Vibration of a Bowed String*

(1st Report - No.1 Monochord Subject to Air Resistance Only)

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Steady self-excited vibrations of a bowed string, which is perfectly flexible and subject to negative damping due to solid frictional force of general characteristic and to positive damping due to air resistance only, are studied by means of a Fourier series method utilizing series transformation.

It is shown that in the limiting case of air resistance tending to zero the obtained solution reproduces the celebrated Helmholtz mode of vibration for a bowed string. The solution also substantiates Raman's results about the necessary bow force to maintain vibration.

Key Words : Theory of Vibration, Self-excited Vibration of Continuous Body, Bowed String

1. Introduction

Since 10 years ago, the authors have been devoting themselves to an exact analysis as possible of steady vibrations of a bowed string as those of a violin, considering them as maintained self-excited vibrations resulting from energy balance between negative damping due to the solid frictional force exerted by the bow on the string, on the one hand, and positive damping due to air resistance, internal friction and sound energy delivery through the bridge etc., on the other.

The history of theoretical study on the vibrations of a bowed string is generally traced to Helmholtz⁽¹⁾. Investigating the motion of an arbitrary point on a bowed string by means of Lissajous' figure observed by inspecting through a kind of mechanical oscilloscope named by him a vibration microscope, he discovered that the vibration figure of an arbitrary point on the string has a two-step zig-zag wave form and identified this mode of vibration with one of those modes of undamped free vibration of a string given by

$$y(x, t) = \frac{2l^2 v_0}{\pi^2 a^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi x}{l} \sin \frac{n\pi a}{l} t \dots (1)$$

where the notations agree with those defined in the next section, only x being different as it is concerned with any point on the whole string.

In the present century, Raman⁽²⁾ utilized ingeniously the velocity wave to construct an "a priori" theory, so called by him, of vibrations of a bowed string. He

succeeded in reproducing theoretically the greater part of the beautiful wave forms obtained experimentally by Krigar-Menzel and Raps⁽³⁾ on a monochord, namely a string with both ends fixed. In 1930's, when rapid developments of electronic measuring technique excited a great many experimental studies⁽⁴⁾, processes of transient build-up and convergence to limit cycle of self-excited vibrations of a bowed string were elucidated by Witt applying the point mapping method⁽⁵⁾. Further, in 1950's Keller⁽⁶⁾ and Friedlander⁽⁷⁾ arrived almost simultaneously at substantially the same theory. This theory traces the build-up process from the initial state of rest to the ultimate steady state of vibration by applying recurrence formulas for successive steps and conditions of periodicity.

Fundamental assumptions for these and other theoretical studies, excepting Miyajima's regrettably unfinished one⁽⁸⁾, all rest on the one-dimensional wave equation of a string without damping. These assumptions may be adequate for the explanations of transient build-up stage of vibrations. However, once the steady vibration is established, no damping means no energy delivery and the string loses its status as an active member of a musical instrument.

In order to atone for these shortcomings of the available theories, the authors published several papers⁽⁹⁾⁽¹⁰⁾⁽¹¹⁾ and reports⁽¹²⁾⁽¹³⁾. First two papers assume

Fig.1 Frictional Force Characteristic with Constant Dynamic Friction

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piecewise-linear characteristic for frictional force with constant dynamic friction as shown in Fig.1 and adopt air resistance as the only source of damping.

In this paper, our theory is extended to include the general frictional force characteristic as shown in Fig.2. In virtue of this extension, it is shown that our old solution for the case of piecewise-linear characteristic is also valid for the case of general characteristic in the limit of air resistance tending to zero. Within this limit our solution coincides with Helmholtz's solution. Thus, Helmholtz's mode of vibration of a bowed string Eq.(1) is substantiated as that of a self-excited maintained vibration with very small positive damping.

Our method of analysis can also be easily applied to investigate the various effects of the bending stiffness, internal friction, torsional vibration, damping and compliance of bowing action and sound energy delivery through the bridge⁽¹¹⁾⁽¹²⁾⁽¹³⁾.

Meanwhile, a powerful new theory of a bowed string capable of treating transient vibrations as well as steady ones⁽¹⁴⁾⁽¹⁵⁾ was introduced. This is an application of nonlinear integral equation method which proved to be effective in the case of analysis of self-excited vibrations in air column of wind instrument. The heuristics underlying this theory is, at least for the case of steady vibrations, the same as that for our theory. It is the recognition of the fact that the velocity of the bow point is a superposition of linear frequency responses to various component harmonics composing the periodic bow force, on the one hand, and this velocity and this force are related to each other according to the frictional force characteristic between the bow and the bow point, on the other. These two conditions yield two key equations for analysis.

While basic heuristics are the same, the paths of approach are quite dissimilar. Here lies the reason for the preparation of this paper.

2. Equations of Motion and Their Formal Solutions

First, basic assumptions of our analysis are presented.

(1) Both ends of the string are fixed, accordingly energy flow through each end is nil.

(2) The string is perfectly flexible, namely it has no bending stiffness. It has no internal friction either.

Fig.2 General Frictional Force Characteristic

(3) The bow has no breadth, so that the bow point on the string is a mathematical point.

(4) The string undergoes no torsion.

(5) The characteristic of solid frictional force between the bow and the bow point is of general type as shown in Fig.2. Namely, it is composed of three branches, one branch for static friction of a straight line segment with very large slope and two symmetric branches for dynamic friction of hyperbola-like curves with negative slope. We do not exclude discontinuous jumps at boundaries between the domain of static friction and that of dynamic friction.

Fig.3 shows a bowed string with its bow. The string has two fixed ends A and B. The bow with a constant velocity v_0 is applied perpendicularly to the string at the bow point C whose distances from the left end A and the right end B are ξ and $\bar{\xi}$. We assume $\xi \leq \bar{\xi}$. This assumption does not detract any generality.

Next, the various notations used below are defined as follows:

- l : Length of the string
- $\xi, \bar{\xi}$: Distances of the bow point C from the left end A and from the right end B ($\xi \leq \bar{\xi}$)
- $x, \bar{x}$: Distance of an arbitrary point P on the left half from the left end A and distance of an arbitrary point $\bar{P}$ on the right half from the right end B
- $y(x, t), \bar{y}(\bar{x}, t)$: Displacements of the point P and the point $\bar{P}$
- μ : Line density of the string
- T : Tension of the string
- $a = \sqrt{T/\mu}$: Propagation velocity of lateral wave
- v_0 : Constant velocity of the bow
- $\nu = \pi a/l$: Undamped circular natural frequency of the string for fundamental tone
- $2c$: Coefficient for air resistance
- ω : Circular frequency of steady maintained vibration
- $\Omega = \omega/\nu$: Frequency ratio
- $\gamma = c/\nu$: Damping ratio for air resistance
- t : Time
- $\theta = \omega t$: Dimensionless time with its origin at the middle point of the stick interval
- $\varphi = \theta - \pi$: Dimensionless time with its origin at the middle point of the slip interval
- θ_0 : Length of the stick interval
- $\varphi_0 = 2\pi - \theta_0$: Length of the slip interval
- $\beta = \varphi_0/(2\pi) = 1 - \theta_0/(2\pi)$: Slip interval ratio
- v : Relative velocity of the bow to the bow point

Fig.3 Bowed String and Bow

$f(v)$: Frictional force corresponding to v

The equations of motion for the left half and the right half of the string will be

$$\left. \begin{aligned} \frac{\partial^2 y}{\partial t^2} + 2c \frac{\partial y}{\partial t} &= a^2 \frac{\partial^2 y}{\partial x^2} \\ \frac{\partial^2 \bar{y}}{\partial t^2} + 2c \frac{\partial \bar{y}}{\partial t} &= a^2 \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} \end{aligned} \right\} \dots\dots\dots (2)$$

The boundary conditions at the left end and at the right end become

$$y(0, t) = 0, \quad \bar{y}(l, t) = 0 \dots\dots\dots (3)$$

At the bow point we have two conditions, namely

$$y(\xi, t) = \bar{y}(\xi, t) \dots\dots\dots (4)$$

$$T\left(\frac{\partial y}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}\right) = B(\theta) = f(v_0 - \dot{y}_e) \dots\dots\dots (5)$$

where $B(\theta)$ represents the driving frictional force exerted by the bow on the bow point, to be designated hereafter as the bow force and the expression

$$\dot{y}_e = \frac{\partial y(\xi, t)}{\partial t} \dots\dots\dots (6)$$

represents the velocity of the bow point.

In this paper, we consider only the most simple type of steady vibration whose whole period consists of one stick interval and one slip interval. The bow point closely follows the bow while attached to it during the stick interval and recoils back while slipping relative to the bow during the slip interval. The velocity of recoil can not be assumed beforehand to be constant. However, the mean relative velocity during the slip interval must be

$$v_s = v_0 - \{(1 - \beta)v_0/\beta\} = v_0/\beta \dots\dots\dots (7)$$

The bow force, a periodic function of θ with the period 2π , can be divided into two parts, namely the static part of constant magnitude equal to the dynamic friction $f(v_0/\beta)$ corresponding to the mean relative velocity v_0/β and the variable part $G(\theta)$ which we call hereafter the additional bow force. Since our steady vibration can be considered a steady forced vibration excited by the bow force $B(\theta)$ once this force is determined, the displacements $y(x, t)$ and $\bar{y}(\bar{x}, t)$ can each be divided into two parts, corresponding to the above division of $B(\theta)$, namely into the static deflections $y_s(x)$ and $\bar{y}_s(\bar{x})$ due to the constant load $F_0 = f(v_0/\beta)$ and the variable vibrational parts $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ due to the additional bow force $G(\theta)$ respectively, so that we have

$$B(\theta) = f(v_0/\beta) + G(\theta) \dots\dots\dots (8)$$

$$\left. \begin{aligned} y(x, t) &= y_s(x) + \eta(x, t) \\ \bar{y}(\bar{x}, t) &= \bar{y}_s(\bar{x}) + \bar{\eta}(\bar{x}, t) \end{aligned} \right\} \dots\dots\dots (9)$$

where

$$\left. \begin{aligned} y_s(x) &= f(v_0/\beta) \xi x / (Tl) \\ \bar{y}_s(\bar{x}) &= f(v_0/\beta) \xi \bar{x} / (Tl) \end{aligned} \right\} \dots\dots\dots (10)$$

The equations of motion and the boundary conditions at both ends for the vibrational displacements $\eta(x, t)$ and $\bar{\eta}(\bar{x}, t)$ are of the same form as those of Eq. (2) and Eq. (3). And the boundary conditions at the bow point are composed of an equation of the same form as that of Eq. (4) and the other corresponding to Eq. (5) given by

$$T\left(\frac{\partial \eta}{\partial x} + \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) = G(\theta) = f(v_0 - \dot{\eta}_e) - f(v_0/\beta) \dots\dots (11)$$

The final expressions of Eq. (11) are different according to the different expressions of characteristic between the stick interval and the slip interval.

Namely, for $-\theta_0/2 \leq \theta \leq \theta_0/2$, we have

$$T\left(\frac{\partial \eta}{\partial x} + \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) = G(\theta) = h(v_0 - \dot{\eta}_e) - f(v_0/\beta) \dots\dots (12)$$

and, for $\theta_0/2 < \theta < 2\pi - \theta_0/2$, we have

$$\begin{aligned} T\left(\frac{\partial \eta}{\partial x} + \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) &= G(\theta) \\ &= -\left(\frac{df}{dv}\right)_{v=v_0/\beta} \left(\frac{1-\beta}{\beta} v_0 + \dot{\eta}_e\right) \\ &\quad + \frac{1}{2!} \left(\frac{d^2 f}{dv^2}\right)_{v=v_0/\beta} \left(\frac{1-\beta}{\beta} v_0 + \dot{\eta}_e\right)^2 + \dots \dots (13) \end{aligned}$$

Hereafter, we make the expressions for displacement, velocity and force dimensionless by adopting v_0/ν , v_0 and $\{Tl/(\xi\xi)\}v_0/\nu$ as the units for them and distinguish them by adding primes ' to their expressions.

The additional bow force $G'(\theta)$, being a periodic function of θ of the period 2π , can be expanded into a Fourier series as

$$\begin{aligned} G'(\theta) &= a'_0 + \sum_{n=1}^{\infty} (a'_n \cos n\theta + b'_n \sin n\theta) \\ &= R \sum_{n=0}^{\infty} (a'_n - jb'_n) e^{jn\omega t} \quad (j = \sqrt{-1}) \dots (14) \end{aligned}$$

where R expresses the real part of the complex number following it. The Fourier coefficients a'_0, a'_n, b'_n ($n=1, 2, \dots$) are unknowns at the present. However, a formal solution for the steady vibration due to the additional bow force $G'(\theta)$ can be obtained as a superposition of linear frequency responses to individual harmonics composing the Fourier expansion of $G'(\theta)$ as

$$\left. \begin{aligned} \eta'(x, t) &= R \sum_{n=0}^{\infty} N_n(x) (a'_n - jb'_n) e^{j(n\omega t - \bar{x}_n(x))} \\ \bar{\eta}'(\bar{x}, t) &= R \sum_{n=0}^{\infty} \bar{N}_n(\bar{x}) (a'_n - jb'_n) e^{j(n\omega t - \bar{x}_n(\bar{x}))} \end{aligned} \right\} \dots (15)$$

where

$$\left. \begin{aligned} N_n(x) e^{-j\bar{x}_n(x)} &= \frac{l^2}{\xi\xi} \frac{\sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} x \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi}{\frac{\bar{\omega}_n - j\bar{c}_n}{a} l \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} l} \\ \bar{N}_n(\bar{x}) e^{-j\bar{x}_n(\bar{x})} &= \frac{l^2}{\xi\xi} \frac{\sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \bar{x}}{\frac{\bar{\omega}_n - j\bar{c}_n}{a} l \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} l} \end{aligned} \right\} \dots\dots\dots (16)$$

and

$$\left. \begin{aligned} \bar{\omega}_n &= \sqrt{n^4 \omega^4 + 4c^2 n^2 \omega^2 + n^2 \omega^2} / \sqrt{2} \\ \bar{c}_n &= \sqrt{n^4 \omega^4 + 4c^2 n^2 \omega^2 - n^2 \omega^2} / \sqrt{2} \end{aligned} \right\} \dots\dots (17)$$

Especially, for the bow point ($x = \xi$ or $\bar{x} = \xi$), we have

$$\begin{aligned} \eta'_e &= \eta'(\xi, t) = \bar{\eta}'(\xi, t) \\ &= R \sum_{n=0}^{\infty} M_n (a'_n - jb'_n) e^{j(n\omega t - \theta_n)} \dots\dots\dots (18) \end{aligned}$$

where

$$\begin{aligned} M_n e^{-j\theta_n} &= N_n(\xi) e^{-j\bar{x}_n(\xi)} = \bar{N}_n(\xi) e^{-j\bar{x}_n(\xi)} \\ &= \frac{l^2}{\xi\xi} \frac{\sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} \xi}{\frac{\bar{\omega}_n - j\bar{c}_n}{a} l \sin \frac{\bar{\omega}_n - j\bar{c}_n}{a} l} \dots\dots (19) \end{aligned}$$

These formal solutions satisfy the left halves of the boundary conditions Eq.(12) and Eq.(13).

3. Fourier Series Solution by Series Transformation Method

The remaining tasks before us are to determine the unknown Fourier coefficients a'_0, a'_n, b'_n ($n=1, 2, \dots$) to satisfy the remaining right halves of the boundary conditions Eq.(12) and Eq.(13). To this end recourse is had to the method of series transformation which was introduced before by one of the authors to obtain a Fourier series solution for steady forced vibrations in piecewise-linear systems⁽¹⁶⁾. We transform the series for $G'(\theta)$ into a sum of two series $K'(\theta)$ and $H'(\varphi)$ as

$$\left. \begin{aligned} G'(\theta) &= K'(\theta) + H'(\varphi), \quad K'(\theta) = \sum_{m=1}^{\infty} (\alpha'_{2m-1} \cos \mu_m \theta + \beta'_{2m-1} \sin \mu_m \theta) \\ H'(\varphi) &= \sum_{m=1}^{\infty} (\bar{\alpha}'_{2m-1} \cos \bar{\mu}_m \varphi + \bar{\beta}'_{2m-1} \sin \bar{\mu}_m \varphi) \end{aligned} \right\} \dots\dots\dots (20)$$

where

$$\mu_m = (2m-1)\pi/\theta_0, \quad \bar{\mu}_m = (2m-1)\pi/\varphi_0 \dots\dots\dots (21)$$

and $K'(\theta)$ is a Fourier series only valid in the stick interval, being zero elsewhere and $H'(\varphi)$ is similarly a Fourier series only valid in the slip interval, being zero elsewhere.

Then, the Fourier coefficients a'_0, a'_n, b'_n ($n=1, 2, \dots$) of the old series can be calculated from those of the new ones $\alpha'_{2m-1}, \beta'_{2m-1}, \bar{\alpha}'_{2m-1}, \bar{\beta}'_{2m-1}$ by means of the transformation formulas given by

$$\left. \begin{aligned} a'_0 &= \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\mu_m} \alpha'_{2m-1} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\bar{\mu}_m} \bar{\alpha}'_{2m-1}, \quad (n=1, 2, \dots) \\ a'_n &= \frac{2}{\pi} \cos \frac{n\theta_0}{2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\mu_m}{\mu_m^2 - n^2} \alpha'_{2m-1} + (-1)^n \frac{2}{\pi} \cos \frac{n\varphi_0}{2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\bar{\mu}_m}{\bar{\mu}_m^2 - n^2} \bar{\alpha}'_{2m-1} \\ b'_n &= \frac{2}{\pi} \cos \frac{n\theta_0}{2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{n}{\mu_m^2 - n^2} \beta'_{2m-1} + (-1)^n \frac{2}{\pi} \cos \frac{n\varphi_0}{2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{n}{\bar{\mu}_m^2 - n^2} \bar{\beta}'_{2m-1} \end{aligned} \right\} \dots\dots\dots (22)$$

Also in the stick interval we are given the following Fourier expansions

$$\left. \begin{aligned} \cos n\theta &= \frac{4}{\theta_0} \cos \frac{n\theta_0}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\mu_k}{\mu_k^2 - n^2} \cos \mu_k \theta \\ \sin n\theta &= \frac{4}{\theta_0} \cos \frac{n\theta_0}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{n}{\mu_k^2 - n^2} \sin \mu_k \theta \\ \mu_k &= (2k-1)\pi/\theta_0 \quad (n=1, 2, \dots) \end{aligned} \right\} \dots\dots\dots (23)$$

Introducing the expressions Eq.(22) and Eq.(23) into the formula for the velocity of the bow point

$$\dot{\eta}' = \sum_{n=1}^{\infty} n \Omega M_n \{ (a'_n \sin \phi_n + b'_n \cos \phi_n) \cos n\theta - (a'_n \cos \phi_n - b'_n \sin \phi_n) \sin n\theta \} \dots\dots\dots (24)$$

which can be obtained by differentiating with respect to t the expression Eq.(18), and interchanging the order of summations with respect to k, m and n (assuming this interchange is allowable), we obtain a new Fourier series for the velocity of the bow point $\dot{\eta}'$ valid in the stick interval as

$$\begin{aligned} \dot{\eta}' &= \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{\infty} (C_{km} \alpha'_{2m-1} + D_{km} \beta'_{2m-1} + \bar{H}_{km} \bar{\alpha}'_{2m-1} + \bar{I}_{km} \bar{\beta}'_{2m-1}) \right\} \cos \mu_k \theta \\ &+ \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{\infty} (E_{km} \alpha'_{2m-1} + F_{km} \beta'_{2m-1} + \bar{J}_{km} \bar{\alpha}'_{2m-1} + \bar{K}_{km} \bar{\beta}'_{2m-1}) \right\} \sin \mu_k \theta \dots\dots\dots (25) \end{aligned}$$

Substituting this result for $\dot{\eta}'$ in the right half of the boundary condition Eq.(12) and considering the expansion

$$1 = \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1} \cos \mu_k \theta \dots\dots\dots (26)$$

We obtain by establishing the harmonic balance between the left side and the right side of the resulting equation, an infinite set of simultaneous linear equations for coefficients $\alpha'_{2m-1}, \beta'_{2m-1}, \bar{\alpha}'_{2m-1}, \bar{\beta}'_{2m-1}$ as an infinite number of unknowns, namely

$$\left. \begin{aligned} \sum_{m=1}^{\infty} \left\{ \left(C_{km} + \frac{l^2}{\xi \xi} \sigma \delta_{km} \right) \alpha'_{2m-1} + D_{km} \beta'_{2m-1} + \bar{H}_{km} \bar{\alpha}'_{2m-1} + \bar{I}_{km} \bar{\beta}'_{2m-1} \right\} &= \{ 1 - f(v_0/\beta) / (h v_0) \} \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1} \\ \sum_{m=1}^{\infty} \left\{ E_{km} \alpha'_{2m-1} + \left(F_{km} + \frac{l^2}{\xi \xi} \sigma \delta_{km} \right) \beta'_{2m-1} + \bar{J}_{km} \bar{\alpha}'_{2m-1} + \bar{K}_{km} \bar{\beta}'_{2m-1} \right\} &= 0, \quad \sigma = \frac{T}{\pi h a} \end{aligned} \right\} \dots\dots\dots (27)$$

($k=1, 2, \dots$)

where δ_{km} is Kronecker's delta notation and $\delta_{km}=0$ for $k \neq m$ while $\delta_{km}=1$ for $k=m$ and coefficients $C_{km}, D_{km}, E_{km}, F_{km}, \bar{H}_{km}, \bar{I}_{km}, \bar{J}_{km}$ and $\bar{K}_{km}$ are given by the following expressions,

$$\left. \begin{aligned} C_{km} &= \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \mu_m \sum_{n=1}^{\infty} \frac{n(1 + \cos n\theta_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} M_n \sin \phi_n \\ D_{km} &= -E_{km} = \frac{4}{\pi \theta_0} (-1)^{k+m} \Omega \mu_k \sum_{n=1}^{\infty} \frac{n^2(1 + \cos n\theta_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} M_n \cos \phi_n \end{aligned} \right\} \dots\dots\dots (28)$$

$$\left. \begin{aligned}
F_{km} &= \frac{4}{\pi\theta_0} (-1)^{k+m} Q \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \sin \phi_n \\
\bar{H}_{km} &= \frac{4}{\pi\theta_0} (-1)^{k+m} Q \mu_k \bar{\mu}_m \sum_{n=1}^{\infty} \frac{n(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \sin \phi_n \\
\bar{I}_{km} &= \frac{4}{\pi\theta_0} (-1)^{k+m} Q \mu_k \sum_{n=1}^{\infty} \frac{n^2(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \cos \phi_n \\
\bar{J}_{km} &= -\frac{4}{\pi\theta_0} (-1)^{k+m} Q \bar{\mu}_m \sum_{n=1}^{\infty} \frac{n^2(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \cos \phi_n \\
\bar{K}_{km} &= \frac{4}{\pi\theta_0} (-1)^{k+m} Q \sum_{n=1}^{\infty} \frac{n^3(1+\cos n\theta_0)}{(\mu_k^2 - n^2)(\bar{\mu}_m^2 - n^2)} M_n \sin \phi_n
\end{aligned} \right\} \dots\dots\dots (28)$$

(k, m=1, 2, ...)

On the other hand, in the slip interval, we are given the Fourier expansions which result from Eq.(23) with substitutions $\theta \rightarrow \varphi$, $\theta_0 \rightarrow \varphi_0$, and $\mu_k \rightarrow \bar{\mu}_k$, and the expression for $\dot{\eta}'_i$ from Eq.(24) by substituting φ for θ after multiplying by $(-1)^n$. Consequently, we have an expression of $\dot{\eta}'_i$ valid in the slip interval obtained by substituting in Eq.(25) H_{km} , I_{km} , J_{km} , K_{km} for C_{km} , D_{km} , E_{km} , F_{km} and $\bar{C}_{km}$, $\bar{D}_{km}$, $\bar{E}_{km}$, $\bar{F}_{km}$ for $\bar{H}_{km}$, $\bar{I}_{km}$, $\bar{J}_{km}$, $\bar{K}_{km}$, where $\bar{C}_{km}$, $\bar{D}_{km}$, $\bar{E}_{km}$, $\bar{F}_{km}$ are obtained from C_{km} , D_{km} , E_{km} , F_{km} in Eq.(28) and H_{km} , I_{km} , J_{km} , K_{km} are obtained from $\bar{H}_{km}$, $\bar{I}_{km}$, $\bar{J}_{km}$, $\bar{K}_{km}$ in the same equation by substitution $\theta_0 \rightarrow \varphi_0$.

Introducing this expression for $\dot{\eta}'_i$ into the right half of Eq.(13) and establishing the harmonic balance between the left and the right sides, we obtain, in general, an infinite set of simultaneous nonlinear equations for the unknown coefficients α'_{2m-1} , β'_{2m-1} , $\bar{\alpha}'_{2m-1}$, $\bar{\beta}'_{2m-1}$ ($m=1, 2, \dots$) as an infinite number of unknowns. Neglecting the terms of the order higher than 2nd of the Taylor expansion in Eq.(13), this simultaneous equations become

$$\left. \begin{aligned}
&\sum_{m=1}^{\infty} \left\{ -A' H_{km} \alpha'_{2m-1} - A' I_{km} \beta'_{2m-1} + (\delta_{km} - A' \bar{C}_{km}) \bar{\alpha}'_{2m-1} - A' \bar{D}_{km} \bar{\beta}'_{2m-1} \right\} \\
&\quad - A'' \left[\frac{1-\beta}{\beta} c_k + \frac{1}{2} \sum_{l=1}^{\infty} \{ c_l (c_{l+k} + c_{l-k}) + d_l (d_{l+k} - d_{l-k}) \} \right] \\
&= \left[A' \frac{1-\beta}{\beta} + A'' \left\{ \left(\frac{1-\beta}{\beta} \right)^2 + \frac{1}{2} \sum_{m=1}^{\infty} (c_m^2 + d_m^2) \right\} \right] \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1} \\
&\sum_{m=1}^{\infty} \left\{ -A' J_{km} \alpha'_{2m-1} - A' K_{km} \beta'_{2m-1} - A' \bar{E}_{km} \bar{\alpha}'_{2m-1} + (\delta_{km} - A' \bar{F}_{km}) \bar{\beta}'_{2m-1} \right\} \\
&\quad - A'' \left[\frac{1-\beta}{\beta} d_k + \frac{1}{2} \sum_{l=1}^{\infty} \{ c_l (d_{l+k} + d_{l-k}) - d_l (c_{l+k} - c_{l-k}) \} \right] = 0
\end{aligned} \right\} \dots\dots\dots (29)$$

(k=1, 2, ...)

where

$$\left. \begin{aligned}
c_l &= \sum_{m=1}^{\infty} (\bar{C}_{lm} \bar{\alpha}'_{2m-1} + \bar{D}_{lm} \bar{\beta}'_{2m-1} + H_{lm} \alpha'_{2m-1} + I_{lm} \beta'_{2m-1}) \\
d_l &= \sum_{m=1}^{\infty} (\bar{E}_{lm} \bar{\alpha}'_{2m-1} + \bar{F}_{lm} \bar{\beta}'_{2m-1} + J_{lm} \alpha'_{2m-1} + K_{lm} \beta'_{2m-1}) \quad (l=1, 2, \dots)
\end{aligned} \right\} \dots\dots\dots (30)$$

and

$$A' = - \left(\frac{df}{dv} \right)_{v=v_0/\beta} \frac{\xi \bar{\xi}}{l^2} \frac{\pi a}{T}, \quad A'' = \frac{1}{2!} \left(\frac{d^2 f}{dv^2} \right)_{v=v_0/\beta} \frac{\xi \bar{\xi}}{l^2} \frac{\pi a v_0}{T} \dots\dots\dots (31)$$

For a given string, namely for given l , μ , T and $2c$ and for a given bow with a given speed applied at the given point, namely for given ξ , $\bar{\xi}$ and v_0 , all the coefficients C_{km} etc., $\bar{H}_{km}$ etc., $\bar{C}_{km}$ etc., and H_{km} etc. in these simultaneous equations (27) and (29) depend only on the frequency ratio Q and the stick interval ratio $\theta_0/(2\pi)$.

In addition to the boundary conditions already mentioned, we have two further conditions to be satisfied, namely conditions of capture and release of the string by and from the bow. In the case of no hysteresis in the frictional force characteristic, these conditions are given respectively by

$$\left. \begin{aligned}
G(\theta_0/2) &= F_s - f(v_0/\beta) \\
G(-\theta_0/2) &= F_s - f(v_0/\beta)
\end{aligned} \right\} \dots\dots\dots (32)$$

Lastly, the obtained solution must satisfy the following compatibility condition, that the necessary bow force $B(\theta)$ to maintain the vibration ought not exceed in absolute value the maximum static friction F_s , namely for $-\theta_0/2 < \theta < \theta_0/2$

$$-F_s - f(v_0/\beta) < G(\theta) < F_s - f(v_0/\beta) \dots\dots\dots (33)$$

If the obtained solution contradicts this condition, it must be discarded as an incompatible one.

4. Solutions for Small Air Resistance

If α'_{2m-1} , β'_{2m-1} obtained from the remaining equations (27), where $\bar{\alpha}'_{2m-1}$, $\bar{\beta}'_{2m-1}$ are put equal to zero, give the bow force such that the velocity of recoil in the slip interval is constant, namely that the equation $\dot{\eta}_s + (1-\beta)v_0/\beta = 0$ is always true for the slip interval, we have the bow force $H(\varphi) = 0$, as seen from the right half of Eq.(13). Namely, in this case, the set of these α'_{2m-1} , β'_{2m-1} , $\bar{\alpha}'_{2m-1} = 0$, $\bar{\beta}'_{2m-1} = 0$ ($m=1, 2, \dots$) gives a complete solution for Eq.(27) and Eq.(29).

The assumption in the foregoing papers (9,10) that the dynamic frictional force is constant, gives $A' = A'' = \dots = 0$, so that necessarily $\bar{\alpha}'_{2m-1}$ and $\bar{\beta}'_{2m-1}$ equal to zero.

Consequently, if any solution in these old papers gives a constant velocity of recoil in the slip interval, it is also a complete solution for the case of general frictional force characteristic as given in Fig.2.

Physically speaking, the constancy of the velocity of recoil means that all the states of motion in the slip interval are represented by only one point P in Fig.2, so that behaviors of the dynamic friction curves in the neighbourhood of the point P can not have any effect whatever. Several examples that follow may illustrate these circumstances.

When air resistance is small and γ is a small quantity of the first order, by neglecting γ^2 relative to 1, we have for the frequency ratio Q equal to 1.0,

$$\left. \begin{aligned} M_n \sin \phi_n &= \frac{1}{L} \frac{2}{n} \sin^2 n\pi \frac{\xi}{l} \\ M_n \cos \phi_n &= \frac{\gamma}{L} \left(\frac{\sin^2 n\pi \frac{\xi}{l}}{n^2} + \frac{\pi}{n} \frac{\xi - \xi}{l} \sin 2n\pi \frac{\xi}{l} \right) \\ L &= \frac{\pi^2 \gamma}{2} \frac{4\xi\xi}{l^2} \quad (n=1, 2, \dots) \end{aligned} \right\} \dots\dots\dots(34)$$

Thus, the remaining equations above mentioned will degenerate into

$$\sum_{n=1}^{\infty} C_{km} \alpha'_{2m-1} = \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1}, \quad \beta'_{2m-1} = 0 \dots\dots(35)$$

Here, for simplicity, it is assumed the slope of static friction characteristic $k \rightarrow \infty$.

Further, if $\theta_0/2\pi \leq \xi/l$, we have $C_{km} = \delta_{km}/L$ and obtain

$$G'(\theta) = L, \quad -\theta_0/2 < \theta < \theta_0/2 \dots\dots\dots(36)$$

(cf. Appendix A). The following two examples (I) and (II) belong to this case. Here the final results only are presented. For their deduction the readers are referred to the foregoing paper⁽¹⁰⁾.

(I) $\xi/l = 1/2, Q = 1.0, \theta_0 = 180^\circ$ (Fig.4)

$$G'(\theta) = L \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^{n-1}}{2n-1} \cos(2n-1)\theta \right\} \dots\dots\dots(37)$$

$$\dot{\eta}'_{\xi} = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{n-1}}{2n-1} \cos(2n-1)\theta \dots\dots\dots(38)$$

(II) $\xi/l = 1/9, Q = 1.0, \theta_0 = 40^\circ$ (Fig.5)

Fig.4 Bow Force, Velocity and Displacement of Bow Point for $\xi/l = 1/2, \Omega = 1.0$ and $\theta_0 = 180^\circ$ (by theory for $\gamma > 0$)

$$G'(\theta) = L \left(\frac{\theta_0}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\theta_0}{2}}{n} \cos n\theta \right) \dots\dots(39)$$

$$\dot{\eta}'_{\xi} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2 \frac{n\theta_0}{2}}{n} \cos n\theta \dots\dots\dots(40)$$

(III) $\xi/l = 1/9, Q = 1.0, \theta_0 = 320^\circ$

(Fig.6 and Fig.7), (cf. Appendix B)

We have for $k \neq m$

$$C_{km} = \frac{1}{L} \frac{2(-1)^{k+m}}{\theta_0(\mu_k^2 - \mu_m^2)} (\mu_k \sin 2\mu_m \pi - \mu_m \sin 2\mu_k \pi) \dots\dots\dots(41)$$

and for $k = m$

$$C_{kk} = \frac{1}{L} \frac{2}{\theta_0} \left\{ \frac{\sin 2\mu_k \pi}{2\mu_k} + \frac{\theta_0}{2} - (\pi - \theta_0) \cos 2\mu_k \pi \right\} \dots\dots\dots(42)$$

A solution to the set of simultaneous equations (35) gives

$$\alpha'_{2m-1} = L \left\{ 4 + \frac{1}{2} \sin^2 \frac{(2m-1)\pi}{16} \right\} \frac{4}{\pi} \frac{(-1)^{m-1}}{2m-1} \dots\dots\dots(43)$$

This solution can be constructed by means of superposing eight appropriate solutions of the example (II) as given in the previous paper⁽¹¹⁾.

Fig.7 shows numerical computation results for a string with $\gamma = 0.001$ bowed at $\xi/l = 1/9$. The frequency ratio and the stick interval ratio are selected such as to give Helmholtz point of $Q = 1.0$ and $\theta_0 = (\xi/l)2\pi = 320^\circ$. As seen from the figure, they reproduce almost perfectly theoretical result of Raman⁽²⁾. Similar results have been obtained for the case of (I), namely for $\xi/l = 1/2, Q = 1.0$ and $\theta_0 = 180^\circ$.

Helmholtz' formula (1) gives for the bow point $x = \xi$ the displacement

$$y_i = y(\xi, t) / (v_0 / \nu) = \frac{2l}{\pi \xi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin \frac{n\pi \xi}{l} \sin \frac{n\pi a}{l} t \dots\dots(44)$$

This expression gives exactly the Fourier series of vibration figures in Fig.4 for $\xi/l = 1/2$ and those in Fig.6 for $\xi/l = 1/9$. In addition, according to Helmholtz, when $\xi/l = 1/n$ where n is a simple integer, harmonics one of whose nodal points coincides with the bow point must be deleted from the summation Σ of Eq.(44). Then, vibration figures at points other than these nodal points will show many small crumples.

Fig.5 Bow force, Velocity and Displacement of Bow Point for $\xi/l = 1/9, \Omega = 1.0$ and $\theta_0 = 40^\circ$ (by theory for $\gamma > 0$)

Our computational results reproduce this fact automatically as shown in an example in Fig.8, representing the vibration figure at a point $x/l=0.06$ for a string with $\gamma=0.001$, $\xi/l=1/10$ at $\Omega=1.0$, and $\theta_0=324^\circ$.

Thus, our solution for a maintained self-excited vibration of a bowed string with general frictional characteristic subject to small air resistance is shown to reproduce the celebrated Helmholtz mode of vibration. The Helmholtz mode is for a bowed string as the simple harmonic motion of amplitude 2 is for a van der Pol oscillator.

5. Conclusions

Steady self-excited vibrations of a bowed string, which is perfectly flexible and subject to negative damping due to solid frictional force of general characteristic and to positive damping due to air resistance only, are studied by means of a Fourier series method utilizing series transformation.

It is shown that in the limiting case of air resistance tending to zero the obtained solution reproduces the celebrated Helmholtz mode of vibration for a bowed string. The solution also substantiates Raman's result for the necessary bow force.

This method of analysis can also be easily applied to investigate the various effects of the bending stiffness, internal friction, torsion of the string, damping and compliance of bowing action and sound energy delivery through bridge, as far as these effects are described by linear equations. These works are in progress⁽¹¹⁾⁽¹²⁾⁽¹³⁾. Detailed Discussions of these problems will be a subject of our coming papers.

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Appendix A : Deduction of Eq.(36)

Denoting

$$\phi_0 = 2\pi\xi/l \quad \dots\dots\dots(45)$$

we obtain from Eq.(28) and Eq.(34)

Fig.6 Bow Force, Velocity and Displacement of Bow Point for $\xi/l=1/9$, $\Omega=1.0$ and $\theta_0=320^\circ$ (by theory for $\gamma \rightarrow 0$)

$$C_{km} = \frac{1}{L} \frac{4}{\pi\theta_0} (-1)^{k+m} \mu_k \mu_m \times \sum_{n=1}^{\infty} \frac{(1+\cos n\theta_0)(1-\cos n\phi_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} \quad \dots\dots\dots(46)$$

and we are also given a Fourier expansion for μ not equal to any integer

$$F(\mu, \theta) \equiv \sum_{n=1}^{\infty} \frac{1 - \cos n\theta}{\mu^2 - n^2} = \frac{\pi \cos \mu\pi - \cos \mu(\pi - \theta)}{2\mu \sin \mu\pi} \quad (0 \leq \theta \leq 2\pi) \quad \dots\dots\dots(47)$$

Now, a partial fraction expansion yields

$$\sum_{n=1}^{\infty} \frac{(1+\cos n\theta_0)(1-\cos n\phi_0)}{(\mu_k^2 - n^2)(\mu_m^2 - n^2)} = \frac{1}{\mu_k^2 - \mu_m^2} \sum_{n=1}^{\infty} \left\{ \frac{(1+\cos n\theta_0)(1-\cos n\phi_0)}{\mu_m^2 - n^2} - \frac{(1+\cos n\theta_0)(1-\cos n\phi_0)}{\mu_k^2 - n^2} \right\} \quad \dots\dots\dots(48)$$

and

$$\sum_{n=1}^{\infty} \frac{(1+\cos n\theta_0)(1-\cos n\phi_0)}{\mu_k^2 - n^2} = F(\mu_k, \phi_0) - F(\mu_k, \theta_0) + \frac{1}{2} F(\mu_k, \phi_0 - \theta_0) + \frac{1}{2} F(\mu_k, \phi_0 + \theta_0) \quad \dots\dots\dots(49)$$

and considering that

$$\mu_k = (2k-1)\pi/\theta_0 \quad \dots\dots\dots(50)$$

Fig.7 Bow Force and Displacement of Bow Point for $\gamma=0.001$, $\xi/l=1/9$, $\Omega=1.0$ and $\theta_0=320^\circ$ (computational results, truncation number for n and $k=m$: 500 and 100)

Fig.8 Computational Results for Vibration Figure at Point $x/l=0.06$ for $\gamma=0.001$, $\xi/l=1/10$, $\Omega=1.0$ and $\theta_0=324^\circ$

we have

$$F(\mu_k, \psi_0) = \frac{\pi \cos \mu_k \pi - \cos \mu_k (\pi - \psi_0)}{2\mu_k \sin \mu_k \pi}$$

$$F(\mu_k, \theta_0) = \frac{\pi \cos \mu_k \pi}{\mu_k \sin \mu_k \pi}$$

$$F(\mu_k, \psi_0 - \theta_0) = \frac{\pi \cos \mu_k \pi + \cos \mu_k (\pi - \psi_0)}{2\mu_k \sin \mu_k \pi}$$

$$F(\mu_k, \psi_0 + \theta_0) = \frac{\pi \cos \mu_k \pi + \cos \mu_k (\pi - \psi_0)}{2\mu_k \sin \mu_k \pi}$$

thus we can establish

$$C_{km} = 0 \quad (k \neq m)$$

Next, we obtain, for μ not equal to any integer, by differentiating Eq.(47) with respect to μ

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1 - \cos n\theta}{(\mu^2 - n^2)^2} \\ &= \frac{\pi}{4\mu^2} \left(1 + \frac{\mu\pi}{\tan \mu\pi} \right) \frac{\cos \mu\pi - \cos \mu(\pi - \theta)}{\sin \mu\pi} \\ &+ \frac{\pi}{4\mu^2} \frac{\pi \sin \mu\pi - (\pi - \theta) \sin \mu(\pi - \theta)}{\sin \mu\pi} \\ & \quad (0 \leq \theta \leq 2\pi) \dots \dots \dots (51) \end{aligned}$$

Performing similar partial fraction expansion about Eq.(49) and considering Eq.(50), we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(1 + \cos n\theta_0)(1 - \cos n\psi_0)}{(\mu_k^2 - n^2)^2} \\ &= \frac{\pi\theta_0}{4\mu_k^2} \quad (\theta_0 \leq \psi_0) \dots \dots \dots (52) \end{aligned}$$

and consequently, we establish

$$C_{kk} = 1/L$$

Then it follows from Eq.(35) that

$$\alpha'_{i,k-1} = \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1} L, \quad \beta'_{i,k-1} = 0$$

By virtue of $H'(\varphi)$ being equal to zero at the present, Eq.(20) gives

$$\begin{aligned} G'(\theta) &= K'(\theta) \\ &= \sum_{m=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{m-1}}{2m-1} L \cos \mu_m \theta = L \\ & \quad (-\theta_0/2 < \theta < \theta_0/2) \dots \dots \dots (36) \end{aligned}$$

Appendix B : Deduction of Eqs.(41) and (42)

This time, we are given $\theta_0 > \psi_0$ so that $\psi_0 - \theta_0 < 0$. For $-2\pi < \theta < 0$ we have instead of Eq.(47)

$$\begin{aligned} F(\mu, \theta) &= \sum_{n=1}^{\infty} \frac{1 - \cos n\theta}{\mu^2 - n^2} \\ &= \frac{\pi \cos \mu\pi - \cos \mu(\pi + \theta)}{2\mu \sin \mu\pi} \\ & \quad (-2\pi \leq \theta \leq 0) \dots \dots \dots (53) \end{aligned}$$

and consequently the result Eq.(41).

Eq.(51) is subject to similar modifications and yields Eq.(42).

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