Kinematic Analysis of Stephenson Six-Link Mechanisms*

(1st Report, Discrimination of Composition Loops)

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In this paper, a simple method of displacement analysis of Stephenson six-link mechanisms of three kinds which have two closed five-link loops including the fixed and driving links is developed in the form of a solution of a sixth order equation.

Moreover, their composition loops to arise from inversions of link chains for a set of kinematic constants are discriminated by means of the domains on the angular displacement curve of the component four-link chain; the domains are separated by the points which correspond to the limits of rotation of the driving link of the six-link mechanism and the positions of the longest or shortest distance between the end pairing points of the component two-link chain and are distinguished by the sign of the sine of the relative displacement angle of the latter.

Key Words : Mechanism, Kinematics, Six-Bar Linkage, Displacement Analysis, Geometrical Inversion.

1. Introduction

The six-link mechanisms of eight kinds are obtained from two kinematic chains of Watt-type and Stephenson-type. The displacement characteristics of the driven link of six-link mechanisms whose driving link can make a complete rotation have been analyzed in detail [1], and the algebraic, geometric and kinematic properties have been derived for the curves generated by pairing points of two adjacent coupler links of six-link mechanisms of three kinds [2]. Moreover, a systematic method of displacement analysis of plane multi-link mechanisms has been presented [3].

However, in the case of Stephenson six-link mechanisms of three kinds which have two closed five-link loops including the fixed and driving links, a method for discriminating the composition loops to arise from inversions of link chains for a set of kinematic constants is not established yet, so that analytical methods of synthesis are difficult to apply for these mechanisms. Thus, in this paper, a simple method of displacement analysis of above-mentioned mechanisms is developed in the form of a solution of a sixth order equation. Moreover, the geometrically inverted composition loops of these mechanisms are discriminated by means of the domains on the angular displacement curve of the component four-link chain.

2. Displacement analysis

There are the Stephenson six-link mechanisms of five kinds. The displacement analysis and the discrimination of composition loops of six-link mechanisms of two kinds whose fixed and driving links are included in a closed four-link loop are performed by the same procedure as in the plane four-link mechanism. The six-link mechanisms which have two closed five-link loops including the fixed and driving links are of three kinds as shown in Fig. 1. Where the marks $\odot$ and $\circ$ denote the fixed and moving turning pairs respectively, and AB or AB_1B_2 the driving link. In this paper, they are referred to as complex six-link mechanisms and designated by S22F, S23F and S32F respectively, where the capital letter S means that they are obtained from the six-link chain of Stephenson-type, two arabic numerals 22, 23 and 32 are the numbers of pairing elements of their fixed and driving links and the capital letter F means that they have two closed five-link loops including the fixed and driving links.

2.1 Analysis of S22F mechanism

Let the kinematic constants of the S22F mechanism arranged in the stationary rectangular coordinate system O-xy as shown in Fig. 1-(a) be the following thirteen constants :

the distances between two pairing points (the lengths of links) $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7$;

the angles of intersection between two lines passing through pairing points of ternary links α_5, α_7 ;

the coordinates of fixed pairing points x_A, y_A, x_D, y_D .

Then, let the angular displacement of the oriented line passing through two

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pairing points be denoted by the Greek letter γ_k holding the subscript k of the length of the link Z_k in common. This angle is measured counterclockwise from the x-axis and is referred to as the angular displacement of the link.

Equating respectively the components of the x- and y-axes in both sides of two closed loop equations in a vector form :

$$\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BE_1} + \overrightarrow{E_1C_1} = \overrightarrow{DC_1} \dots\dots\dots (1)$$

$$\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BE_2} + \overrightarrow{E_2C_2} = \overrightarrow{DC_2} \dots\dots\dots (2)$$

we have the following basic equations of displacement analysis.

$$x_A - x_D + Z_1 \cos \gamma_1 + Z_2 \cos \gamma_2 + Z_3 \cos \gamma_3 = Z_4 \cos \gamma_4 \dots\dots\dots (3)$$

$$y_A - y_D + Z_1 \sin \gamma_1 + Z_2 \sin \gamma_2 + Z_3 \sin \gamma_3 = Z_4 \sin \gamma_4 \dots\dots\dots (4)$$

$$x_A - x_D + Z_1 \cos \gamma_1 + Z_5 \cos(\gamma_2 + \alpha_5) + Z_6 \cos \gamma_6 = Z_7 \cos(\gamma_4 + \alpha_7) \dots\dots\dots (5)$$

$$y_A - y_D + Z_1 \sin \gamma_1 + Z_5 \sin(\gamma_2 + \alpha_5) + Z_6 \sin \gamma_6 = Z_7 \sin(\gamma_4 + \alpha_7) \dots\dots\dots (6)$$

Eliminating γ_3 from Eqs.(3) and (4), we have

$$(a_1 + a_2 \cos \gamma_2 + a_3 \sin \gamma_2) \cos \gamma_4 + (a_4 + a_5 \cos \gamma_2 + a_6 \sin \gamma_2) \sin \gamma_4 = a_7 + a_8 \cos \gamma_2 + a_9 \sin \gamma_2 \dots\dots\dots (7)$$

where

$$\begin{aligned} a_1 &= 2Z_4(x_A - x_D + Z_1 \cos \gamma_1) \\ a_2 &= 2Z_2Z_4, \quad a_3 = 0 \\ a_4 &= 2Z_4(y_A - y_D + Z_1 \sin \gamma_1) \\ a_5 &= 0, \quad a_6 = 2Z_2Z_4 \\ a_7 &= (x_A - x_D + Z_1 \cos \gamma_1)^2 + (y_A - y_D + Z_1 \sin \gamma_1)^2 + Z_2^2 - Z_3^2 + Z_4^2 \\ a_8 &= 2Z_2(x_A - x_D + Z_1 \cos \gamma_1) \\ a_9 &= 2Z_2(y_A - y_D + Z_1 \sin \gamma_1) \end{aligned}$$

Moreover, eliminating γ_6 from Eqs.(5) and (6), we have

$$\begin{aligned} &(b_1 + b_2 \cos \gamma_2 + b_3 \sin \gamma_2) \cos \gamma_4 \\ &+ (b_4 + b_5 \cos \gamma_2 + b_6 \sin \gamma_2) \sin \gamma_4 \\ &= b_7 + b_8 \cos \gamma_2 + b_9 \sin \gamma_2 \dots\dots\dots (8) \end{aligned}$$

where

$$\begin{aligned} b_1 &= 2Z_7\{(x_A - x_D + Z_1 \cos \gamma_1) \cos \alpha_7 \\ &+ (y_A - y_D + Z_1 \sin \gamma_1) \sin \alpha_7\} \\ b_2 &= 2Z_3Z_7 \cos(\alpha_7 - \alpha_5) \end{aligned}$$

$$\begin{aligned} b_3 &= 2Z_5Z_7 \sin(\alpha_7 - \alpha_5) \\ b_4 &= -2Z_7\{(x_A - x_D + Z_1 \cos \gamma_1) \sin \alpha_7 \\ &- (y_A - y_D + Z_1 \sin \gamma_1) \cos \alpha_7\} \\ b_5 &= -2Z_5Z_7 \sin(\alpha_7 - \alpha_5) \\ b_6 &= 2Z_5Z_7 \cos(\alpha_7 - \alpha_5) \\ b_7 &= (x_A - x_D + Z_1 \cos \gamma_1)^2 + (y_A - y_D + Z_1 \sin \gamma_1)^2 \\ &+ Z_5^2 - Z_6^2 + Z_7^2 \\ b_8 &= 2Z_5\{(x_A - x_D + Z_1 \cos \gamma_1) \cos \alpha_5 \\ &+ (y_A - y_D + Z_1 \sin \gamma_1) \sin \alpha_5\} \\ b_9 &= -2Z_5\{(x_A - x_D + Z_1 \cos \gamma_1) \sin \alpha_5 \\ &- (y_A - y_D + Z_1 \sin \gamma_1) \cos \alpha_5\} \end{aligned}$$

As for coefficients of Eqs.(7) and (8), the equalities :

$$a_2 = a_8, \quad a_3 = -a_5, \quad b_2 = b_8, \quad b_3 = -b_5 \dots\dots\dots (9)$$

hold, so that the resultant equation obtained by eliminating γ_4 from Eqs.(7) and (8) is expressed as follows.

$$\begin{aligned} &f_1 \cos^3 \gamma_2 + f_2 \cos^2 \gamma_2 + f_3 \cos \gamma_2 + f_4 \\ &+ (f_5 \cos^2 \gamma_2 + f_6 \cos \gamma_2 + f_7) \sin \gamma_2 = 0 \end{aligned} \dots\dots\dots (10)$$

where

$$\begin{aligned} f_1 &= 2(C_2C_4 - C_3C_5 + C_5d_2 + C_4d_3) \\ f_2 &= C_2^2 - C_3^2 + C_5^2 + 2C_1C_4 + d_2^2 - d_3^2 + C_4^2 \\ &+ 2C_5d_1 - e_2^2 + e_3^2 \\ f_3 &= 2(C_1C_2 + C_3C_5 + d_1d_2 - C_4d_3 - e_1e_2) \\ f_4 &= C_1^2 + C_3^2 + d_1^2 + d_2^2 - e_1^2 - e_2^2 \\ f_5 &= 2(C_2C_3 + C_3C_4 - C_4d_2 + C_5d_3) \\ f_6 &= 2(C_1C_3 + C_2C_3 - C_4d_1 + d_2d_3 - e_2e_3) \\ f_7 &= 2(C_1C_3 + d_1d_3 - e_1e_3) \\ C_1 &= a_4b_7 - a_7b_4 + a_6b_9 - a_9b_6 \\ C_2 &= a_4b_8 - a_8b_4 + a_5b_7 - a_7b_5 \\ C_3 &= a_4b_9 - a_9b_4 + a_6b_7 - a_7b_8 \\ C_4 &= a_5b_8 - a_8b_5 - a_6b_9 + a_9b_6 \\ C_5 &= a_5b_9 - a_9b_5 + a_6b_8 - a_8b_6 \\ d_1 &= a_1b_7 - a_7b_1 - a_5b_9 + a_9b_5 \\ d_2 &= a_1b_8 - a_8b_1 + a_6b_7 - a_7b_6 \\ d_3 &= a_1b_9 - a_9b_1 - a_5b_7 + a_7b_5 \\ e_1 &= a_4b_1 - a_1b_4 - a_6b_5 + a_5b_8 \\ e_2 &= a_4b_6 - a_6b_4 + a_5b_1 - a_1b_5 \\ e_3 &= -a_4b_5 + a_5b_4 + a_6b_1 - a_1b_6 \end{aligned}$$

Equation (10) is a functional relation between the angular displacements of

Fig. 1 Stephenson six-link mechanisms which have two closed five-link loops including the fixed and driving links.

driving and coupler links, namely, γ_1 and γ_2 . Putting $t = \tan(\gamma_2/2)$, this equation becomes the following one of sixth order:

$$h_0 t^6 + h_1 t^5 + h_2 t^4 + h_3 t^3 + h_4 t^2 + h_5 t + h_6 = 0 \quad \dots\dots\dots(11)$$

where

$$\begin{aligned} h_0 &= -f_1 + f_2 - f_3 + f_4, & h_1 &= 2(f_5 - f_6 + f_7) \\ h_2 &= 3(f_1 + f_4) - f_2 - f_3, & h_3 &= -4(f_5 - f_7) \\ h_4 &= -3(f_1 - f_4) - f_2 + f_3, & h_5 &= 2(f_5 + f_6 + f_7) \\ h_6 &= f_1 + f_2 + f_3 + f_4 \end{aligned}$$

Therefore, for any given value of γ_1 , the values of γ_2 are determined as the real roots of Eq.(11). Then, the values of γ_4 , γ_3 and γ_6 for each value of γ_2 are calculated by using Eqs.(7), (8), Eqs.(3), (4) and Eqs.(5), (6), respectively.

The above-mentioned procedure which gives the only solutions corresponding to the possible composition loops of the S22F mechanism is remarkably simpler than conventional ones.

2.2 Analysis of S23F and S32F mechanisms

Let the pairing points, the kinematic constants, the angular displacements of links be denoted by the symbols shown in Figs. 1-(b) and (c). Then, in the cases of S23F and S32F mechanisms, the four equations obtained by equating respectively the components of the x- and y-axes in both sides of the two closed loop equations in a vector form can lead to an equation of sixth order with respect to the angular displacement of the coupler link γ_2 . Therefore, the displacement analysis of these mechanisms can be performed in the same procedure as in the section 2.1.

3. Discrimination of composition loops

The functional relations between displacements of links of a mechanism generally are prescribed by closed loop equations in a vector form. In such a case, with a set of the values of kinematic constants, more than two mechanisms which perform different motions may be composed. In this paper, these mechanisms, which are described as mutually in the state of geometrical inversion [4], are regarded as the same mechanisms which have different composition loops.

The geometrical inversion of the plane four-link mechanism and the plane six-link mechanisms whose fixed and driving links are included in a closed four-link loop is caused by the fact that unknown pairing points are determined as the points of intersection of two circles whose centers and radii are the already known pairing points and the lengths of adjacent links respectively. For example, in the Watt six-link mechanism (W22) whose fixed and driving links are binary links, there are four composition loops as shown in Fig. 2.

Therefore, the composition loops can be discriminated by means of relative displacement of the newly determined pairing points with respect to an oriented line passing through the known pairing points

In such a special case that two circles which determine unknown pairing points are in contact with each other, two composition loops in the state of geometrical inversion coincide. At this position, the driving link can move in only one way, because two links of either of component two-link chains lie on a straight line. In this paper, we call this position the limit position of motion.

3.1 Basic characteristics

The complex six-link mechanisms, which have two closed five-link loops including the fixed and driving links, may be disassembled into a component four-link chain either of whose ternary links is specified as the fixed link (hereinafter referred to briefly as the four-link mechanism) and a component two-link chain whose first and second links have fixed and free pairing points respectively. Inversely, in composition of these mechanisms, we must determine the points of intersection between the curve traced by the third pairing point on the coupler link of the four-link mechanism, namely, the coupler curve and the circle traced by the free pairing point of the two-link chain when the relative angular displacement of two links is prescribed in the case of the S22F mechanism and the angular displacement of the first link is prescribed in the cases of S23F and S32F mechanisms, respectively. Hereinafter, these circles are called end pairing-point circles.

Because the coupler curve of the plane four-link mechanism is a tricircular curve of sixth order [5], the number of points of intersection between the coupler curve and the end pairing-point circle is even less than six. The procedure of displacement analysis established in the chapter 2 is equivalent to the above-mentioned fact.

In such a special case that the coupler curve is in contact with the end

Fig. 2 Composition loops in the state of geometrical inversion.

pairing-point circle, the composition loops in the state of geometrical inversion also coincide. At this position, the driving link can move in only one way by similar reasonings as in the simple six-link mechanism. The tangent of the coupler curve at this position is perpendicular to the line passing through its point of contact and the instantaneous center of the coupler link with respect to the fixed link. Therefore, the limit position of motion is constructed such that the coupler point, the instantaneous center of the coupler link with respect to the fixed link and the center of the end pairing-point circle exist on a straight line [6]. The driving link moves continuously in only one domain separated by adjacent limit positions of motion obtained in the above-mentioned manner but can not transfer the other domains without taking apart some turning pairs. As an example, the four-link mechanism and the two-link chain of the S22F mechanism are shown in Fig. 3, and the points on the coupler curve at the limit positions of motion are shown by the mark $\bullet$ in Fig. 4.

The coupler curve of the four-link mechanism is a tricircular curve of sixth order. Hence, the discrimination of the composition loops by means of the domains on the coupler curve separated by points corresponding to limit positions of motion, which is necessary to classify the

Fig. 3 The four-link mechanism and the two link chain of the S22F mechanism.

Fig. 4 Points on the coupler curve of the four-link mechanism which correspond to limit positions of rotation of the driving link of the S22F mechanism (limit points).

curve of sixth order, is impracticable.

On the other hand, any two of angular displacements of the driving, coupler and driven links of the four-link mechanism are related mutually by two valued functions and the point on the angular displacement curve corresponds one to one to the point on the coupler curve. Moreover, in order to discriminate the domains on this curve separated by points corresponding to limit positions of motion, what is necessary are the classification of the angular displacement curve and the discrimination of the composition loops of the plane four-link mechanism.

Therefore, applying these motion characteristics, we can discriminate easily the composition loops of the complex six-link mechanism.

3.2 Limit points

We call the points on the coupler curve and moreover the angular displacement curve of the four-link mechanism corresponding one to one to the limit positions of motion of the driving link of the six-link mechanism the limit points.

3.2.1 In the case of the S22F mechanism

The limit points of the S22F mechanism are determined as such sets of values of angular displacements of the driving and driven links of the four-link mechanism as shown in Fig. 4 that the coupler point B, the instantaneous center of the coupler link with respect to the fixed link I and the center of the end pairing-point circle F exist on a straight line.

For the purpose of generalization of discussions, we redefine the symbols of pairing points, kinematic constants and angular displacements of the four-link mechanism as shown in Fig. 5, and denote the rectangular coordinate system whose origin and x-axis coincide with the pairing point A and the fixed link DA respectively by A-xy. Moreover, let (x_P, y_P) , (x_I, y_I) and (x_F, y_F) denote the coordinates of the coupler point P, the instantaneous center I and the center of the end pairing-point circle F in the coordinate system A-xy.

Elimination of θ from two resultant equations obtained by equating respectively the components of the x- and y-axes in both sides of the closed loop equation in a vector form of the four-link mechanism :

Fig. 5 The geometrical condition for limit points of the S22F mechanism.

$$\overline{DA} + \overline{AB} + \overline{BC} = \overline{DC} \quad (12)$$

yields the functional relation between ϕ and ψ as follows.

$$a_1 \cos \phi + a_2 \sin \phi + a_3 = 0 \quad (13)$$

where

$$a_1 = Z_1 Z_4 + Z_2 Z_3 \cos \phi$$

$$a_2 = Z_2 Z_3 \sin \phi$$

$$a_3 = -(Z_1^2 + Z_2^2 - Z_3^2 + Z_4^2)/2 - Z_1 Z_2 \cos \phi$$

Moreover, the coordinates of the coupler point P and the instantaneous center I are calculated by the following equations.

$$x_P = Z_2 \cos \phi - (Z_3/Z_1)\{Z_1 \cos \alpha + Z_2 \cos(\phi - \alpha) - Z_4 \cos(\psi - \alpha)\} \quad (14)$$

$$y_P = Z_2 \sin \phi + (Z_3/Z_1)\{Z_1 \sin \alpha - Z_2 \sin(\phi - \alpha) + Z_4 \sin(\psi - \alpha)\} \quad (15)$$

$$x_I = Z_1 \cos \phi \sin \psi / \sin(\phi - \psi) \quad (16)$$

$$y_I = Z_1 \sin \phi \sin \psi / \sin(\phi - \psi) \quad (17)$$

Therefore, expanding the resultant equation obtained by substitution of Eqs. (14)~(17) into the condition that three points (x_P, y_P) , (x_I, y_I) and (x_F, y_F) exist on a straight line:

$$\begin{vmatrix} x_P & y_P & 1 \\ x_I & y_I & 1 \\ x_F & y_F & 1 \end{vmatrix} = 0 \quad (18)$$

we have the following equation for the purpose of determination of limit points.

$$b_1 \cos^2 \psi + b_2 \cos \psi + b_3 + (b_4 \cos \psi + b_5) \sin \psi = 0 \quad (19)$$

where

$$b_1 = -(Z_4 Z_3/Z_1)\{(x_F + Z_1) \cos(\phi + \alpha) - y_F \sin(\phi + \alpha)\}$$

$$b_2 = x_F \sin \phi \{(Z_2 Z_3/Z_1) \sin(\phi - \alpha) - Z_2 \sin \phi - (Z_1 Z_3/Z_1) \sin \alpha\} - y_F \sin \phi \{(Z_2 Z_3/Z_1) \times \cos(\phi - \alpha) - Z_2 \cos \phi + (Z_1 Z_3/Z_1) \cos \alpha\}$$

$$b_3 = (Z_4 Z_3/Z_1)\{Z_1 \cos(\phi + \alpha) + (x_F \cos \alpha - y_F \sin \alpha) \cos \phi\}$$

$$b_4 = -(Z_4 Z_3/Z_1)\{(x_F + Z_1) \sin(\phi + \alpha) + y_F \cos(\phi + \alpha)\}$$

$$b_5 = (Z_1 Z_3/Z_1)\{\sin(\phi + \alpha) + Z_2 \sin \alpha + Z_1(x_F \sin \phi - y_F \cos \phi) - x_F \cos \phi \times \{(Z_2 Z_3/Z_1) \sin(\phi - \alpha) - Z_2 \sin \phi - (Z_1 Z_3/Z_1) \sin \alpha\} + y_F \cos \phi \{(Z_2 Z_3/Z_1) \times \cos(\phi - \alpha) - Z_2 \cos \phi + (Z_1 Z_3/Z_1) \cos \alpha\}$$

Putting $t = \tan(\psi/2)$, Eqs. (13) and (19) become respectively, a quadratic and a quartic equation with respect to t . Then the resultant equation obtained by elimination of t from the above-mentioned equations becomes an equation of eighth order with respect to $\cos \phi$ and $\sin \phi$. Hence, it is found that the number of limit points of the S22F mechanism is even less than sixteen.

3.2.2 In the cases of S23F and S32F mechanisms In the cases of S23F and S32F mechanisms shown in Figs. 1-(b) and (c), to be obtained the four-link mechanisms by fixing the ternary links $B_1 B_2$ and $D_2 D_1$ of the component four-link chains $B_2 E_2 E_1 B_1$ and $D_1 C_1 C_2 D_2$ respectively, the end pairing-point circles become circles

whose radii are the lengths of links Z_4 and Z_2 respectively and whose centers exist on the circles of the center A.

We redefine the symbols of pairing points, kinematic constants and angular displacements of the four-link mechanism and the two-link chain as shown in Fig. 6 and denote the rectangular coordinate system whose origin and x-axis coincide with the pairing point A and the fixed link DA respectively by A-xy. Then the limit points of S23F and S32F mechanisms are determined as such sets of values of angular displacements of the driving and driven links that the coupler point P, the instantaneous center I and the center F of the end pairing-point circle passing through the coupler point P exist on a straight line. Since the procedures of calculations of the limit points of S23F and S32F mechanisms are similar, we perform the discussions with respect to the latter.

The coordinates (x_P, y_P) of the coupler point P and (x_I, y_I) of the instantaneous center I in the coordinate system A-xy are calculated by using Eqs. (14), (15) and Eqs. (16), (17) respectively. Moreover, letting (x_G, y_G) denote the coordinates of the fixed pairing point G of the two-link chain, the coordinates of points of intersection of two circles whose radii and centers are Z_6 , (x_G, y_G) and Z_7 , (x_P, y_P) are calculated as the solution of the following simultaneous equation.

$$\begin{cases} (x - x_G)^2 + (y - y_G)^2 = Z_6^2 \\ (x - x_P)^2 + (y - y_P)^2 = Z_7^2 \end{cases} \quad (20)$$

The equation obtained by subtracting the second equation from the first one of Eq. (20):

$$(x_F - x_G)x + (y_F - y_G)y + (x_G^2 + y_G^2 - x_P^2 - y_P^2 - Z_6^2 + Z_7^2)/2 = 0 \quad (21)$$

is an equation of a straight line passing through the points of intersection of two circles. The coordinates of the point of intersection of the above-mentioned straight line and the one passing through two points I and P:

$$\begin{vmatrix} x & y & 1 \\ x_I & y_I & 1 \\ x_P & y_P & 1 \end{vmatrix} = 0 \quad (22)$$

are calculated by the following equations.

Fig. 6 The geometrical condition for limit points of the S32F mechanism.

$$x = -\{(x_1 y_P - y_1 x_P)(y_P - y_G) - (x_P^2 + y_P^2 - x_G^2 - y_G^2 + Z_s^2 - Z_0^2)(x_1 - x_P)/2\} / \{(x_1 - x_P) \times (x_P - x_G) + (y_1 - y_P)(y_P - y_G)\} \quad (23)$$

$$y = \{(x_1 y_P - y_1 x_P)(x_P - x_G) + (x_P^2 + y_P^2 - x_G^2 - y_G^2 + Z_s^2 - Z_0^2)(y_1 - y_P)/2\} / \{(x_1 - x_P) \times (x_P - x_G) + (y_1 - y_P)(y_P - y_G)\} \quad (24)$$

Therefore, multiplying both sides of the resultant equation obtained by substitution of Eqs. (23) and (24) into the first or second equation of Eq. (20) by

$$(x_1 - x_P)(x_P - x_G) + (y_1 - y_P)(y_P - y_G)$$

the required condition is expressed as follows.

$$\begin{aligned} & X_1^2(X_8 - 2X_5 + Z_s^2) + (X_8 + Z_s^2 - Z_0^2 - Z_1^2)^2 \\ & \times (X_7 - 2X_1 + X_8)/4 - (Z_s^2 - Z_0^2) \\ & \times (X_1 - X_3 + X_5 - X_8)^2 \\ & - X_2(X_8 + Z_s^2 - Z_0^2 - Z_1^2) \\ & \times (X_2 - X_4 + X_6) - \{2X_2X_6 + (X_3 - X_5) \\ & \times (X_8 + Z_s^2 - Z_0^2 - Z_1^2)\}(X_1 - X_3 + X_5 - X_8) = 0 \end{aligned} \quad (25)$$

where Z_8 is the distance between the origin A and the fixed pairing point G of the two-link chain, and X_i ($i=1 \sim 8$) are new variables defined by the following equations.

$$\left. \begin{aligned} X_1 &= x_1 x_P + y_1 y_P, & X_2 &= x_1 y_P - y_1 x_P \\ X_3 &= x_1 x_G + y_1 y_G, & X_4 &= x_1 y_G - y_1 x_G \\ X_5 &= x_P x_G + y_P y_G, & X_6 &= x_P y_G - y_P x_G \\ X_7 &= x_1^2 + y_1^2, & X_8 &= x_P^2 + y_P^2 \end{aligned} \right\} \quad (26)$$

The resultant equation obtained by substitution of Eqs. (14)~(17) into Eq. (25) is expressed as follows.

$$\begin{aligned} & b_1 \cos^5 \psi + b_2 \cos^4 \psi + b_3 \cos^3 \psi + b_4 \cos^2 \psi \\ & + b_5 \cos \psi + b_6 + (b_7 \cos^4 \psi + b_8 \cos^3 \psi \\ & + b_9 \cos^2 \psi + b_{10} \cos \psi + b_{11}) \sin \psi = 0 \end{aligned} \quad (27)$$

where b_i ($i=1 \sim 11$) are expressions of fifth order with respect to $\cos \phi$ and $\sin \phi$.

Then the resultant equation obtained by elimination of ψ from Eqs. (13) and (27) becomes an equation of twentieth order with respect to $\cos \phi$ and $\sin \phi$. Hence it is found that the numbers of limit points of S23F and S32F mechanisms are even less than forty.

3.3 Farthest and closest points

The complex six-link mechanism may be composed only when the distance between third pairing points of the fixed and coupler links of the four-link mechanism is less than the longest distance between free and fixed pairing points of the two-link chain, namely the sum Z_s of two lengths of links and is greater than the shortest distance between them, namely the absolute value Z_0 of the subtract of two lengths of links. That is, the points of intersection of the coupler curve of the four-link mechanism and the end pairing-point circle exist in a domain bounded by two concentric circles of radii Z_s and Z_0 whose center is the fixed pairing point of the two-link chain.

Hence, in this paper, we call the

points of intersection of the coupler curve and two concentric circles of radii Z_s and Z_0 , and moreover the points on the angular displacement curve corresponding to them, the farthest and closest points respectively.

Because the sets of values of angular displacements of the driving and driven links (ϕ, ψ) corresponding to the points of intersection of the coupler curve and the circle are determined as the real roots of the equation of sixth order, the numbers of the farthest and closest points are even less than six.

3.4 Procedure for discrimination

The angular displacements (ϕ, ψ) of the four-link mechanism take such values at two positions that the values of the relative angular displacement γ_r of the two-link chain (hereinafter referred to as the relative displacement angle) become mutually equal in magnitude but opposite in sign. The relative displacement angles of the two-link chains of S22F, S23F and S32F mechanisms measured counterclockwise from the first link to the second one as shown in Figs. 7-(a), (b) and (c) respectively are given by the following equations.

$$\gamma_r = \begin{cases} \gamma_1 - \alpha_0 & (\text{S22F mechanism}) \\ \gamma_4 - \alpha_0 + \pi & (\text{S23F mechanism}) \\ \gamma_2 - \gamma_1 & (\text{S32F mechanism}) \end{cases} \quad (28)$$

where

$$\begin{aligned} \alpha_0 &= \tan^{-1} \{(y_A - y_D)/(x_A - x_D)\} \\ \cos \alpha_0 &= (x_A - x_D)/Z_0, \quad \sin \alpha_0 = (y_A - y_D)/Z_0 \\ Z_0 &= \sqrt{(x_A - x_D)^2 + (y_A - y_D)^2} \end{aligned}$$

Hence, in the case of the S22F mechanism whose limit points are determined regardless of the relative displacement angle, two composition loops correspond to each domain on the angular displacement curve. On the other hand, in the cases of S23F and S32F mechanisms whose limit points are determined together with the value of the relative displacement angle at limit positions of motion, the composition loops correspond one to one to the domains on the angular displacement curve which are distinguished by the sign of

Fig. 7 Relative displacement angles of two-link chains.

$\sin\gamma r$. If there exists the farthest or closest point, at which γr becomes 0° or 180° respectively, on the angular displacement curve, two domains distinguished by the sign of $\sin\gamma r$ are connected at this position.

Hereupon, it is found that the numbers of the composition loops corresponding to the domains in which the driving link moves continuously are even less than 32, 40 and 40 in the cases of S22F, S23F and S32F mechanisms respectively. Then, even less than six domains among them correspond to the composition loops which are mutually in the state of geometrical inversion. In this paper, we call the domains of the angular displacement of the driving link, the end points of which correspond to adjacent limit positions of motion, the domains of motion, and we regard the composition loops corresponding to them as independent composition loops.

Now, the end points of domains of motion on the angular displacement curve are of three kinds as follows:

- (a) limit points,
- (b) limit point and farthest or closest point,
- (c) farthest and closest points.

The domains separated by the above-mentioned points may be modified for the purpose of correspondence one to one to the composition loops according to their signs of $\sin\gamma r$. The procedures for modification of domains and discrimination of composition loops are as follows:

(1) Discriminate the kind of the angular displacement curve.

(2) Determine the limit points and the farthest and closest points together with their signs of $\sin\gamma r$.

(3) Regard the angular displacement curve as a bicurve which has two parts corresponding to the plus and minus signs of $\sin\gamma r$ respectively, and assign the composition-loop numbers to the domains on the bicurve separated by the limit points and the farthest and closest points. Here the identical numbers are assigned to the domains which hold the farthest or closest point in common.

(4) Calculate the value of the angular displacement of each link of the complex six-link mechanism with a given value γ_1 of the angular displacement of the driving link. Then, determine the set of values (ϕ, ψ) of the angular displacement of the four-link mechanism and the value γr of the relative displacement angle of the two-link chain.

(5) Discriminate a domain on the bicurve, where the point (ϕ, ψ) distinguished by the sign of $\sin\gamma r$ is included, by means of the kinds and values of both end points of the domains, and identify the composition-loop number.

3.5 Models of domains of motion

Two values of ψ for any given value of ϕ may be discriminated by the sign of $\sin\delta$; the angle δ is measured counter-clockwise from the vector $\overrightarrow{DB}$ to $\overrightarrow{DC}$.

Therefore, plot the limit points, the farthest and closest points and the end points of the domain of ϕ (hereinafter

referred to as the limit points of the four-link mechanism) on four parallel number lines which are discriminated by the signs of $\sin\delta$ and $\sin\gamma r$. Then connect by a line two farthest or closest points and two limit points of the four-link mechanism whose coordinates are equal and have the same signs of $\sin\delta$ and $\sin\gamma r$ respectively, so that we can model the domains of motion as such on number lines as in (a)~(e) of Fig. 8 or combinations of some of them. Where $\bullet$, $\circ$ and $\ominus$ are the limit point, the farthest or closest point and the limit point of the four-link mechanism respectively. The set of signs of $(\sin\delta, \sin\gamma r)$ of the first, second, third and fourth number lines from the top are $(+, +)$, $(-, +)$, $(+, -)$, $(-, -)$ respectively. The arabic numerals above the domains are the assigned composition-loop numbers.

4. Examples of displacement analysis

Figures 9 and 10 are the coupler curve and the angular displacement curve of the four-link mechanism of the S22F mechanism whose kinematic constants are as follows.

$$Z_1 = Z_2 = 50, \quad Z_3 = Z_4 = 35.355\,339,$$

$$Z_5 = Z_6 = Z_7 = 70.710\,678, \quad \alpha_5 = 315^\circ, \quad \alpha_7 = 0,$$

$$(x_A, y_A) = (0, 0), \quad (x_D, y_D) = (-50, 0)$$

The marks $\bullet$, $\ominus$ and $\circ$ in Figs. 9 and 10 are the limit point, the limit point of one above another and the farthest point respectively. The number of independent composition loops is ten, and they corre-

Fig. 8 Models of domain on the angular displacement curve.

Fig. 9 Limit points and farthest points on the coupler curve of the four-link mechanism.

Fig. 10 Limit points and farthest points on the angular displacement curve.

spond each to the domains 1~10 on the angular displacement curve regarded as a bicurve. These domains correspond to the parts of the angular displacement curve of γ_2 to which the numbers 1~10 are given as shown in Fig. 11.

By using the method of displacement analysis established in this paper, the values of γ_2 for any given value of γ_1 are obtained as the real roots of the equation of sixth order and their composition loops are discriminated analytically. For example, the six values 275.96° , 219.49° , 162.46° , -52.46° , -109.49° , -165.96° of γ_2 for 110° of γ_1 are yielded together with the composition-loop numbers 1~6 respectively in numerical computations.

5. Conclusions

The displacement analysis and the discrimination of the composition loops of the Stephenson six-link mechanisms which have two closed five-link loops including the fixed and driving links have been developed. The results are summarized as follows.

(1) The displacement analysis of the mechanism, in which the four nonlinear

Fig. 11 The angular displacement curve of the S22F mechanism having composition-loop numbers.

equations obtained from two closed loop equations in a vector form with respect to the four angular displacements of the coupler and driven links are reduced to an equation of sixth order with respect to the one among them, gives the only solutions corresponding to the possible composition loops.

(2) The composition loops and the domains in which the driving link moves continuously can be discriminated by means of the domains on the angular displacement curve of the component four-link chain separated by the limit points and the farthest and closest points and the sign of the sine of the relative angular displacement of the component two link chain. The numbers of the independent composition loops corresponding to the domains of motion of the driving link of the six-link mechanism are even less than 32, 40 and 40 in the cases of S22F, S23F and S32F mechanisms respectively.

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