Pulsating Flow in Curved Pipes *

(1st Report, Numerical and Approximate Analyses)

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A pulsating laminar flow in a circular curved pipe is analyzed numerically, changing a flow rate ratio and a nondimensional frequency parameter. A time-dependence of internal mechanism is made clear, such as velocity profiles, distributions of wall shear stresses and others. A pressure gradient, important in engineering, is calculated on the basis of these results. Also, the pressure gradient of the pulsating flow is calculated approximately by assuming the quasi-steady state and taking an unsteady inertia force into account. The approximate result is compared with the numerical result for a mean value, an amplitude value and a phase difference of the pulsating pressure gradient.

Key Words: Fluid Mechanics, Unsteady Flow, Pulsating Flow, Quasi-steady Flow, Curved Pipe, Numerical Analysis

1. Introduction

Curved pipes are utilized actively for piping system of oil pipelines, heat exchangers and chemical plants etc. For an internal flow of curved pipes, therefore, many investigations have been performed in practical uses and scientific interests, and many theoretical and experimental results have been reported. However, most of them were concerned with steady flows through curved pipes. On the other hand, studies have become more important on unsteady flows at the start-stop or the undesirable accident of pumps and blowers because recent fluid turbomachineries have larger scales and faster transfer rate. Also, in field of medical engineering, many attempts have been made intently in order to solve problems on blood circulation through a heart and an artery from view points of hydraulics.

However, not so many reports have been published on unsteady pipe flows. Especially on unsteady flows in curved pipes, only a few results have been reported because of its complicated phenomena. Lyne⁽¹⁾ analyzed theoretically an oscillating flow in a curved pipe under the condition of Womersley number $\alpha \gg 1$, and pointed out that a secondary flow, different from Dean's type vortices⁽²⁾ of steady flows, circulated inversely in a core of pipe. Zalosh et al.⁽³⁾ and Mullin et al.⁽⁴⁾ calculated an oscillating flow with a small amplitude by the perturbation method like Dean's. Their flow patterns of secondary vortices were experimentally demonstrated by the flow visualization

with aluminium powder⁽⁵⁾ and dye⁽⁶⁾ Smith⁽⁷⁾ analyzed the unsteady flow in the curved pipe with an arbitrary cross section under a pulsating pressure gradient, and obtained approximate solutions for $\alpha \ll 1$ and $\alpha \gg 1$. Lin et al.^{(8),(9)} computed the pulsating flow in a curved pipe numerically, and experimented under the condition of $2.5 < \alpha < 5.5$ and $Re < 1100$. They concluded that a certain kind of resonant phenomenon occurred between the axial flow and the secondary vortices. However as the internal flow mechanism is not made clear, more careful investigations are needed in order to apply their conclusion to practical fields and blood circulation.

On the basis of above mentioned matters, at first in the present paper, the velocity profile of primary flow, the stream line of secondary flow, the distribution of wall shear stresses and the pulsating pressure gradient are computed by the numerical analysis, and the effect of the nondimensional parameters α and β is made clear on the behavior of the pulsating flow through the curved pipe. Second, a simple expression for the pulsating pressure gradient is presented by the approximate analysis under the assumption of the quasi-steady state, and the approximate solution is compared with the numerical solution in order to demonstrate its validity.

2. Nomenclature

- a : radius of a curved pipe
- D : Dean number $= Re \sqrt{a/r_c}$
- P : nondimensional pressure $= pa^2/\rho U^2$
- R, θ , Z : nondimensional torus coordinates $= r/a, \theta, z/a$
- Rc : nondimensional curvature radius of a curved pipe $= r_c/a$
- Re : Reynolds number $= 2U$

* Received 22nd October, 1982.

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- T : nondimensional time $= t\nu/a^2$
 Tw : nondimensional wall shear stress $= \tau_w a^2/\mu\nu^2$
 U : nondimensional cross-sectional average velocity $= ua/\nu$
 W : nondimensional axial velocity $= wa/\nu$
 α : Womersley number (nondimensional frequency) $= a\sqrt{\omega/\nu}$
 γ : pressure gradient ratio $= (-dP/dZ)/(-dP/dZ)_s$
 η : flow rate ratio $= \bar{U}/\bar{U}$
 η_p : amplitude ratio of pressure gradient $= (-dP/dZ)/(-dP/dZ)_s$
 θ : phase $= \alpha^2 T$
 ν, ρ : kinematic viscosity and density of fluid
 σ : amplitude ratio $= \eta/\eta_p$
 Φ : phase difference
 Ψ : nondimensional stream function of secondary flow $= \psi/\nu$
 ω : angular frequency
 Ω : nondimensional vorticity of secondary flow $= \xi a^2/\nu$

Subscripts

- $-$: time-mean value
 $\sim$: amplitude value
 s : steady flow
 qs : quasi-steady flow

3. Analysis

3.1 Numerical analysis The coordinate system for the numerical analysis is shown in Fig.1. Basic equations for a nondimensional axial velocity W , a nondimensional vorticity Ω and a nondimensional stream function Ψ of the secondary flow reduce respectively to

$$\frac{\partial W}{\partial T} = \frac{\partial^2 W}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 W}{\partial \theta^2} + \left(A + \frac{1}{R} + \frac{1}{HR} \frac{\partial \Psi}{\partial \theta} \right) \frac{\partial W}{\partial R} + \left(B - \frac{1}{H} \frac{\partial \Psi}{\partial R} \right) \frac{1}{R} \frac{\partial W}{\partial \theta} - \left\{ \frac{1}{H^2} + \left(\frac{B}{H} \frac{\partial \Psi}{\partial R} - \frac{A}{H} \frac{1}{R} \frac{\partial \Psi}{\partial \theta} \right) \right\} W - \frac{R_c}{H} \frac{dP}{dZ} \quad (1)$$

$$\frac{\partial \Omega}{\partial T} = \frac{\partial^2 \Omega}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 \Omega}{\partial \theta^2} + \left(A + \frac{1}{R} + \frac{1}{HR} \frac{\partial \Psi}{\partial \theta} \right) \frac{\partial \Omega}{\partial R} + \left(B - \frac{1}{H} \frac{\partial \Psi}{\partial R} \right) \frac{1}{R} \frac{\partial \Omega}{\partial \theta} + \left\{ \frac{1}{H^2} + \left(\frac{B}{H} \frac{\partial \Psi}{\partial R} - \frac{A}{H} \frac{1}{R} \frac{\partial \Psi}{\partial \theta} \right) \right\} \Omega + 2W \left(B \frac{\partial W}{\partial R} - \frac{A}{R} \frac{\partial W}{\partial \theta} \right) \quad (2)$$

$$H\Omega = \frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \theta^2} + \left(\frac{1}{R} - A \right) \frac{\partial \Psi}{\partial R} - \frac{B}{R} \frac{\partial \Psi}{\partial \theta} \quad (3)$$

These equations are derived from the unsteady 3-dimensional Navier-Stokes

Fig.1 :Coordinate System

equations for a fully-developed laminar flow of an incompressible fluid through the curved pipe. Here,

$$\left. \begin{aligned} H &= R_c + R \sin \theta \\ A &= \sin \theta / H, \quad B = \cos \theta / H \end{aligned} \right\} \quad (4)$$

From physical considerations, boundary conditions are given as follows;

$$\left. \begin{aligned} \theta = \pm \frac{\pi}{2} &: \frac{\partial W}{\partial \theta} = 0, \quad \Omega = 0, \quad \Psi = 0 \\ R = 1 &: W = 0, \quad \Omega = \frac{1}{H} \frac{\partial^2 \Psi}{\partial R^2}, \quad \Psi = 0 \\ R = 0 = \theta &: \frac{\partial W}{\partial R} = 0 \end{aligned} \right\} \quad (5)$$

Considering that the curved pipe flow is symmetric about the central plane of pipe, Eqs.(1), (2) and (3) are differentiated only in a half section of pipe to solve numerically. As a large velocity gradient is predicted to appear near the pipe wall, 18 grid points are arranged at geometric intervals in the R -direction and 21 points at arithmetic intervals in the θ -direction. In the present analysis, when the pulsating flow rate is given, numerical solutions, such as velocity profiles etc., are obtained by the unsteady explicit method (10).

The iteration of numerical calculation was stopped if the following convergence criterion

$$\frac{\sum_{i=1}^n \sum_{j=1}^m \left(\left| 1 - \frac{W_{ij}^{n+1}}{W_{ij}^n} \right| + \left| 1 - \frac{\Omega_{ij}^{n+1}}{\Omega_{ij}^n} \right| + \left| 1 - \frac{\Psi_{ij}^{n+1}}{\Psi_{ij}^n} \right| \right)}{3mn} < 10^{-4} \quad (6)$$

, which was defined by relative residuals in each period, was satisfied. CPU time was about 5~9 min (using an optimum option: level 2) with HITAC M-200H in Hiroshima Univ. Information Processing Center.

Integrating Eq.(1) over the cross section of pipe in order to evaluate the pressure drop which is important in the industrial engineering, the following equation is derived.

$$\pi \frac{dU}{dT} + T_c = C_p \left(-\frac{dP}{dZ} \right) - T_w \quad (7)$$

Namely,

$$-\frac{dP}{dZ} = \frac{\pi}{C_p} \frac{dU}{dT} + \frac{T_c + T_w}{C_p} \quad (7)'$$

In the above equation, the nondimensional average velocity U , the nondimensional convective term T_c , the nondimensional wall shear stress T_w and the coefficient C_p can be written as follows;

$$U = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \int_0^1 (RW) dR d\theta \quad (8)$$

$$T_c = 4 \int_{-\pi/2}^{\pi/2} \int_0^1 \frac{\Psi}{H} \left(A \frac{\partial W}{\partial \theta} - RB \frac{\partial W}{\partial R} \right) dR d\theta \quad (9)$$

$$T_w = 2 \int_{-\pi/2}^{\pi/2} \left(-\frac{\partial W}{\partial R} \right) \Big|_{R=1} d\theta \quad (10)$$

$$C_p = 2\pi R_c (R_c - \sqrt{R_c^2 - 1}) \quad (11)$$

Each value of Eqs.(8), (9) and (10) can be estimated by the numerical integration.

The error between the steady pressure gradient $(-dP/dZ)_s$ obtained by the present calculation and the following semi-theoretical expression

$$\left(-\frac{dP}{dZ} \right)_s = 4R_c C_0 D^{1/2} \{ 1 + C_1 D^{-1/2} + C_2 D^{-1} \}$$

$$+C_3 D^{-3/2} + C_4 D^{-2} + O(D^{-5/2}) \dots\dots\dots(12)$$

was within about 1%. Expression (12) was derived for steady flows through the curved pipe under the boundary layer approximation by Ito (11). Here, the coefficients are

$$C_0=0.1008, C_1=3.945, C_2=7.732$$

$$C_3=9.097, C_4=5.608.$$

Because convective terms are non-linear, there appear high frequency components in the wave form of the pressure gradient, estimated from Eq. (7). Therefore, in the present paper, the pulsating pressure gradient is expressed as the following Fourier's series

$$-\frac{dP}{dZ} = \left(-\frac{dP}{dZ}\right)_0 + \sum_{n=1}^{\infty} \left(-\frac{dP}{dZ}\right)_n \sin(n\theta + \phi_n) \dots\dots\dots(13)$$

by analyzing its numerical data for two periods. Here, $\theta = \alpha^2 T$ and subscript n means the n -th component.

The above is an outline of numerical calculation. Considering the convergence of solution and the cost of computation, however, it is desirable in practical uses to present a simple expression for the pulsating pressure gradient. So, by assuming a quasi-steady state which means that the steady relation can be applied to the relation between a pulsating flow rate and a pulsating pressure gradient, and by taking the unsteady inertia force into account, a simple expression can be derived as follows.

3.2 Approximate analysis When the pulsating flow rate is given as

$$U = \bar{U}(1 + \eta \sin \theta) \dots\dots\dots(14)$$

Reynolds number and Dean number are expressed as follows;

$$\left. \begin{aligned} R_e &= \bar{R}_e(1 + \eta \sin \theta) \\ D &= \bar{D}(1 + \eta \sin \theta) \end{aligned} \right\} \dots\dots\dots(15)$$

Here, $\bar{R}_e = 2\bar{U}\bar{a}/\nu$, $\bar{D} = \bar{R}_e \sqrt{a/r_c}$ and η is the flow rate ratio of periodic flow to steady flow. Substituting Eq. (15) into Eq. (12) and expanding each term into a binomial series, Eq. (12) reduces to

$$\begin{aligned} \left(-\frac{dP}{dZ}\right)_{qs} &= \left(\frac{T_s + T_u}{C_p}\right)_{qs} \\ &= \left(-\frac{dP}{dZ}\right)_0 (g_0 + g_1 \sin \theta + g_2 \cos 2\theta + \dots) \end{aligned} \dots\dots\dots(16)$$

The above expression indicates the pressure gradient of the quasi-steady state. Here,

$$\left. \begin{aligned} g_0 &= 1 + (f_1/f_0)\eta^2 + (f_2/f_0)\eta^4 + O(\eta^6) \\ g_1 &= (f_3/f_0)\eta + (f_4/f_0)\eta^3 + O(\eta^5) \\ g_2 &= (f_5/f_0)\eta^2 + (f_6/f_0)\eta^4 + O(\eta^6) \end{aligned} \right\} \dots\dots\dots(17)$$

$$\begin{aligned} f_0 &= 1 + C_1 \bar{D}^{-1/2} + C_2 \bar{D}^{-1} + C_3 \bar{D}^{-3/2} + C_4 \bar{D}^{-2} \\ &\quad + O(\bar{D}^{-5/2}) \\ f_1 &= \frac{3}{16} - \frac{1}{16} C_2 \bar{D}^{-1} + \frac{3}{16} C_3 \bar{D}^{-2} + O(\bar{D}^{-3}) \\ f_2 &= \frac{9}{1024} - \frac{15}{1024} C_2 \bar{D}^{-1} + \frac{105}{1024} C_3 \bar{D}^{-2} \\ &\quad + O(\bar{D}^{-3}) \\ f_3 &= \frac{3}{2} + C_1 \bar{D}^{-1/2} + \frac{1}{2} C_2 \bar{D}^{-1} - \frac{1}{2} C_3 \bar{D}^{-2} \\ &\quad + O(\bar{D}^{-5/2}) \end{aligned}$$

$$\left. \begin{aligned} f_4 &= -\frac{3}{64} + \frac{3}{64} C_2 \bar{D}^{-1} - \frac{15}{64} C_3 \bar{D}^{-2} \\ &\quad + O(\bar{D}^{-5/2}) \\ f_5 &= -\frac{3}{16} + \frac{1}{16} C_2 \bar{D}^{-1} - \frac{3}{16} C_3 \bar{D}^{-2} + O(\bar{D}^{-3}) \\ f_6 &= -\frac{3}{256} + \frac{5}{256} C_2 \bar{D}^{-1} - \frac{35}{256} C_3 \bar{D}^{-2} \\ &\quad + O(\bar{D}^{-3}) \end{aligned} \right\} \dots\dots\dots(18)$$

and

$$\left(-\frac{dP}{dZ}\right)_n = 4\bar{R}_e C_0 \bar{D}^{1/2} f_0 \dots\dots\dots(19)$$

On the other hand, substituting Eq. (14) into the first term on the left hand of Eq. (7), the following relation is obtained.

$$\pi \frac{dU}{dT} = \pi \alpha^2 \frac{\bar{R}_e}{2} \eta \cos \theta \dots\dots\dots(20)$$

Here, α^2 means the ratio of the unsteady inertia force to the viscous force. Replacing the first term on the right hand of Eq. (7') by Eq. (20) and the second term by Eq. (16), the pressure gradient can be approximated as

$$\begin{aligned} -\frac{dP}{dZ} &= \left(-\frac{dP}{dZ}\right)_0 \{g_0 + g'_1 \sin(\theta + \phi) \\ &\quad + g_2 \cos 2\theta + \dots\} \dots\dots\dots(21) \end{aligned}$$

Here,

$$g'_1 = \sqrt{g_1^2 + \left(\frac{1}{8} \frac{\pi \alpha^2 \eta}{C_p C_0 \bar{D}^{1/2} f_0}\right)^2} \dots\dots\dots(22)$$

$$\phi = \tan^{-1} \left(\frac{1}{8} \frac{\pi \alpha^2 \eta}{C_p C_0 \bar{D}^{1/2} f_0 g_1} \right) \dots\dots\dots(23)$$

ϕ is a phase difference between the pressure gradient and the flow rate.

4. Results and Discussion

Numerical calculation was executed under the condition of the nondimensional curvature radius $R_c=30$, the mean Dean number $\bar{D}=400$, the flow rate ratio $\eta=0.1 \sim 1.0$ and Womersley number (the nondimensional frequency parameter) $\alpha=2.0 \sim 15.0$ in order to mainly study effects of η and α on behaviors of the pulsating flow through the curved pipe.

4.1 Velocity profiles and distribution of wall shear stresses In order to study general characteristics of the pulsating flow through the curved pipe, at first, analytical results for one period are displayed in case of a middle Womersley number ($\alpha=5.6$) and a middle flow rate ratio ($\eta=0.5$). The contour line of axial velocity W and the stream line of stream function ψ are shown in Fig. 2. The upper half of each figure indicates the profile of axial velocity and the lower half indicates the profile of corresponding stream function. Also, Fig. 3 shows velocity profiles at $\theta=\pm\pi/2$ and 0° under the same condition as Fig. 2. The axial velocity W is normalized by the maximum value W_{max} during one period. The flow pattern during one period in case of middle α and is similar to the pattern in the steady flow (see Fig. 4). Namely, the location of

Fig.2 :Flow Patterns over One Period

the maximum velocity is shifted toward the outer wall and the secondary flow, induced by the centrifugal force, circulates clockwise in the lower half of pipe and counterclockwise in the upper half. However, the trace of the vortex center draws a cyclic trajectory. That is to say, its center moves from $\theta = -170^\circ$ neighborhood at the start of the accelerating phase ($\theta = 270^\circ$) to $\theta = -140^\circ$ neighborhood at the finish ($\theta = 90^\circ$) and returns back on the same path during the decelerating phase. For the axial velocity profile, the time-dependent variation of velocity near the outer wall is larger than that near the inner wall as shown in Fig.3(a). This means that most of pulsating flow rates are concentrated on the outside of the curved pipe. Also, the peak phenomenon of velocity, so-called annular effect, appears near the pipe wall (see Fig.3(b)). This phenomenon is also observed at $\theta = -60^\circ \sim -10^\circ$ neighborhood of the steady laminar flow through the curved pipe (see Fig.4).

Fig.3 :Case of Middle Womersley Number and Middle Flow Rate Ratio

Fig.4 :Flow Pattern of Steady Flow in Curved Pipe

Fig.5 :Wall Shear Stress Distributions over One Period

Figure 5 shows a distribution of wall shear stresses T_w in the θ -direction. The ordinate is normalized by the maximum value T_{wmax} during one period. The wall shear stress always changes from the maximum value at the outside wall to the minimum value at the inside wall. Also, it is known that the wall shear stress changes abruptly near $\theta = -60^\circ \sim -40^\circ$.

Secondly, influences of Womersley number α and the flow rate ratio β are studied. Figure 6 shows velocity profiles in case of a low Womersley number ($\alpha = 2.0$) and a high flow rate ratio ($\beta = 1.0$). The annular phenomenon appears near the pipe wall at $\theta = -50^\circ \sim -30^\circ$ during the period from the accelerating phase ($\theta = 0^\circ$) to the decelerating phase ($\theta = 180^\circ$) as shown

(a) Flow Pattern (b) Velocity Profile at $\theta = \pm \pi/2$ (c) Velocity Profile at $\theta = 0^\circ$
Fig.6 :Case of Low Womersley Number and High Flow Rate Ratio

(a) Flow Pattern (b) Velocity Profile at $\theta = \pm \pi/2$ (c) Velocity Profile at $\theta = 0^\circ$
Fig.7 :Case of High Womersley Number and Low Flow Rate Ratio

(a) Flow Pattern (b) Velocity Profile at $\theta = \pm \pi/2$ (c) Velocity Profile at $\theta = 0^\circ$
Fig.8 :Case of High Womersley Number and High Flow Rate Ratio

in Fig.6(c). Also, an area of increased velocity exists near the pipe wall at $\theta = -40^\circ$ as shown in Fig.6(a). This peculiar phenomenon is also observed in the corresponding secondary flow, and two vortices rotate clockwise in the lower half of Fig.6(a).

Figure 7 shows the analytical result in case of a high Womersley number ($\alpha = 10$) and a low flow rate ratio ($\gamma = 0.2$). The time-dependent variation of axial velocity at the inside portion is rather different from that at the outside of the curved pipe (see Fig.7(b)). However, this flow pattern is almost similar to that of the pulsating flow with a middle α and a middle γ . From these results, it is known that, in case of a low flow rate ratio γ , the flow pattern is not so affected by the frequency parameter α and the velocity profile during one period is not so different from that of the steady flow.

Figure 8 shows the result in case of a high Womersley number ($\alpha = 10$) and a high flow rate ratio ($\gamma = 1.0$). Under this condition, the velocity profile changes much complicatedly all over the cross section of a pipe. The velocity at the inside portion grows up to change remarkably, and there appears an inverse flow during the period from the decelerating phase ($\theta = 180^\circ$) to the accelerating phase ($\theta = 310^\circ$) as shown in Fig.8(b). Also, a distinguished difference can be observed between the time-dependent variation of velocity at the inside portion and that at the outside portion, and it results in a very peculiar profile at the accelerating phase ($\theta = 0^\circ$) as shown in Fig.8(b). The axial velocity at $\theta = 0^\circ$ shows a complicated profile. The corresponding secondary flow shows a strange flow pattern. Namely, three rotating vortices, including a counter rotating one, are induced in the lower half of pipe.

Moreover, the velocity profile is omitted in case of a low Womersley number and a low flow rate ratio, because it is hardly different from the steady one of Fig.4.

4.2 Pressure gradient The pressure gradient ratio ζ is defined by the following expression

$$\zeta = \left(\frac{-dP}{dz} \right) / \left(\frac{-dP}{dz} \right)_s \quad (24)$$

Fig.9 :Pressure Gradient Ratio ζ vs. Flow Rate Ratio γ

and it is shown in Fig.9 for the flow rate ratio γ of the abscissa. The solid line in figure shows the result by the numerical analysis and the dotted line shows the result by the approximate analysis. Moreover, the higher terms than the third one can be neglected in calculation of Eq.(21) because their magnitude is much smaller than the first and the second. As shown in the figure, the mean value of the pressure gradient, independent on Womersley number α , increases nearly in proportion to the square of the flow rate ratio γ in the pulsating laminar flow through the curved pipe. The approximate solution by the quasi-steady theory gives a relatively accurate value as a whole though it becomes a little lower than the numerical solution in the range of $\gamma > 0.7$.

An amplitude value of the pressure gradient is studied next. As mentioned in section 3.1, the pressure gradient can be expressed by Eq.(13) in the pulsating flow through the curved pipe. In order to compare the amplitude values with each order, an amplitude ratio of the pressure gradient η_p is defined by the following expression

$$\eta_p = \left(\frac{-dP}{dz} \right)_n / \left(\frac{-dP}{dz} \right) \quad (n=1, 2, \dots) \quad (25)$$

and it is shown in Fig.10 for Womersley number α of the abscissa with the flow rate ratio γ selected as a parameter. The second component is fairly smaller than the first one of the pressure gradient. For example, in case of $\gamma = 0.5$, the second component is about 6% of the first one for a low Womersley number. However, as Womersley number α increases, the second component decreases though the first one increases rapidly. This trend is almost the same at the other flow rate ratio, too. Therefore, it can be permitted to characterize the amplitude value of the pressure gradient with only the first component. This is also demonstrated by the experimental result (12). In the following discussion, therefore, the first component of the pulsating pressure gradient is studied.

In order to indicate the relation between the flow rate and the pressure gradient in the pulsating flow, the amplitude ratio

Fig.10 :Pressure Gradient Amplitude Ratio η_p vs. Womersley Number α

$$\sigma = \frac{\eta}{\eta_p} = \frac{\bar{U}}{(-dP/dZ)} \frac{(-dP/dZ)}{\bar{U}} \dots (26)$$

which is defined by the flow rate ratio η and the amplitude ratio of the pressure gradient η_p , is shown in Fig.11. The phase difference Φ is shown in Fig.12. The solid line shows a numerical solution and the dotted line shows an approximate solution. Also, one-point dotted line shows an exact solution for the pulsating laminar flow through a straight pipe.

The amplitude ratio σ of the curved pipe is lower than that of the straight pipe for a low Womersley number α . As α increases over a certain value, the amplitude ratio σ of the curved pipe becomes larger than that of the straight pipe. Also, in a limiting case of $\alpha \rightarrow 0$, the amplitude ratio approaches not unity but various values depending on the flow rate ratio η in the curved pipe although σ approaches unity in the straight pipe. In another limiting case of $\alpha \rightarrow \infty$, σ approaches zero in the curved pipe as well as in the straight pipe.

The phase difference Φ between the pulsating flow rate and the pulsating pressure gradient has no relation to the flow rate ratio η . Also, the dependence on Womersley numbers is similar to that of the straight pipe flow. That is to say, the phase difference changes from 0° to 90° gradually as Womersley number α increases.

Fig.11 :Amplitude Ratio σ vs. Womersley Number α

Fig.12 :Phase Difference Φ vs. Womersley Number α

However, the phase difference of the curved pipe is always lower than that of the straight pipe.

Moreover, the result by the quasi-steady approximation, which includes the unsteady inertia force, agrees with the result by the numerical analysis fairly well. This means that the approximate expression is not only simple in practical uses but also reliable numerically. Solid points in Fig.12 are quoted from experimental results in a companion report (22). Experimental results agree with both results by the numerical calculation and the extended quasi-steady approximation. It is known that the analytical result has a good accuracy in the present paper.

5. Conclusion

A pulsating laminar flow in a curved pipe is investigated analytically, and the following conclusions are obtained from the results.

- (1) In case of a low flow rate ratio η , the velocity profile is not so different from that of the steady flow, and not so dependent on the Womersley number α . However, as η increases, the velocity profile becomes complicated and the annular phenomenon appears remarkably. Corresponding to its axial velocity profile, the secondary flow shows a peculiar pattern. Especially in case of a high Womersley number and a high flow rate ratio, inversely rotating vortices are induced in the outside portion of the curved pipe.
- (2) In case of a low Womersley number α , the axial velocity changes largely at the outside portion. However, as α increases, the velocity becomes higher at the inside portion. In case of a high Womersley number and a high flow rate ratio, the inverse flow appears remarkably in the axial velocity at the inside portion.
- (3) The mean value of the pulsating pressure gradient is larger than the pressure gradient of the steady curved pipe flow, and the pressure gradient ratio ζ increases depending on the flow rate ratio η .
- (4) In the amplitude values of the pulsating pressure gradient, the higher components over the second are relatively smaller than the first component. The amplitude ratio of the pressure gradient in the curved pipe flow becomes smaller than that in the straight pipe flow in the range beyond a certain Womersley number. Also, the amplitude ratio σ , which is defined by the flow rate ratio η and the amplitude ratio of the pressure gradient η_p , approaches a certain value depending on η in case of a low Womersley number although it decreases to zero as α increases.
- (5) The phase difference Φ between the pulsating flow rate and the pulsating pressure gradient increases to 90° as Womersley number α increases. Its value is always smaller than that of the straight pipe.
- (6) The approximate solution by the quasi-steady theory, which takes the

unsteady inertia effect into account, agrees with the numerical solution relatively well.

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