given in the positive direction, the output wave deflects to the negative direction at first. And, it has an inclination to go up in the right direction and a sharp peak. Then, it approaches zero asymptotically again. This coincides excellently with the results of the semi-ininitely long beam as shown in Fig.2. Figure 7 shows numerical results for the case of the pulse width being 10 ms and the elements of the damping matrix being changed. Though their influences are recognized in absolute amplitudes, the fundamental tendencies of output waves are similar to the case of the semi-ininitely long beam in Fig.1. Besides, response at positions farther from the input position are not indicated because the influences of the reflection arise on account of the small absolute displacement.

From above results, it is known that when the rectangular wave by the forced deflection is given to an actual infinitely long beam, the dispersive phenomena arise after all and obtained that the output wave distorts largely. The numerical solutions obtained by using the finite element method more closely represent the actual phenomena when the influences of reflections are reduced. However, if we investigate only influences of pure dispersion, we can suppose that a semi-ininitely long beam in Chapter 2 is efficient and appropriate.

4. Experiment

4.1 Experimental rig

Figure 8 shows an outline of the experimental rig. An aluminium circular rod of 4.10 is fixed firmly by a vise. The length of the cantilever is 1200 mm and its natural frequencies by forced harmonic excitations are as follows: The first natural frequency is 4.7 Hz (theoretical value is 4.43 Hz), the second is 23 Hz (28.1 Hz), the third is 76 Hz (79.2 Hz) and the fourth is 150 Hz (155.2 Hz).

4.2 Acceleration response for impulse input

An impulse input is given by an impact having a single pulse of an arbitrary period to the coil of the solenoid's iron core. The waves are detected by the acceleration pickup sensors at positions of 500 mm and 1000 mm from the input position. In order to determine the time taken for propagation from the input to the detection, a voltage of 5 V is previously applied to the beam. The record on the transient recorder starts just at the moment when the iron core touches the cantilever. At first, the acceleration pickup was attached to the input position. The delay time of the trigger in the recorder except for the delay time caused by the propagation was measured. It was less than 0.05 ms. The obtained acceleration response waves are indicated in Fig.9. (a) shows the response at the position of 500 mm and (b) shows the one at 1000 mm. Both of them are observed by turns from waves of shorter periods. For the case of (b) at position farther from the input position, the delay time caused by propagations becomes longer. It seems that each frequency separates well when the moving distance is long. Comparing (a) with (b), a tendency is recognized. Here, the band frequency of the acceleration pickup is 1 kHz.

The practical dispersive phenomena can be qualitatively understood by above results. And then, let us discuss them quantitatively in comparison with theoretical values. The waves observed in Fig.9 are produced as composed waves of several frequencies. Therefore, the phase velocity for the single harmonic wave of Eq.(3) can not be applied. Such wave is a so-called wave packet and the propagation velocity of the wave packet is defined as the group velocity. The following relation between the group velocity and the phase velocity is introduced by one dimensional flexural theory of a beam.

$$v_g = 2v_p$$

After reading the periods of wave packets, the times taken to propagate the distances 500 mm or 1000 mm are calculated by obtaining group velocities from Eq.(8). These are looked upon as theoretical values. They are gathered in Table 1 comparing with the times read directly from Fig.9. Theoretical and actual measured values match distinctly as shown in Table 1. It is found that the dispersive phenomena of the flexural wave are also observed quantitatively in this experiment. In Fig.9, the waves after about 0.4 ms in (a) and 0.8 ms in (b) are distorted considerably. These are caused by an interference of reflected waves from the clamped end and the free end. By preventing
reflected waves, the wave packets of longer periods can be also observed.

4.3 Forming process of standing wave

The standing waves, namely natural modes of the flexural vibration are formed by an interference between progressive waves and reflected waves. It has been considered up to date that the standing waves of a low frequency are formed at the same time. However, as the wave packet of a higher frequency has a larger group velocity on account of the dispersion of the flexural wave, it would be natural to think that higher order standing waves are earlier formed. Because the quantitative experiments or reported examples in relation to this were not available, the following experiments were performed as to the forming process of the standing wave.

Experiments are carried out by the same method as in section 4.2. The minimum band pass filters near each natural frequency as shown in Table 2 are located in front of transient recorders so that they decay at a rate of 60 db/oct. Figure 10 shows the acceleration waves detected at positions 500 mm and 1000 mm. The necessary times to form the standing waves differ on account of the dispersion. The higher order standing waves are earlier formed. Though the transient traveling waves are not observed because of small amplitudes, we can guess the process from Fig.10 such that the amplitudes gradually grow to standing waves after interfering between traveling waves and reflected waves at the clamped end or the free end. By both obtaining phase velocities based on frequencies read from Fig.10 and calculating total moving distances based on transient times read from Fig.10 for some frequencies, the number of times of reflections was calculated as shown in Table 2. Then, transient times signify times from an instant of the input to attainment of almost constant amplitude. We suppose that above mentioned forming processes of standing waves are supported by these experimental results. A more detailed mechanism of the transition process from traveling wave to the standing wave remains unknown at present. However, the reflection at the boundary point forms a standing wave finally. And then, the forming of the standing wave has a time constant and it is clear from this experimental results that the time constants for the standing wave of higher order are smaller.

Table 1 Comparison of theoretical value with experimental value for progressive time using group velocity

(a) position at 500 mm from input position

<table>
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<th>No.</th>
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<th>$\frac{\epsilon}{T}$</th>
<th>frequency m/s</th>
<th>wave number</th>
<th>dispersion</th>
<th>group velocity</th>
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(b) position at 1000 mm from input position

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Fig.10 Forming process of standing wave (natural mode)

Table 2 Trial calculation of number of times of reflection using phase velocity

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4.4 Displacement response for impulse input

In order to compare with the numerical solutions by means of the finite element method in section 3.2, the displacement responses are measured in this section. The measurement uses an optical detector instead of the acceleration pickup and each displacement every 240 mm from input positions is detected.

Figure 11 shows response waves. (a) shows the response waves within about 500 ms and (b) shows the ones within 10 ms from an instant of the input using the external trigger. As Fig. 9 shows acceleration responses within about 5 ms from an instant of the input, the halves of displacement waves at positions of 480 mm or 960 mm in Fig. 11 (b) correspond to Fig. 9 (a) and (b). Therefore, the same data are observed by accelerations and displacements. The dispersive phenomena for displacement responses in Fig. 11 (b) are also observed distinctly. When the moving distances become longer, no response times become larger. In addition, comparing numerical solutions in Fig. 5 by means of the finite element method with experimental results in Fig. 11 at same positions, both waves coincide very well with each other to details. Consequently, it implies that the finite element model of the cantilever approximates rightly the experimental system and numerical solutions in the case of the rectangular wave input well simulate practical phenomena.

4.5 Output wave for rectangular wave input

After clamping the solenoid iron coil at the free end of the cantilever with adhesive, rectangular waves were given varying the pulse widths of the pulse generators. Optical displacement detectors were used and the displacements were detected at two positions 100 mm and 1000 mm from the input position. As the damping action is small in this propagation system, the lower order free vibrations arise if the rectangular wave is given in this state and then it is difficult to observe the dispersive phenomena. So, the free vibrations were decayed by an electro-magnetic damper using the eddy current loss. The reflected waves were reduced by this method.

The results of this kind are shown in Fig. 12. On the upper side in both (a) and (b) are shown waves at positions 100 mm from the input position. They are waves with little dispersions, namely they are regarded as input waves. They can be considered approximately as rectangular waves. If rectangular waves of shorter pulse widths are given, they differ from rectangular waves on account of mechanical inputs. In this experiment, the pulse width of 0.4 second was a limit. On the under side in (a) and (b) are shown output waves. Peaks like a projection can be recognized at the start down of the rectangular wave. These are similar to numerical solutions by means of the finite element model. The displacement is negative at the start up of the output wave. Moreover, the displacements at positions 960 mm as shown in Fig. 5 (b) and Fig. 11 (b) by
impulse responses are negative. Therefore, it seems that these sections suffer extreme influences of the dispersion. For the case of an ideal semi-ininitely long beam, Fig. 13 shows the computed dispersive wave at the position 1000 mm when the rectangular wave having the same pulse width as in experiments is approximated by Fourier series expansion of 300 terms. Comparing Fig.13 with Fig.12, we see that both coincide well. Therefore, the forced displacement given by the solenoid propagates in the dispersive medium as a flexural wave and thus the output wave differs from the input wave because of the dispersion in this process.

5. Conclusions
The problem of a flexural wave propagation with dispersion was studied in this paper. Attending to the dispersive wave when a rectangular wave was given mainly, theoretical and experimental investigations were performed. The conclusions are summarized as follows:

(1) In the case of the rectangular wave input, the output waves with the dispersion of the ideal semi-ininitely long beam and the output waves obtained by the experiments coincided well qualitatively with each other. Therefore, the wave distortion caused by the dispersion arises actually when a wave is given to the beam as a flexural wave.

(2) When the pulse width becomes shorter for the rectangular wave, its dispersion becomes larger and the difference between the input wave and the output wave (so-called the distortion rate) increases. The distinctive features of the dispersive wave for the rectangular wave input are as follows: First, its displacement arises in the negative side, next the wave shape goes up to the right and has one sharp peak. After this, it takes an opposite pattern, namely it goes down to the right and comes back to the initial state.

(3) After expanding the input wave in Fourier series, the method of the recomposition in consideration of the dispersion coincides with the numerical solution by means of the finite element model in the case of noreflection. So, if the influence of pure dispersion is to be investigated, the method of the recomposition will be more convenient and efficient.

(4) It is clear from the experimental results that a standing wave, namely the natural mode of the flexural vibration is formed by the interference between the traveling wave and the reflected wave at the ends. Moreover, it is evident that the natural modes of different frequencies are not formed at the same time and ones of higher order frequencies are earlier formed.

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References