Vibrations of Conical Shells with Variable Thickness (continued)

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In this paper, free vibrations of truncated conical shells with variable thickness are studied by two methods. One is the method to use the stress-strain relations of the classical theory and the other is to use the three dimensional Hooke's law on the assumption that displacements are linear functions of thickness of shells. In both studies, Lagrangians of vibrations are obtained, equations of vibrations and boundary conditions are deduced from stationary conditions of Lagrangians and strict solutions are calculated in a form of series. These are, also, compared with equations and numerical results of the improved theory published by authors before.

Key Words: Vibration, Conical Shell, Variable Thickness Shell, Classical Theory, Improved Theory

1. Introduction

There are many papers on the vibration of a truncated conical shell. On its non-axisymmetric vibration, Federhofer⁽¹⁾ has formerly researched. Authors⁽²⁾ have also studied the vibration of a conical shell with variable thickness by the improved theory. In those days Irie and others⁽³⁾ have published their research, in which many literatures are cited. One of them is a paper of Penzes and Padovan⁽⁴⁾ but authors have failed to notice it in their previous paper. In Irie's paper a matrix method is explained to solve vibration problems of a conical shell with variable thickness by the classical theory.

To study the vibration of a shell, we generally employ the method based on the classical theory or the improved one. Using the spherical coordinates, we can denote any point on the middle surface of a conical shell by (r, ϕ, θ_0) , where θ_0 is the semi-vertex angle. We also denote the distance from the middle surface to any point in the shell by z .

In both classical and improved theories, it is assumed that displacements are linear with respect to z . In the classical theory, it is also assumed that the normal and tangential stresses along z are zero and stress-strain relations obey two-dimensional Hooke's law. In the improved theory, the rotatory inertia and shear deformation are considered, the normal stress along z is zero and the tangential stress is expressed by the shear coefficient. From an other point of view, the classical theory stands on the assumption that the possible displacements, which are functions of x and ϕ only, are three, while the improved theory assumes that they are five or six. To solve the vibration problem of thick shells more accurately perhaps we must use polynomials of z as displacements. Authors have thought that the

case of linear displacements of z should be studied before polynomials.

In this paper, authors have obtained the Lagrangian of a conical shell with variable thickness by the classical theory. They have deduced equations of vibration and boundary conditions from the Lagrangian and showed a rigorous solution in the form of series. Then, assuming five displacements of linear functions of z , which are similar to those in their previous paper of the improved theory, and using three-dimensional Hooke's law as stress-strain relations, they have obtained another Lagrangian in which terms of the rotatory inertia and the shear deformation are contained. From the Lagrangian, they have also deduced equations of vibration and boundary conditions, which are compared with those in the previous paper, and calculated the rigorous solution.

From the numerical calculations of both theories mentioned above, they have studied effects of boundary conditions and some parameters on the frequency, compared them with those obtained by the improved theory and showed the available range of the classical theory. For comparison of the three theories they have, in numerical calculations, considered relatively thicker conical shells than those appearing in other papers.

Authors guess that the improved theory is the most accurate of the three theories.

2. Lagrangian, Equations of Vibration and Boundary Conditions (Classical Theory)

Fig.1

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We develop the Lagrangian, equations of vibration and boundary conditions of free vibration of a conical shell by the classical theory. We denote the vertex of the middle surface of a conical shell by O , its axis by OC and any point on the middle surface by P (Fig.1(a)). The position of P is $(R\sin\theta, \theta, \phi)$ in the spherical coordinates, where R is a representative length and θ_0 is the semi-vertex angle. An axis z is the outward perpendicular to the middle surface. We denote displacements at P along the x , ϕ and z directions by $(u, v$ and $w)\sin pt$, where p and t are a circular frequency and time respectively. We assume that u, v and w are independent of z and functions of x and ϕ only. We define $y = \phi \sin \theta_0$ and $k = \cot \theta_0$. Denoting strains and stresses by $(\epsilon_x, \epsilon_y, \gamma_{xy})\sin pt$ and $(\sigma_x, \sigma_y, \tau_{xy})\sin pt$, we obtain the following relations by Love⁽⁵⁾

$$\left. \begin{aligned} \Phi_x &= \partial u / \partial x, \quad \Phi_y = u/x + (1/x) \partial v / \partial y - kw/x, \\ \Phi_z &= (1/x) \partial u / \partial y + \partial v / \partial x - v/x, \\ \Psi_x &= \partial^2 w / \partial x^2, \\ \Psi_y &= (1/x) \partial w / \partial x + (1/x^2) \partial^2 w / \partial y^2 \\ &\quad + (k/x^2) \partial v / \partial y, \\ \Psi_z &= (\partial / \partial x) \{ (1/x) \partial w / \partial y \} + k(\partial / \partial x)(v/x), \\ \epsilon_x &= (\Phi_x - z\Psi_x/R)/R, \\ \epsilon_y &= (\Phi_y - z\Psi_y/R)/R, \\ \gamma_{xy} &= (\Phi_z - 2z\Psi_z/R)/R \end{aligned} \right\} \dots\dots\dots(1)$$

$$\left. \begin{aligned} \sigma_x &= \{E/(1-\nu^2)\}(\epsilon_x + \nu\epsilon_y), \\ \sigma_y &= \{E/(1-\nu^2)\}(\nu\epsilon_x + \epsilon_y), \\ \tau_{xy} &= \{E/2(1+\nu)\}\gamma_{xy} \end{aligned} \right\} \dots\dots\dots(2)$$

In above equations, E and ν are Young's modulus and Poisson's ratio respectively.

We obtain the Lagrangian of a conical shell in vibration for a period as follows:

$$L = (1/2) \iiint (\gamma \rho^2/g) (u^2 + v^2 + w^2) - \sigma_x \epsilon_x - \sigma_y \epsilon_y - \tau_{xy} \gamma_{xy} R^2 x dx dy dz \dots\dots(3)$$

where γ/g is the density. We denote the representative thickness and the thickness at any point on the middle surface by h_0 and h respectively. We use the following notations: $H = h/h_0$, $\beta = 12R^2/h_0^2$, $D = Eh_0^3/12(1-\nu^2)$, $\alpha^4 = \gamma h_0^2 R^4/gD$ and $\alpha_0 (\alpha_0^4 = \alpha^4/\beta)$, where D is the flexural rigidity of plate and α and α_0 are nondimensional frequencies. Authors have shown many data of α_0 in their previous papers and so, in this paper, α_0 is calculated for numerical comparisons. Substituting Eqs.(1) and (2) into Eq.(3), we integrate it with respect to z from $(-h/2)$ to $(h/2)$,

$$\begin{aligned} -L/(D/R^2) &= \iint \{ -\alpha^4 H(u^2 + v^2 + w^2) \\ &\quad + \beta H(\Phi_x + \Phi_y)^2 + H^3(\Psi_x + \Psi_y)^2 \\ &\quad - 2(1-\nu)\beta H\Phi_x\Phi_y - 2(1-\nu)H^3\Psi_x\Psi_y \\ &\quad + (1/2)(1-\nu)\beta H\Phi_z^2 + 2(1-\nu)H^3\Psi_z^2 \} x dx dy \end{aligned} \dots\dots\dots(4)$$

From the partial integral of the first variation of Eq.(4), we obtain

$$\begin{aligned} \delta L/(D/R^2) &= \iint \{ E_1 \delta u + E_2 \delta v + E_3 \delta w \} x dx dy \\ &\quad - \int \{ T_1 \delta u + T_2 \delta v + T_3 \delta w \\ &\quad + T_4 \delta(\partial u / \partial x) \} x dx dy \\ &\quad - \int \{ T_5 \delta u + T_6 \delta v + T_7 \delta w \\ &\quad + T_8 \delta(\partial w / \partial y) \} x dx dy \\ &\quad - 2(1-\nu)H^3\Psi_x\delta w|_{x=0}^{x=h/2} = 0 \end{aligned} \dots\dots\dots(5)$$

where

$$\left. \begin{aligned} E_1 &= \alpha^4 Hxu + (\partial/\partial x)\{\beta Hx(\Phi_x + \nu\Phi_y)\} + (\partial/\partial y) \\ &\quad \times \{(1/2)(1-\nu)\beta H\Phi_x\} - \beta H(\nu\Phi_x + \Phi_y), \\ E_2 &= \alpha^4 Hxv + (\partial/\partial x)\{(1/2)(1-\nu)\beta Hx\Phi_x \\ &\quad + 2k(1-\nu)H^3\Psi_x\} + (\partial/\partial y)\{\beta H(\nu\Phi_x + \Phi_y) \\ &\quad + (kH^3/x)(\nu\Psi_x + \Psi_y)\} + (1/2)(1-\nu)\beta H\Phi_x \\ &\quad + 2(1-\nu)(k/x)H^3\Psi_x, \\ E_3 &= \alpha^4 Hxw - (\partial^2/\partial x^2)\{H^3x(\Psi_x + \nu\Psi_y)\} \\ &\quad - (\partial^2/\partial y^2)\{(H^3/x)(\nu\Psi_x + \Psi_y)\} \\ &\quad - (\partial^2/\partial x\partial y)\{2(1-\nu)H^3\Psi_x\} \\ &\quad + (\partial/\partial x)\{H^3(\nu\Psi_x + \Psi_y)\} \\ &\quad - (\partial/\partial y)\{(2/x)(1-\nu)H^3\Psi_x\} \\ &\quad + k\beta H(\nu\Phi_x + \Phi_y) \end{aligned} \right\} \dots\dots\dots(6)$$

$$\left. \begin{aligned} T_1 &= \beta Hx(\Phi_x + \nu\Phi_y), \\ T_2 &= (1/2)(1-\nu)\beta Hx\Phi_x + 2k(1-\nu)H^3\Psi_x, \\ T_3 &= -(\partial/\partial x)\{H^3x(\Psi_x + \nu\Psi_y)\} - (\partial/\partial y) \\ &\quad \times \{2(1-\nu)H^3\Psi_x\} + H^3(\nu\Psi_x + \Psi_y), \\ T_4 &= H^3x(\Psi_x + \nu\Psi_y), \\ T_5 &= (1/2)(1-\nu)\beta H\Phi_x, \\ T_6 &= \beta H(\nu\Phi_x + \Phi_y) + (kH^3/x)(\nu\Psi_x + \Psi_y), \\ T_7 &= -(\partial/\partial x)\{2(1-\nu)H^3\Psi_x\} - (\partial/\partial y) \\ &\quad \times \{(H^3/x)(\nu\Psi_x + \Psi_y)\} - (2/x)(1-\nu)H^3\Psi_x, \\ T_8 &= (H^3/x)(\nu\Psi_x + \Psi_y) \end{aligned} \right\} \dots\dots\dots(7)$$

In above equations, x_1, x_0 and y_1, y_0 are upper and lower limits of x and y respectively. From Eq.(5), it is clear that equations of vibration are $E_1=E_2=E_3=0$ and boundary conditions at the large and small ends are $T_1u=T_2v=T_3w=T_4\partial w/\partial x=0$. The third term in Eq.(5) is boundary conditions with respect to y or ϕ . In this paper, we treat the vibration of a truncated conical shell, that is the case that $\phi = -\pi \sim \pi$, and these conditions about ϕ are naturally satisfied. The fourth term in Eq.(5) is also zero. Putting $H=1$ in these equations, we obtain the Lagrangian, equations of vibration and boundary conditions of a conical shell with uniform thickness shown in the Handbook of Vibration Engineering⁽⁶⁾.

3. Solution

To compare the frequency obtained by the classical theory with one by the improved theory shown in the previous paper, we solve equations of a vibration of a conical shell, of which the thickness varies linearly as $H=x$. The number of nodal generators is 2. We use notations $K=n/\sin\theta_0$ and $P=q+2m$, where q is an unknown constant and $m=0, 1, 2, \dots$. We denote the summation from $m=0$ to infinity by Σ . We also denote unknown coefficients by A_m, B_m and C_m . We assume that displacements u, v and w are as follows;

$$\left. \begin{aligned} u &= (\Sigma A_m x^{P-0.5}) \sin n\phi, \\ v &= (\Sigma B_m x^{P-0.5}) \cos n\phi, \\ w &= (\Sigma C_m x^{P-0.5}) \sin n\phi \end{aligned} \right\} \dots\dots\dots(8)$$

Substituting Eq.(8) into $E_1=E_2=E_3=0$, we obtain

$$\left. \begin{aligned} \Sigma \{ P_1 A_m + P_2 B_m + P_3 C_m \\ + (\alpha^4/\beta) A_{m-1} \} x^{P-0.5} &= 0, \\ \Sigma \{ Q_1 A_m + Q_2 B_m + Q_3 C_m \\ + (\alpha^4/\beta) B_{m-1} \} x^{P-0.5} &= 0, \\ \Sigma \{ R_1 A_m + R_2 B_m + R_3 C_m \\ + (\alpha^4/\beta) C_{m-1} \} x^{P-0.5} &= 0 \end{aligned} \right\} \dots\dots\dots(9)$$

$$\begin{aligned}
P_1 &= P^2 - H_A, \\
P_2 &= K(-H_A P + H_C), \\
P_3 &= k(-\nu P + H_0), \\
Q_1 &= K(H_A P + H_C), \\
Q_2 &= H_A P^2 - H_F, \\
Q_3 &= kK(-H_C P^2 + H_H P - H_I), \\
R_1 &= k(\nu P + H_0), \\
R_2 &= kK(-H_C P^2 - H_H P - H_I), \\
R_3 &= -P^4/\beta + H_I P^2 - H_K, \\
H_A &= 5/4 - \nu + (1/2)(1 - \nu)K^2, \\
H_B &= (1/2)(1 + \nu), \\
H_C &= (7 - 5\nu)/4, \\
H_D &= 1 - \nu/2, \\
H_E &= (1/2)(1 - \nu)(1 + 4k^2/\beta), \\
H_F &= 9(1 - \nu)(1/8 + k^2/2\beta) + K^2(1 + k^2/\beta), \\
H_G &= (-2 + \nu)/\beta, \\
H_H &= (1 - 2\nu)/\beta, \\
H_I &= 1 + (20 - 21\nu)/4\beta + K^2/\beta, \\
H_J &= (3.5 - 3\nu + 2K^2)/\beta, \\
H_K &= 0.8125 - 0.75\nu \\
&\quad + (5.5 - 6\nu + K^2)K^2/\beta + k^2
\end{aligned}$$

(10)

By putting expressions in brackets in Eq.(9) as zero, we obtain three reduction formulae to describe A_m , B_m and C_m by A_{m-1} , B_{m-1} and C_{m-1} , where $m=1, 2, 3, \dots$. When $m=0$, $A_{m-1}=B_{m-1}=C_{m-1}=0$ and we can obtain a characteristic equation of 3×3 determinants from the condition that at least, one of A_0 , B_0 and C_0 is not zero. Each element of the determinant is coefficient of A_0 , B_0 or C_0 . Expanding the determinant, we obtain

$$\begin{aligned}
b_0 q^3 &+ b_1 q^2 + b_2 q + b_3 = 0, \\
b_0 &= H_E/\beta, \\
b_1 &= -H_A H_I + (K^2 H_B^2 - H_A H_E - H_F)/\beta \\
&\quad + k^2 K^2 H_A^2, \\
b_2 &= H_E H_K + H_A H_F/\beta + H_I (H_A H_E + H_F) \\
&\quad - \nu^2 k^2 H_E - K^2 (H_B^2/\beta + H_B^2 H_I) + k^2 K^2 \\
&\quad \times (-H_A H_B^2 - 2\nu H_A H_C + 2H_C H_I - H_B^2), \\
b_3 &= -H_A H_F H_I - H_K (H_A H_E + H_F) \\
&\quad + k^2 (\nu^2 H_F + H_B^2 H_E) + K^2 (H_B^2 H_K + H_B^2 H_I) \\
&\quad + k^2 K^2 (2H_C (H_A H_H + H_C H_G) \\
&\quad - 2\nu (H_B H_I + H_C H_H) + H_A H_B^2 \\
&\quad - 2H_A H_C H_I + H_I^2), \\
b_0 &= H_A H_F H_K - k^2 H_B^2 H_F - K^2 H_B^2 H_K \\
&\quad + k^2 K^2 (2H_C H_0 H_I - H_A H_I^2)
\end{aligned}$$

(11)

one of A_0 , B_0 and C_0 is arbitrary. After calculating q in Eq.(11) and putting $C_0=1$, we obtain A_0 and B_0 . Generally, q is a complex, and so A_0 , B_0 and A_m , B_m , C_m are also complex. We denote solutions corresponding to eight roots of Eq.(11) by $(u_1, v_1, w_1), \dots, (u_8, v_8, w_8)$. The solution to equations of vibration is that $u=a_1 u_1 + a_2 u_2 + \dots + a_8 u_8$, $v=a_1 v_1 + \dots + a_8 v_8$, $w=a_1 w_1 + \dots + a_8 w_8$ where a_1, a_8 are arbitrary constants. From eight boundary conditions at both ends, we can obtain a frequency equation and $a_1, a_2, \dots, a_8$. The frequency equation is 8×8 determinants consisting of coefficients of a_1, a_8 .

4. Approximation by Three-dimensional Hooke's Law

In the classical theory, we have, in chapt.2, assumed that the stress-strain relations obey two dimensional Hooke's law. In this chapter, we study the vibration of a conical shell on the assumption of three-dimensional Hooke's law and compare it with those obtained by the improved theory of the previous paper and by the classical one.

We denote any point in a conical shell by Q , as shown in Fig.1(b), in which OP is the middle surface. The length from P to Q is denoted by z . We denote displacements at Q along OP , PQ and ϕ by $(\bar{u}, \bar{w}, \bar{v}) \sin \theta$. As before, $y = \phi \sin \theta_0$. In general, displacements are functions of three coordinates (x, y, z) . But to compare with the classical theory in the previous chapter and the improved theory in the previous paper, we assume that $\bar{u} = u + z u_0/R$, $\bar{v} = v + z v_0/R$ and $\bar{w} = w$ and σ_θ (normal stress along z axis) = 0, where u , v , w , u_0 and v_0 are independent of z and dependent on x and y only. We also assume in this paper that stress-strain relations obey three-dimensional Hooke's law and that strain-displacement relations and the Lagrangian which contains terms of the rotatory inertia and the shear deformation are same as in the previous paper. We denote normal and tangential strains by ϵ_r and $\gamma_{r\theta}$, these stresses by σ_r and $\tau_{r\theta}$, Lamé's constants by λ and μ and the expansion by ϵ , respectively.

$$\left. \begin{aligned}
(\sigma_r, \sigma_\theta) &= \lambda' \epsilon + 2\mu(\epsilon_r, \epsilon_\theta), \\
(\tau_{r\theta}, \tau_{\theta\theta}, \tau_{rr}) &= \mu(\gamma_{r\theta}, \gamma_{\theta\theta}, \gamma_{rr})
\end{aligned} \right\} \dots\dots\dots (12)$$

The volumetric element is $(Rr+kz)Rdxdydz$ as shown in the previous paper. After expressing the Lagrangian of free vibration for a period by displacements, we integrate it with respect to z from $(-h/2)$ to $(h/2)$ and neglect terms of higher orders than h^5 . Calculating the first variation and the partial integral, we obtain

$$\begin{aligned}
\delta L/\mu R &= \iint (E' \delta u + E' \delta v + E' \delta w + E' \delta u_0 \\
&\quad + E' \delta v_0) dx dy - \int (T' \delta u + T' \delta v + T' \delta w \\
&\quad + M' \delta u_0 + M' \delta v_0) \frac{1}{R} dx dy \\
&\quad - \int (T' \delta u + T' \delta v + T' \delta w + M' \delta u_0 \\
&\quad + M' \delta v_0) \frac{1}{R} (\sin \theta_0) dx = 0 \dots\dots\dots (13)
\end{aligned}$$

$$\begin{aligned}
\lambda &= \lambda'/\mu = 2\nu/(1-2\nu), \quad \alpha'' = \gamma p^2 R^2/\mu g, \\
\bar{H} &= H'/\beta, \quad H' = H/x + k^2 \bar{H}/x^2, \\
T'_1 &= (2+\lambda)(Hx(\partial u/\partial x) + k\bar{H}(\partial u_0/\partial x)) \\
&\quad + \lambda H(u + \partial v/\partial y + kw), \\
T'_2 &= H(x(\partial v/\partial x) + \partial u/\partial y - v) + k\bar{H}(\partial v_0/\partial x), \\
T'_3 &= Hx(\partial w/\partial x + u_0), \\
M'_1 &= \bar{H}[(2+\lambda)(k(\partial u/\partial x) + x(\partial u_0/\partial x)) \\
&\quad + \lambda(u_0 + \partial v_0/\partial y)], \\
M'_2 &= \bar{H}(k(\partial v/\partial x) + x(\partial v_0/\partial x) + \partial u_0/\partial y - v_0), \\
T'_4 &= H(\partial v/\partial x) + H'(\partial u/\partial y - v) \\
&\quad - k\bar{H}(\partial u_0/\partial y - v_0)/x^2, \\
T'_5 &= \lambda H(\partial u/\partial x) + (2+\lambda)H'(u + \partial v/\partial y + kw) \\
&\quad - (2+\lambda)k\bar{H}(u_0 + \partial v_0/\partial y)/x^2, \\
T'_6 &= H'(-kv + \partial w/\partial y + xv_0), \\
M'_3 &= \bar{H}(k(v - \partial u/\partial y)/x^2 + \partial v_0/\partial x \\
&\quad + (\partial u_0/\partial y - v_0)/x), \\
M'_4 &= \bar{H}(-(2+\lambda)k(u + \partial v/\partial y + kw)/x^2 \\
&\quad + \lambda(\partial u_0/\partial x) + (2+\lambda)(u_0 + \partial v_0/\partial y)/x),
\end{aligned}$$

(14)

$$\begin{aligned}
E_1' &= \partial T_1' / \partial x + \partial T_1' / \partial y - T_1' \\
&\quad + \alpha'' (Hx + kHx_0), \\
E_2' &= \partial T_2' / \partial x + \partial T_2' / \partial y + T_2' + kT_2' \\
&\quad + \alpha'' (Hx + kHx_0), \\
E_3' &= \partial T_3' / \partial x + \partial T_3' / \partial y - kT_3' + \alpha'' Hx, \\
E_4' &= \partial M_1' / \partial x + \partial M_1' / \partial y - T_3' - M_1' \\
&\quad + \alpha'' \bar{H} (ku + xv_0), \\
E_5' &= \partial M_2' / \partial x + \partial M_2' / \partial y + M_2' - xT_3' \\
&\quad + \alpha'' \bar{H} (kv + xv_0)
\end{aligned}
\tag{15}$$

Equations of vibration are $E_1' = E_2' = E_3' = E_4' = E_5' = 0$ and boundary conditions at both ends are $T_1' = T_2' = T_3' = T_4' = T_5' = 0$. Eq. (15) is identical to $\bar{E}_1 \sim \bar{E}_5$ in Eq. (17) in the previous paper. We denote $T_1 R / h_0$ by T_1'' , where T_1 is given in Eq. (17) in the previous paper, and denote the shear coefficient by k' ,

$$\begin{aligned}
T_1'' &= Hx(\partial u / \partial x) + kH(\partial u_0 / \partial x) \\
&\quad + \nu H(u + \partial v / \partial y + kv) + \bar{H}[(1 - k'(1 - \nu)) \\
&\quad \times \{-2(\partial u / \partial x) + k(\partial v / \partial x + u_0)\} / x \\
&\quad - \nu(u + \partial v / \partial y + kv) / x^2]
\end{aligned}$$

Two expressions T_1' in Eq. (14) and T_1'' have a resemblance. The ratio of coefficients of the first and second terms in T_1' is such that $\lambda / (2 + \lambda) = \nu / (1 - \nu) = \nu + \nu^2 + \nu^3 + \dots$,

Considering T_1'' , we find that the corresponding ratio is ν and that T_1'' has an extra term of $\bar{H}$ []. Though full particulars will be given later, authors consider from the difference between T_1' and T_1'' that, the latter is more accurate than the former. When a shell becomes thin, $\bar{H}$ is very small and the difference becomes zero. Similar results can be introduced by comparing $T_2', T_3', \dots$ with those in the previous paper.

We consider the case that $H=x$ as in the previous chapter. We assume as follows:

$$\begin{aligned}
(ku, w, xv_0) \\
&= \sum [(A_n, C_n, D_n) x^{n-0.5}] \sin n\phi, \\
(kv, xv_0) &= \sum [(B_n, E_n) x^{n-0.5}] \cos n\phi
\end{aligned}
\tag{16}$$

where $P = q + 2m$. Substituting above equations into $E_1' = E_2' = \dots = E_5' = 0$, we obtain,

$$\begin{aligned}
&\sum [(G_1 P^2 + G_{12}) A_n / T + (G_{12} P + G_{13}) B_n / T \\
&\quad + (\lambda P + G_{14}) C_n \\
&\quad + (G_2 P^2 - G_2 P + G_{23}) D_n + G_{20} E_n \\
&\quad + \alpha'' (A_{n-1} / T + D_{n-1} / \beta)] x^{n-0.5} = 0, \\
&\sum [(-G_{12} P + G_{13}) A_n / T + (P^2 + G_{17}) B_n / T \\
&\quad + G_{23} C_n + G_{20} D_n \\
&\quad + (G_3 P^2 - G_3 P + G_{33}) E_n \\
&\quad + \alpha'' (B_{n-1} / T + E_{n-1} / \beta)] x^{n-0.5} = 0, \\
&\sum [(-\lambda P + G_{14}) A_n + G_{23} B_n + (P^2 \\
&\quad + G_{20}) C_n + (P + G_{21}) D_n \\
&\quad + G_{31} E_n + \alpha'' C_{n-1}] x^{n-0.5} = 0,
\end{aligned}$$

Table 1

Values of x_0 (length from vertex to small end) in the case that $n=2$, $\beta=4000$ and $\theta_0=30^\circ$. The large end of a conical shell is built-in. Values in the first column are α_0 . The first row indicates boundary conditions at the small end, where symbol $uTTw'$ is that $u=T_2=T_3=\partial w/\partial x=0$ and $TTTT$, $T_1=T_2=T_3=T_4=0$. In column (II), in the secondary vibration with built-in or free small end is given.

α_0	$uTTw'$	$TTTT$	(II)	$uTTT$	$uTTT$	$TuTT$	$TuTw'$	uTw'
2.2	0.85916	0.85938	0.88405	0.71988	0.88387	0.88264	0.88277	0.85705
2.0	0.84228	0.84266	0.86920	0.68242	0.86888	0.86637	0.86659	0.83838
1.8	0.81955	0.82026	0.84863	0.62715	0.84800	0.84182	0.84216	0.81115
1.7	0.80467	0.80571	0.83476	0.58725	0.83382	0.82280	0.82322	0.79074
1.6	0.78611	0.78770	0.81693	0.53433	0.81543	0.79259	0.79305	0.76035
1.5	0.76195	0.76458	0.79287	0.46552	0.79021	0.72625	0.72626	0.70541
1.45	0.74676	0.75027	0.77730	0.42455	0.77380	0.66116	0.66222	0.65902
1.40	0.72862	0.73344	0.75840	0.37806	0.75307	0.60320	0.60577	0.60232
1.35	0.70853	0.71339	0.73522	0.32352	0.72726	0.55620	0.55969	0.55564
α_0	uTw'	(II)	$TuTw'$	$TuTw'$	uTw'	uTw'	$TTuT$	$TTuT$
2.2	0.73752	0.57725	0.73754	0.77231	0.77231	0.77753	0.77802	0.74383
2.0	0.70883	0.52926	0.70888	0.74555	0.74555	0.75230	0.75290	0.71481
1.8	0.67132	0.46269	0.67147	0.70955	0.70951	0.71973	0.72057	0.67970
1.7	0.64754	0.41754	0.64781	0.68608	0.68596	0.69923	0.70027	0.65747
1.6	0.61880	0.35989	0.61929	0.65701	0.65670	0.67453	0.67585	0.63090
1.5	0.58304	0.28567	0.58393	0.61986	0.61908	0.64380	0.64553	0.59811
1.45	0.56156	0.24157	0.56275	0.59711	0.59592	0.62524	0.62724	0.57844
1.40	0.53892	0.19276	0.53852	0.57074	0.56897	0.60377	0.60609	0.55580
1.35	0.50836	0.13872	0.51050	0.53998	0.53744	0.57850	0.58119	0.52936

$$\left. \begin{aligned} &\Sigma((G_2P^2+G_2P+G_{15})A_m+G_{20}B_m \\ &+(-P+G_{31})C_m+(G_2P^2+G_{15})D_m \\ &+(G_{33}P+G_{36})E_m+(\alpha'/\beta)(A_{m-1} \\ &+D_{m-1}))x^{P-0.5}=0, \\ &\Sigma(G_{20}A_m+(G_2P^2+G_2P+G_{15})B_m \\ &+G_{31}C_m+(-G_{33}P+G_{36})D_m+(G_2P^2 \\ &+G_{33})E_m+(\alpha'/\beta)(B_{m-1}+E_{m-1}))x^{P-0.5}=0 \end{aligned} \right\} \dots\dots\dots(17)$$

where $T=\cot^2\theta_0$. Comparing the symmetrical terms about the diagonal in the above equations, we find that coefficients of P have opposite signs to each others, as in the previous chapter and paper. These constants $G_1 \sim G_{59}$ (there are eleven missing numbers) consist of $n, \lambda, \beta, \sin\theta_0$ and $\cot\theta_0$, and they are given in Appendix.

As in the previous paper, we can obtain simultaneous linear equations of A_m, B_m, C_m, D_m and E_m by equating expressions in brackets [] in Eq.(17) to zero. From these equations, we can obtain $A_m \sim B_m$, when $m=1,2,3,\dots$. When $m=0$, they are homogeneous simultaneous linear equations with five unknown. From the determinant of their coefficient matrix (the characteristic equation), we obtain the value

of q (in general, complex) and then assuming that $C_0=1$, we can calculate A_0, B_0, D_0 and E_0 . The characteristic equation is of five orders about q^2 , solutions of equations of vibration are 10 and boundary conditions at both ends are also 10, as seen in Eq.(14). Coefficients of q^2, q^4 and q^6 in the characteristic equation are so complicated that authors have calculated them by the method shown in Appendix 2 in the previous paper. Moments and forces in boundary conditions are expanded as follows:

Fig.4 Comparison of three theories

Fig.2 Relation between frequency and β (a standard of thickness) and comparison of the classical, improved and approximate three-dimensional theories. (both built-in ends)

Fig.3 The same as in Fig.2 (Built-in at large end and free at small end)

Fig.5 Relation between frequency and n (wave number along periphery)

Fig.6 The same as Fig.5

$$\begin{aligned}
 X &= (x h_0 / R) \times (\sin n\phi \text{ または } \cos n\phi) \\
 kT_1/X &= \sum \{ [G_1(P-0.5) + \lambda] A_m + G_{17} B_m \\
 &\quad + \lambda T C_m + G_2 T(P-1.5) D_m \} x^{P-0.5}, \\
 kT_2/X &= \sum \{ G_6 A_m + (P-1.5) B_m \\
 &\quad + G_8 T(P-1.5) E_m \} x^{P-0.5}, \\
 T_3/X &= \sum \{ (P-0.5) C_m + D_m \} x^{P-0.5}, \\
 M_1/xX &= \sum \{ G_3(P-0.5) A_m + (G_2(P-0.5) \\
 &\quad + G_9) D_m + G_{10} E_m \} x^{P-0.5}, \\
 M_2/xX &= \sum \{ G_4(P-0.5) B_m + G_{11} D_m \\
 &\quad + G_5(P-2.5) E_m \} x^{P-0.5}
 \end{aligned}
 \tag{18}$$

5. Numerical Calculations

In this chapter, solutions in the classical and the approximate three-dimensional theories are calculated numerically. The approximate three-dimensional theory is an abbreviated form of the theory in which, it is assumed that displacements are linear expressions of z and stress-strain relations obey three dimensional Hooke's law. The representative length is one along a generator from the vertex to the large and ($x_1=1$). For values of n , β and θ_0 , roots of the characteristic equation have been calculated where Poisson's ratio is $\nu=0.3$. For appropriate values of α , Coefficients A_m , B_m etc are calculated. It is assumed in this chapter, that the large end is built-in. For various boundary conditions at the small end, the frequency equation is calculated to obtain the value of x_0 (the length from vertex to the small end). Number of each coefficients A_m etc. enough to converge is 20~40 terms in the classical theory and 40~60 terms in the approximate three-dimensional one, for values of n , β , θ_0 and α_0 used in this chapter. The maximum terms in the series mentioned above appears at $m=1 \sim 10$ and the ratio of the last

term to that is smaller than $10^{-4} \sim 10^{-5}$. The convergency of the approximate three-dimensional theory is similar to that of the improved theory. The convergency of the classical theory is better than that of the formers. The capacity of machines has expelled authors to reduce complex calculations to real ones on paper before the operation of computer. In the calculation of solutions in the classical theory, they have used OKITAC system 50 in Yamagata University and ACOS system 1000 in Tohoku University for the approximate three-dimensional theory. In the following discussion are shown vibration modes and effects of boundary conditions and parameters (β , n , θ_0) on the frequency and convergency.

It is necessary in the classical theory to satisfy 16 boundary conditions at each end. In Table 1, the relation between α_0 and x_0 of a conical shell with a built-in large end, is given for 16 boundary conditions at the small end, where $n=2$, $\beta=4000$ and $\theta_0=30^\circ$. Frequency for the condition $w=0$ differs widely from the one $T_3=0$, as in the improved theory. The condition $v=0$ or $T_2=0$, has some effect on the frequency, but u and T_1 have little effect. For reference, the second frequencies in the case that the smaller end is built-in or free are shown on column (II) in the table. Though numerical results are omitted in this paper, the effect of boundary conditions on the frequency by the approximate three-dimensional theory is identical to that in the improved theory. The difference of boundary conditions has little effect on the convergency. The convergency becomes worse as the wave number along a generator becomes larger.

In Figs. 2 and 3, are shown effects of β on frequencies and the difference of the classical, improved or approximate three-dimensional theories, where $n=2$ and $\theta_0=30^\circ$. Six curves on the right denote the fundamental

Fig. 7 Relation between frequency and θ_0 (semi-vertex angle)

Fig. 8 Vibration modes

Fig. 9 Vibration modes

frequency and left six do the second ones. In the classical theory, $\beta=10^4$ may be the upper limit of convergency of solutions obtained in the preceding chapter and it may be 10^3 in the approximate three-dimensional theory as in the improved one. When β becomes smaller, the convergency becomes better. The frequency for large β is smaller than the one for small β in the most part of various boundary conditions, n and θ_0 . Frequencies obtained by the approximate three-dimensional theory come always between those of the classical and improved ones. In the approximate three-dimensional theory, not only the relation between frequencies and boundary conditions, as mentioned above, but also effects of n , β and θ_0 on frequencies are similar to those in the improved theory and so their details are omitted in this paper.

In Fig.5 is made a comparison of frequencies in the cases that $n=2$ and 5 in the classical theory, where I, II and III indicate the fundamental, the second and the third mode ($\beta=4000$, $\theta_0=40^\circ$). In the second and the third modes, solutions make slow convergency and so frequency ranges obtained are restricted. Not only in the case that a conical shell is built-in or free at the small end, as shown in the graph, but also in the case of other boundary conditions, α_0 becomes large with n , when $\beta < 10^4$. Another example is Fig.6. As n becomes large, the convergency becomes poor. The effect of n on the frequency is complicated and is influenced by α_0 and θ_0 . Generally speaking, frequencies when $n=3$ and 4 are smaller than when $n=2$ for large β . For example, α_0 of a conical shell of which $x_0=0.6$, $\beta=10^4$: $\theta_0=30^\circ$ and $n=2, 3, 4$ and 5, are 1.378, 1.337, 1.371 and 1.473 respectively.

In Fig.7 is shown the relation between θ_0 and α_0 in the classical theory. As θ_0 becomes small, α_0 becomes large rapidly. The convergency is good when $\theta_0=20^\circ \sim 60^\circ$, but for a larger or smaller θ_0 , it is poor. Due to careless programming, authors have not been able to calculate the frequency in the case that $\theta_0=90^\circ$ ($k \rightarrow \infty$).

Some examples of vibration modes are given in Figs.8 and 9. They are obtained by the classical theory, where $n=2$, $\beta=4000$ and $\alpha_0=1.8$.

6. Conclusions

Comparing frequencies obtained by the classical and the approximate three-dimensional theories with one by the improved theory of the previous paper, we find that the frequency of the classical theory is always the highest and one of the improved theory is the lowest. Authors guess from some experiences that the theory by which a lower frequency is obtained, is a more accurate one. Though they make every effort to verify it in future, they think at present that the improved theory is the most accurate of the three theories. Of course, it may be possible to lower the frequency by varying the shear coefficient but it seems that decrement is very small (an example in the case of plate vibration is shown in author's paper⁽⁷⁾). When vibration problems are treated under the assumption that displacements are linear with respect to z

(measurement of thickness), authors think that Mirsky's method is the most reasonable. He has determined the shear coefficient by wave theory to obtain new stress-strain relations. Under the assumption of z , it may produce more accurate results to use three-dimensional Hooke's law. Before long, research of three-dimensional theory will be done, and the study of the approximate three-dimensional theory in this paper will be found useful for comparison.

Other results are as follows:

(1) It seems from the comparison of the classical and the improved theories that the frequency and mode obtained by the classical theory are sufficiently accurate when $\beta > 10^3$ (β is a standard of shell thickness).

(2) The frequency increases with n , the wave number along periphery. But when $\beta > 10^4$, frequencies in the case that $n=3$ and 4 are smaller than in other cases, as Irie and others have suggested.

(3) The frequency becomes small as β or the semi-vertex angle becomes large.

(4) In the classical theory, there are 16 boundary conditions to each end. The condition that the displacement or the force perpendicular to the middle surface is zero has an important effect on frequency, while the condition along a generator has little effect.

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Appendix ($G_1 \sim G_{33}$)

$$\begin{aligned} G_1 &= 2 + \lambda, & G_2 &= G_1/\beta, & G_3 &= \lambda/\beta, \\ G_4 &= 1 + T/\beta, & G_5 &= G_1 G_6, & G_6 &= 1/\beta, \\ G_7 &= -n\lambda/\sin \theta_0, & G_8 &= -G_7/\lambda, & G_9 &= G_1 - G_7, \\ G_{10} &= -G_7 G_6, & G_{11} &= G_8 G_6, & G_{12} &= G_7 G_6, \\ G_{13} &= -G_7 G_6, & G_{14} &= G_7 G_6, & G_{15} &= \lambda - G_1 - G_8 G_{12}, \\ G_{16} &= G_7 - G_8, & G_{17} &= G_7 + G_1 G_6 - G_{12}, \\ G_{18} &= \lambda - G_8, & G_{19} &= G_8 G_{11}, & G_{20} &= -G_{11} - G_8 G_6, \\ G_{21} &= 1 + (1/\beta) \sin^2 \theta_0, & G_{22} &= G_8 + G_{12} + G_1 G_6, \\ G_{23} &= -1 + G_8 G_{12} - G_1/\sin^2 \theta_0, & G_{24} &= G_{12} + G_8 G_6, \\ G_{25} &= -G_8 + G_{11} + G_1 G_{11}, & G_{26} &= G_1 + G_8 G_{11}, \\ G_{27} &= -G_8 G_{12} - G_1 T, & G_{28} &= 1 + G_1 T, \\ G_{29} &= -G_1 G_8 + G_{11} T, & G_{30} &= G_1 T, \\ G_{31} &= 3G_7 - G_1 + G_8, \\ G_{32} &= -1 - G_1 + G_1 + 2G_8 - G_8 G_{11}, & G_{33} &= G_{12} - G_8 G_6, \\ G_{34} &= 2G_{12} + G_{11} + 2G_8 G_6, & G_{35} &= G_{11} + G_8 G_{11}, \\ G_{36} &= 3G_{11} - G_8 G_6, & G_{37} &= -3G_8 - G_{11} - G_1 G_{11}, \\ G_{38} &= G_{11} - G_1/4, & G_{39} &= G_{11} - G_{11}/2, \\ G_{40} &= G_{11} - \lambda/2, & G_{41} &= G_{11} + G_1/4, \\ G_{42} &= G_{11} - 1/4, & G_{43} &= G_{11} + G_1/4, \\ G_{44} &= G_{11} - 1/4, & G_{45} &= G_{11} - 1/2, \\ G_{46} &= G_{11} - G_{11}/2 + 3G_1/4, & G_{47} &= G_{11} - G_{11}/2, \\ G_{48} &= G_{11} - G_1/4 \end{aligned}$$

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