The Influence of a Rise Time of Longitudinal Impact on the Propagation of Elastic Waves in a Bar

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The purpose of this paper is to examine not only qualitatively but also quantitatively the influence of a rise time of longitudinal impact on the propagation of elastic waves in a bar. The adequacy of application of Love's theory for the propagation of longitudinal elastic waves to this problem is discussed by means of the simultaneous measuring of both axial and radial strains at the same stations along a bar. Theoretical results based on the Love theory predicted that, as the rise time of impact velocity increases, the effect of lateral inertia on longitudinal waves decreases and then the oscillatory waves of it become negligibly small. Experimental results for a steel bar agreed well with the theoretical results. Moreover, as an example of application, the experimental measurement of the dynamic yield stress by using a stress bar was analyzed.


1. Introduction

Although there have been numerous investigations on the propagation of longitudinal elastic waves in a cylindrical bar subjected to a step loading on the end, little investigation is examined for the effect of a rise time of impact loading(1)(2). Thus, the influence of a rise time of longitudinal impact on the propagation of elastic waves along a bar may be known qualitatively but not quantitatively.

The Hopkinson bar technique, which is one of the experimental methods to measure a dynamic load(3), utilizes one-dimensional elastic waves in a bar. In this experimental measurement, at a low rate of loading the one-dimensional waves may be observed on a stress bar, but at a rapid loading the influences of the lateral inertia effect of the bar and the higher modes of wave motion on one-dimensional waves may appear. In view of such a measurement of a dynamic load, it is necessary to know the quantitative relation between the elastic waves at various locations along a bar and the rise time of impact loading or impact velocity.

In this paper, first, we examine the adequacy of application of Love's theory on the propagation of longitudinal elastic waves to this problem by means of the simultaneous measuring of both axial and radial strains at the same stations along a bar. Second, we shall show quantitatively as well as qualitatively the influence of a rise time of impact on the wave propagation in a bar by both experimental and theoretical examinations of relation between elastic waves at various stations and a rise time of impact. Sequentially, as an example of application, discussion on the experimental measurement of dynamic yield stress by using a stress bar was made.

2. Theoretical Consideration

Let us consider the propagation of longitudinal elastic waves in a cylindrical bar subjected to an impact loading on one end. The first exact solution to a problem of this type was reported by Skalak(4). Equation of motion is that, in spite of the fact that the effects of the higher modes are present in the exact theory of waves in rods, the effect of the higher modes in far-field is not so large but the lateral inertia effect is dominant. In fact, Love's theory on wave propagation in a bar, which incorporates lateral inertia effects in elementary theory, does closely approximates the wave of the lowest mode for a limited region(5). Therefore, if attention is focused on a wave with a long wavelength of the far-field, the Love theory might be utilized instead of more excellent approximate theories such as the Mindlin-Herrmann theory and the Mindlin-McNiven theory(6).

2.1 Love's assumption

Love's assumption is such that the
displacement in the radial displacement $v$ is proportional to the radial coordinate $r$, measured from a centroidal axis, and to the axial strain $\partial u/\partial z$; that is

$$v = -\nu \frac{\partial u}{\partial r}$$ .............................. (1)

where $u$ is the axial displacement in the longitudinal $x$-direction and $\nu$ is Poisson's ratio. By applying Hamilton's principle to expressions of the kinetic and potential energies, the following Love's equation for motion is obtained:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{E\rho} \frac{\partial^2 u}{\partial r^2} + \frac{1}{c_0^2} \frac{\partial^2 u}{\partial z^2} = 0$$ .............................. (2)

where $c_0 = \sqrt{E/\rho}$ is the bar velocity, $E$ is Young's modulus, $\rho$ is the mass density of materials, and $k$ is the polar radius of gyration of the cross section. Now, let us assume that Love's equation is applicable to the boundary condition given by an impact velocity, though in this theory the natural boundary conditions at the end of a bar are given by stress.

2.2 Elastic waves under an impact with a rise time

Usually the end of a bar is considered to be subjected to a step loading in stress or to a step change in velocity. However, even though the experimental input is sudden, the step input used with the theory can not be realized experimentally. In other words, there must be more or less a finite rise time in the experimental impact. We suppose that an impact velocity on one end of a bar with length $l$ and radius $r$ increases exponentially with time while the other end is free from stress. Thus, the boundary conditions assumed are given as follows:

$$\left. \frac{\partial u}{\partial r} \right|_{r=a} = u_0 \left[ 1 - \exp\left( -\frac{t}{t_0} \right) \right]$$ .............................. (3)

where $u_0$ is a constant velocity and $t_0$ is the rise time of impact given on the impacted end. Applying the Laplace transform to Eq. (2) and (3), under the initial conditions, $u(x,0) = \partial u(x,0)/\partial t = 0$, the axial displacement in the transformed domain is obtained as follows:

$$u(x,p) = \frac{u_0}{p^2(1 + ip)} \times \frac{\cosh(p-l) - \sqrt{(u_0^2)p^2 + c_0^2}}{\cosh(p/l)}$$ .............................. (4)

where $p$ is the Laplace transform parameter. By carrying out the inverse Laplace transform

$$u(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u(x,p) e^{pt} \, dp$$ .............................. (5)

the solution in original domain can be found. The image function of Eq. (4) has the following poles

$$p = \pm \left\{ \frac{1}{2} \left[ (2n-1)\sqrt{8l^2 + \left( (2n-1)\pi - u_0 \right)^2} \right] \right\} (n=1,2,3,\ldots)$$ .............................. (6)

so that the inverse Laplace transform of Eq. (5) reduces to the evaluation of residues around poles. Then the axial displacement is obtained as follows:

$$u(x,t) = u_0 \left[ \frac{1}{t_0} \right] \times \exp\left( -\frac{t}{t_0} \right) - \frac{u_0}{c_0 \sqrt{2l}} \times \left[ \frac{1}{\sqrt{1 + (\gamma c_0 t_0^2)} \sinh(\gamma c_0 t_0 - \theta)} \right]$$ .............................. (7)

where

$$\gamma = 1/\sqrt{(u_0^2) + \left( c_0 t_0^2 \right)^2}$$

$$\theta = \tan^{-1}(\gamma c_0 t_0)$$

Consequently we obtain the solution for the axial strain.

$$\frac{\partial u}{\partial x}(l,t) = -\frac{8u_0}{c_0 \sqrt{(2n-1)\pi} \gamma c_0} \times \frac{\cosh\left( \frac{(2n-1)\pi}{2l} \right) \sinh(\gamma c_0 t_0 - \theta)}{\sqrt{1 + (\gamma c_0 t_0^2)}}$$

$$+ \frac{c_0 t_0}{c_0 \gamma c_0 t_0} \sinh(\gamma c_0 t_0 - \theta) \exp\left( -\frac{l}{t_0} \right)$$ .............................. (8)

Eq. (8) becomes the solution to a step impact as $t_0 = 0$ (7), while it does the elementary solution of one-dimensional theory as $k = 0$.

2.3 Theoretical prediction

Fig. 1 shows the time histories of the axial strain for different rise-times of impact at the location $x = 50d$ from the impacted end of a bar having diameter $d$ = 2 cm and length $l$ = 400 cm. A steel bar with $\rho = 0.29$ and $c_0 = 5.17 \times 10^2$ m/sec is supposed. It is clear from Fig. 1 that, as the rise time of impact

![Time histories of longitudinal strain for different rise-times of impact](image)

Fig. 1 Time histories of longitudinal strain for different rise-times of impact.
increases, the effect of lateral inertia on the longitudinal waves decreases and then the oscillatory waves become negligibly small.

Axial strains vs. time at various stations along a bar in the case of \( t = 5 \mu \text{sec} \) are shown in Fig. 2. In this stage, let us remark that, as pointed out by Willi and McComy\(^{(20)}\), with the distance of the wave propagation the period of the oscillation and the rise time of the initial portion of the response become longer in the case of a step impact. Besides these behaviors, in the case of the elastic waves under the impact with a rise time, as shown in Fig. 2, it can be found that the vibratory amplitudes grow with the distance of wave propagation.

3. Experiment

3.1 Experimental verification of Love's assumption

The adequacy of Love's theory has been discussed by a comparison with the exact theory of Pochhammer-Chree with use of dispersion curves; otherwise, by comparing the slopes of the initial portion and the first few oscillations of longitudinal waves predicted by Love's theory with experimental data\(^{(20)}\). As mentioned previously, we shall examine directly Love's assumption, \( \partial \sigma / \partial x = -\rho \partial u / \partial t \) by means of the simultaneous measurements of both axial and radial strain responses at the same stations of a bar.

In the present study, the classical Hopkinson bar apparatus is used. The test bar is mild steel (SGD-D), 400 cm long and has a diameter of 2 cm. The impacting bar has the same diameter but 100 cm in length. At several stations along the test bar, two semiconductor strain gages are cemented in the axial and circumferential directions, respectively. They are cemented on opposite end of a diameter of cross section of the bar in order to eliminate any possible antisymmetric components of strains. The axial and radial strains were measured simultaneously by the strain gages located at the same distance from the impacted end. They were amplified and recorded by a transient-memory and then displayed on an oscilloscope.

Typical results of time histories of strains are demonstrated in Fig. 3, which is obtained by measuring simultaneously the axial and radial strains at the gage stations \( x = 10 \) and 200 cm from the impacted end of the bar. In each figure the upper trace is the radial strain \( \varepsilon_r \) and the lower trace the axial strain \( \varepsilon_a \). The velocity of the impacting bar is about 2 m/sec. These curves are characterized by a rapid rise time and an oscillating wave. The strain waves \( \varepsilon_a \) and \( \varepsilon_r \) at the station close to the end, namely, \( x = 5d(100) \) cm, show irregular oscillations due to the higher modes of wave motion. The comparison of their oscillations indicates poor propriety of Love's assumption. In contrast to this, the strains in the far-field, \( x = 100d(200) \) cm, clearly indicate the adequacy of Love's assumption. A series of experimental measurements at various stations showed that Love's theory might be applicable to the farther distance of about ten diameters from the impacted end, even for the rapid rise time of impact.

3.2 Influence of a rise time on waves

Fig. 4 presents a comparison of the experimental results at \( x = 100d \) shown in Fig. 3 with the theoretical results. The calculated results with a rise time of 3 \( \mu \text{sec} \) agree well with the experimental results. In this connection it is pointed out that the experimental results by Miklowits\(^{(19)}\) quoted in many references, have also a rise time of 3 \( \mu \text{sec} \).
Fig. 4 Comparison of experimental and theoretical strains at $x = 100d$.

Fig. 5 shows the time variation of the axial and radial strains for three different rise-times of impact, measured at the station of $x = 150d$. In each figure the upper trace is the radial strain and the lower trace is the axial strain. The rise time may be longer in the order of (a), (b), (c). The rise time was controlled by changing slightly the degrees of roughness and flatness of the collision surface of the impacting bar.

Fig. 6 shows the theoretical waves at $x = 150d$ for various rise-times. Some of them may correspond to the measured results shown in Fig. 5. For example, it can be presumed that the waves shown in Fig. 5(b) have a rise time of $7 \mu s$.  

4 Measurement of the Phenomenon of Dynamic Yield by Using a Stress Bar

It is known that, as metallic materials are plastically deformed under an impact loading, most of them show an increase in yield stress and a phenomenon of delayed yield, caused by the effect of strain-rate\(^{12}\) \(-^{14}\).

Fig. 7 illustrates a Hopkinson bar type apparatus in which a cylindrical specimen plastically deformed is subjected to longitudinal compressional impact with a stress bar which remains elastic during the experiment. From the elastic response of the stress bar, we are able to observe the elastic-plastic stress and the phenomenon of dynamic yield at the impacted end of the specimen.

Fig. 6 Theoretical axial strains at $x = 150d$ for various rise-times of impact.
Fig. 8(a) shows the typical time variations of elastic-plastic stress at the impacted end of a lead specimen which is subjected to the impact of a steel stress bar with the velocity of 12.7 m/sec. The upper trace is the gages located at 1 bar diameter from the impacted end and the lower trace is the gages located at 5 bar diameters. Because of both gages being located comparatively close to the impacted end, the influence of the higher modes as well as lateral inertia appears in both waves. However, as discussed in foregoing chapters, we can approximately estimate the impact load by reading out the center of oscillation in the waves. In this case the dynamic yield stress at the impacted end of a lead specimen increases to about two times of static one and the delay time of yield is about 80 usec.

The waves shown in Fig. 8(b) are measured by the same experiment as in Fig. 8(a), except for a rise time. The elevation of the dynamic yield stress is smaller than that in Fig. 8(a). Judging from the waves that indicate less effect of lateral inertia and higher modes, those waves would have a longer rise time. Therefore, it is supposed that the increase of the dynamic yield stress strongly depends on a rise time of impact. This fact is also predicted by the strain-rate dependent theory on plastic wave propagation[13].

Fig. 9 is the magnification of Fig. 8(a) in sweep time. As seen in Fig. 9, under the conditions of a rapid rise time of impact and at a location close to the impacted end, the influence of the higher modes of wave motion tends to appear. For such a case, more precise theoretical discussion based on the higher approximate theories such as the Mindlin-McKiven theory[18] or the exact numerical solution would be necessary[17].

5. Conclusions

The results may be summarized as follows:

1) By measuring simultaneously the axial and radial strains at the same locations along a bar, we showed that Love's assumption was able to be directly verified and examined its adequacy and applicable ranges: Love's theory might be applicable to the farther distance of about 10 times of diameter from the impacted end, even for a rapid rate loading; the elastic responses at the station more close to the end, for example, 5 times of diameter, presented irregular oscillations, caused by the higher modes of wave motion.

2) With an increase of the rise time of impact, the effect of lateral inertia on the waves decreases and then the oscillatory waves become negligibly small. The vibratory amplitudes in waves grow up with the distance of wave propagation due to the effect of lateral inertia. However, for over about 10 usec of the rise time of impact, no contribution of lateral inertia appears at the location up to 150 diameters from the impacted end of a bar.
(3) As an example of application, the measurement of the phenomenon of dynamic yield in metallic materials under an impact loading was discussed by using an elastic stress bar. The dynamic yield stress and the delay time in lead specimens were estimated by considering the elastic response of a steel stress bar. In addition, we concluded that the elevation of dynamic yield stress strongly depended on the rise time of impact which had an effect on the behavior of the elastic-plastic stress at the impacted end of a specimen.

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References