Parametric Excitation in a Self-exciting System*

(2nd Report, Behaviors in the Regions of
Resonances of Orders 1/3 and 2)

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In a self-exciting system of Van der Pol type with the restoring force expressed as the product of a non-linear function of deflection and a periodic function of time, self-excitation and parametric excitation induce resonances of higher orders than orders 1 and 1/2. In the present paper firstly, behaviors in the region of subharmonic resonance of order 1/3 are investigated in the phase plane by using the averaging method. Next an approximate solution and the stability of a parametric resonance of order 2 is determined. In the neighborhood of this resonance a vibration with beat character occurs and then its amplitude is easily obtained. It is ascertained from the results of numerical calculation that the approximate solutions have high accuracy.

Key Words : Theory of Vibration, Self-excited Vibration, Parametric Excitation, Subharmonic Vibration, Beat Vibration, Mathieu Equation, Nonlinear Vibration

1. Introduction

In a system which consists of a beam subjected to a periodic axial force and simultaneously to self-excitation by a flow, a parametric vibration occurs together with a self-excited vibration. Such a system is described as a mathematical model by self-excitation of Van der Pol type and parametric excitation expressed by the product of a non-linear function of deflection and a periodic function of time (1)(2). A subharmonic resonance of order 1/2 is induced by parametric excitation in the system, so that the method of its analysis is demonstrated in a previous report (1). Needless to say, a parametric resonance of the n th order (n : natural numbers) also occurs at the frequencies corresponding to the unstable region of the n th order in a linear Mathieu equation. Hereafter a parametric resonance and a subharmonic one are simplified as a resonance without distinction.

In the present paper resonances of orders 1/3 and 2 are investigated because these resonances of higher orders are not yet dealt with by Tondl(2). They are analyzed with an idea of a transformation to the rotating coordinates as employed in Ref.(1) and with the averaging method. Moreover vibrations in the neighborhood of these resonances are also investigated in detail. For convenience sake, the region of frequency where a resonance and a vibration near it occur is called the region of resonance hereafter.

2. Steady Solution in the Region
of Resonance of Order 1/3
and the Stability

The system considered is idealized by the same model as in Ref.(1). Namely self-excitation is described by a non-linear resistance of Van der Pol type. Parametric excitation is described by a restoring force expressed by the product of a non-linear function of deflection and a periodic function of time. Then we obtain a non-dimensional equation of motion as follows:

$$\ddot{y} - (\beta - \delta y^2)\dot{y} + (1 + \gamma y^2)(1 + \mu \cos 2\eta\tau)y = 0;$$

$$\dot{y} = dy/d\tau \quad \dots\dots\dots (1)$$

where the parameters β , δ , γ and μ are quite smaller than unity.

In the region of resonance of order 1/3, we consider a phase trajectory of Eq. (1) in the plane $y-3\dot{y}/\eta$ and express Eq. (1) by a set of differential equations for y_1 and y_2 ($y_1 = y$, $y_2 = 3\dot{y}/\eta$).

$$\left. \begin{aligned} \dot{y}_1 &= \frac{\eta}{3}y_2 \\ \dot{y}_2 &= (\beta - \delta y_1^2)y_2 - \frac{3}{\eta}(1 + \gamma y_1^2)(1 + \mu \cos 2\eta\tau)y_1 \end{aligned} \right\} \dots\dots\dots (2)$$

In the same way as in Ref.(1) the solution is assumed as follows:

$$\left. \begin{aligned} y_1 &= \xi \cos \frac{\eta}{3}\tau + \zeta \sin \frac{\eta}{3}\tau \\ &\quad + \Xi \cos \eta\tau + Z \sin \eta\tau \\ y_2 &= -\xi \sin \frac{\eta}{3}\tau + \zeta \cos \frac{\eta}{3}\tau \\ &\quad - 3\Xi \sin \eta\tau + 3Z \cos \eta\tau \end{aligned} \right\} \dots\dots\dots (3)$$

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Fig.1 Coordinate systems in the region of resonance of order 1/3

Solution (3) implies a transformation from the plane $y-3\dot{y}/\eta$ to the coordinate systems as shown in Fig.1. Figure 1 shows a new coordinate system $\xi-\zeta$ rotating with an angular velocity $\eta/3$ whose origin (Ξ, Z) moves on an ellipse trajectory with an angular velocity η . Substituting Eq.(3) into Eq.(2) and applying the averaging method, differential equations for ξ, ζ, Ξ and Z are obtained.

$$\begin{aligned} \dot{\xi} = & \frac{1}{2} \left\{ \beta - \frac{\delta}{4} (A_{1/3}^2 + 2A_1^2) \right\} \xi \\ & + \frac{3}{2\eta} \left\{ 1 - \left(\frac{\eta}{3} \right)^2 + \frac{3\gamma}{4} (A_{1/3}^2 + 2A_1^2) \right\} \\ & + \frac{3\mu\gamma}{4} (\Xi^2 - Z^2) \left\{ \xi - \frac{1}{4} \left\{ \frac{\delta}{2} (\xi^2 - \zeta^2) \right\} \right. \\ & + \frac{9\gamma}{\eta} \left(1 + \frac{\mu}{2} \right) \xi \zeta \left. \right\} \Xi + \frac{1}{4} \left\{ -\delta \xi \zeta \right. \\ & + \frac{9\gamma}{2\eta} \left(1 - \frac{\mu}{2} \right) (\xi^2 - \zeta^2) \left. \right\} Z \dots\dots\dots (4.a) \end{aligned}$$

$$\begin{aligned} \dot{\zeta} = & -\frac{3}{2\eta} \left\{ 1 - \left(\frac{\eta}{3} \right)^2 + \frac{3\gamma}{4} (A_{1/3}^2 + 2A_1^2) \right\} \\ & + \frac{3\mu\gamma}{4} (\Xi^2 - Z^2) \left\{ \xi + \frac{1}{2} \left\{ \beta \right. \right. \\ & - \frac{\delta}{4} (A_{1/3}^2 + 2A_1^2) \left. \right\} \zeta + \frac{1}{4} \left\{ \delta \xi \zeta \right. \\ & - \frac{9\gamma}{2\eta} \left(1 + \frac{\mu}{2} \right) (\xi^2 - \zeta^2) \left. \right\} \Xi - \frac{1}{4} \left\{ \frac{\delta}{2} (\xi^2 - \zeta^2) \right. \\ & + \frac{9\gamma}{\eta} \left(1 - \frac{\mu}{2} \right) \xi \zeta \left. \right\} Z \dots\dots\dots (4.b) \end{aligned}$$

$$\begin{aligned} \dot{\Xi} = & \frac{\delta}{24} (3\zeta^2 - \xi^2) \xi + \frac{\gamma}{8\eta} (3\xi^2 - \zeta^2) \left(1 - \frac{\mu}{2} \right) \xi \\ & + \frac{1}{2} \left\{ \beta - \frac{\delta}{4} (2A_{1/3}^2 + A_1^2) \right\} \Xi \\ & + \frac{1}{2\eta} \left\{ 1 - \eta^2 + \frac{3\gamma}{4} (2A_{1/3}^2 + A_1^2) \right. \\ & - \frac{\mu}{2} - \frac{\mu\gamma}{4} (3A_{1/3}^2 + 2Z^2) \left. \right\} Z \dots\dots\dots (4.c) \end{aligned}$$

$$\begin{aligned} \dot{Z} = & -\frac{\gamma}{8\eta} (\xi^2 - 3\zeta^2) \left(1 + \frac{\mu}{2} \right) \xi - \frac{\delta}{24} (3\xi^2 - \zeta^2) \zeta \\ & - \frac{1}{2\eta} \left\{ 1 - \eta^2 + \frac{3\gamma}{4} (2A_{1/3}^2 + A_1^2) + \frac{\mu}{2} \right. \\ & + \frac{\mu\gamma}{4} (3A_{1/3}^2 + 2\Xi^2) \left. \right\} \Xi \\ & + \frac{1}{2} \left\{ \beta - \frac{\delta}{4} (2A_{1/3}^2 + A_1^2) \right\} Z \dots\dots\dots (4.d) \end{aligned}$$

where $A_{1/3}^2 = \xi^2 + \zeta^2$, $A_1^2 = \Xi^2 + Z^2$.

Equations (4) contain the following four kinds of steady solutions.

- (i) A trivial solution ($\xi = \zeta = \Xi = Z = 0$)
- (ii) A semi-trivial solution ($\xi = \zeta = 0$)
- (iii) A semi-trivial solution ($\Xi = Z = 0$)
- (iv) A non-trivial solution

We shall consider the possibility of occurrence and the stability in each case.

2.1 The trivial steady solution

Only the stability is investigated. Introducing small disturbances from steady solutions, each characteristic equation for Eqs.(4.a), (4.b) and Eqs.(4.c), (4.d) is obtained as follows:

$$\left. \begin{aligned} \lambda^2 - \beta\lambda + \left(\frac{1}{2}\beta \right)^2 + \left(\frac{3}{2\eta} \right)^2 \left\{ 1 - \left(\frac{\eta}{3} \right)^2 \right\}^2 &= 0 \\ \lambda^2 - \beta\lambda + \left(\frac{1}{2}\beta \right)^2 + \left(\frac{1}{2\eta} \right)^2 \left\{ (1 - \eta^2)^2 - \frac{\mu^2}{4} \right\} &= 0 \end{aligned} \right\} \dots\dots\dots (5)$$

Thus the trivial solution is always unstable because the parameter β is positive.

2.2 The semi-trivial steady solution [(ii) $\xi = \zeta = 0$]

This solution evidently coincides with a resonance of order 1 with a frequency η in Ref. (1). Then a characteristic equation for Eqs.(4.a) and (4.b) is given by

$$\begin{aligned} \lambda^2 - \left(\beta - \frac{\delta}{2} A_{1/3}^2 \right) \lambda + \frac{1}{4} \left(\beta - \frac{\delta}{2} A_{1/3}^2 \right)^2 \\ + \left(\frac{3}{2\eta} \right)^2 \left\{ 1 - \left(\frac{\eta}{3} \right)^2 + \frac{3\gamma}{2} A_{1/3}^2 \right. \\ \left. + \frac{3\mu\gamma}{4} (\Xi^2 - Z^2) \right\}^2 &= 0 \dots\dots\dots (6) \end{aligned}$$

Solutions $\xi = \zeta = 0$ are stable if an inequality $A_{1/3}^2 > 2\beta/\delta$ holds.

2.3 The semi-trivial steady solution [(iii) $\Xi = Z = 0$]

When $\Xi = Z = 0$, differential equations (4.a) and (4.b) are reduced to

$$\left. \begin{aligned} \dot{\xi} = & \frac{1}{2} \left(\beta - \frac{\delta}{4} A_{1/3}^2 \right) \xi + \frac{3}{2\eta} \left\{ 1 - \left(\frac{\eta}{3} \right)^2 + \frac{3\gamma}{4} A_{1/3}^2 \right\} \zeta \\ \dot{\zeta} = & -\frac{3}{2\eta} \left\{ 1 - \left(\frac{\eta}{3} \right)^2 + \frac{3\gamma}{4} A_{1/3}^2 \right\} \xi \\ & + \frac{1}{2} \left(\beta - \frac{\delta}{4} A_{1/3}^2 \right) \zeta \end{aligned} \right\} \dots\dots\dots (7)$$

Then the steady solution (resonance curve) is given by

$$A_{1/3}^2 = 4\beta/\delta, \quad (\eta/3)^2 = 1 + 3\gamma\beta/\delta \dots\dots\dots (8)$$

Since Eq. (8) means a point and contradicts the actual phenomenon, this semi-trivial solution does not hold.

2.4 The non-trivial steady solution

This solution is expected to describe a resonance of order 1/3. Since this solution cannot be easily obtained, we investigate it by the trajectories of Eq. (4) in the phase plane.

3. Solutions in the Neighborhood of Resonance of Order 1/3

In the neighborhood of this resonance, the solution of Eq.(1) by Runge-Kutta-Gill (R.K.G.) method has a vibration with beat character. We consider its solution according to the classification as in Chapt.2.

3.1 A vibration derived from the semi-trivial solution ($\xi=\zeta=0$)

This solution coincides with the solution in the neighborhood of resonance of order 1 analyzed in Ref.(1).

3.2 A vibration derived from the semi-trivial solution ($\bar{\xi}=\bar{\zeta}=0$)

Since a semi-trivial steady solution is not available, this solution cannot hold.

3.3 A vibration derived from the non-trivial solution

This solution is expected to describe a vibration in the neighborhood of resonance of order 1/3. We investigate it by the trajectories of Eq.(4) in both planes $\xi-\zeta$ and $\bar{\xi}-\bar{\zeta}$ as in Sec.2.4.

4. Steady Solution in the Region of Resonance of Order 2 and the Stability

A parametric resonance of order 2 is also expected to occur at the frequencies corresponding to the unstable region of order 2 in a linear Mathieu equation as in Ref.(3). This resonance is considered to have a frequency 2η and a constant component. So in the same way as in Ref.(3), we shall express Eq.(1) as differential equations for y_1 and y_2 where $y_1=y$ and $y_2=\dot{y}/2\eta$

$$\begin{cases} \dot{y}_1 = 2\eta y_2 \\ \dot{y}_2 = (\beta - \delta y_1^2)y_2 - \frac{1}{2\eta}(1 + \gamma y_1^2)(1 + \mu \cos 2\eta\tau)y_1 \end{cases} \quad (9)$$

and then assume the solutions as follows:

$$\begin{cases} y_1 = \xi \cos 2\eta\tau + \zeta \sin 2\eta\tau + z_1(\tau) \\ y_2 = -\xi \sin 2\eta\tau + \zeta \cos 2\eta\tau + z_2(\tau) \end{cases} \quad (10)$$

Applying the averaging method and assuming that the components z_1^2 and z_2^2 are much smaller than the component A^2 ($=\xi^2 + \zeta^2$), we obtain a set of differential equations for ξ , ζ , z_1 and z_2 .

$$\begin{aligned} \dot{\xi} &= \frac{1}{2}\left(\beta - \frac{\delta}{4}A^2\right)\xi + \frac{1}{4\eta}\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\}\zeta \\ &+ \frac{3\mu\gamma}{8\eta}\xi\zeta z_1 \quad (11.a) \end{aligned}$$

$$\begin{aligned} \dot{\zeta} &= -\frac{1}{4\eta}\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\}\xi \\ &+ \frac{1}{2}\left(\beta - \frac{\delta}{4}A^2\right)\zeta - \frac{\mu}{4\eta}\left\{1 + \frac{3\gamma}{4}(3\xi^2 + \zeta^2)\right\}z_1 \quad (11.b) \end{aligned}$$

$$\dot{z}_1 = 2\eta z_2 \quad (11.c)$$

$$\dot{z}_2 = -\frac{\mu}{4\eta}\left\{1 + \frac{3\gamma}{4}A^2\right\}\xi - \frac{1}{2\eta}\left\{1 + \frac{3\gamma}{4}A^2\right\}z_1$$

$$+ \left(\beta - \frac{\delta}{2}A^2\right)z_2 \quad (11.d)$$

Using the relations $\xi = A \cos \varphi$ and $\zeta = A \sin \varphi$, differential equations in the polar coordinates (A , φ) are obtained.

$$\dot{A} = \frac{1}{2}\left(\beta - \frac{\delta}{4}A^2\right)A - \frac{\mu z_1}{4\eta}\left(1 + \frac{3\gamma}{4}A^2\right)\sin \varphi \quad (12)$$

$$\begin{aligned} \dot{\varphi} &= -\frac{1}{4\eta}\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\} \\ &- \frac{\mu z_1}{4\eta A}\left(1 + \frac{9\gamma}{4}A^2\right)\cos \varphi \quad (13) \end{aligned}$$

Equations (11.c) and (11.d) are reduced to

$$\begin{aligned} \dot{z}_1 &+ \left(\frac{\delta}{2}A^2 - \beta\right)z_1 + \left(1 + \frac{3\gamma}{4}A^2\right)z_1 \\ &= -\frac{\mu}{2}\left(1 + \frac{3\gamma}{4}A^2\right)\xi \quad (14) \end{aligned}$$

Steady solutions of Eqs.(12) to (14) are given by

$$\begin{aligned} \eta^4 &+ \left\{\frac{1}{4}\frac{\left(\beta - \frac{\delta}{4}A^2\right)^2\left(1 + \frac{9\gamma}{4}A^2\right)^2}{\left(1 + \frac{3\gamma}{4}A^2\right)^2} - \frac{1}{2}\left(1 + \frac{3\gamma}{4}A^2\right)\right. \\ &+ \left.\frac{\mu^2}{8}\frac{\left(1 + \frac{3\gamma}{4}A^2\right)\left(1 + \frac{9\gamma}{4}A^2\right)}{1 + \frac{3\gamma}{2}A^2}\right\}\eta^2 \\ &+ \frac{1}{16}\left(1 + \frac{3\gamma}{4}A^2\right)^2 \\ &- \frac{\mu^2}{32}\frac{\left(1 + \frac{3\gamma}{4}A^2\right)^2\left(1 + \frac{9\gamma}{4}A^2\right)}{1 + \frac{3\gamma}{2}A^2} = 0 \quad (15) \end{aligned}$$

and by

$$z_1^2 = \frac{A^2}{2} \frac{\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\}\left(1 + \frac{3\gamma}{4}A^2\right)}{\left(1 + \frac{9\gamma}{4}A^2\right)\left(1 + \frac{3\gamma}{2}A^2\right)} \quad (16)$$

Then a resonance curve is shown by $A + z_1$ and $A - z_1$ where z_1 is a constant component. The phase φ is determined by

$$\tan \varphi = -2\eta \frac{\left(\beta - \frac{\delta}{4}A^2\right)\left(1 + \frac{9\gamma}{4}A^2\right)}{\left(1 + \frac{3\gamma}{4}A^2\right)\left\{1 - (2\eta)^2 + \frac{3\gamma}{4}A^2\right\}} \quad (17)$$

When the subscript s stands for a steady solution, ξ_s and ζ_s components are given by $\xi_s = A_s \cos \varphi_s$ and $\zeta_s = A_s \sin \varphi_s$.

The stability of steady solutions (ξ_s , ζ_s , z_{1s}) is analyzed in the same way as in Chapt.2. According to the theorem of Routh-Hurwitz, stability criteria are as follows:

$$\begin{cases} \text{i) } \beta - \frac{\delta}{2}A_s^2 - c\gamma - f\eta < 0 \\ \text{ii) } ae - bd + (cd - af)h + g(bf - ce) > 0 \end{cases} \quad (18)$$

where

$$\begin{aligned}
a &= \frac{1}{2} \left\{ \beta - \frac{\delta}{4} (A_0^2 + 2\xi_s^2) + \frac{3\gamma}{4\eta} (\xi_s \zeta_s + \mu z_{1s} \zeta_s) \right\}, \\
b &= \frac{1}{4\eta} \left\{ 1 - (2\eta)^2 + \frac{3\gamma}{4} (A_0^2 + 2\xi_s^2 + 2\mu z_{1s} \xi_s) \right\} - \frac{\delta}{4} \xi_s \zeta_s, \\
c &= \frac{3\mu\gamma}{8\eta} \xi_s \zeta_s, \\
d &= -\frac{1}{4\eta} \left\{ 1 - (2\eta)^2 + \frac{3\gamma}{4} (A_0^2 + 2\xi_s^2 + 6\mu z_{1s} \xi_s) \right\} - \frac{\delta}{4} \xi_s \zeta_s, \\
e &= \frac{1}{2} \left\{ \beta - \frac{\delta}{4} (A_0^2 + 2\xi_s^2) - \frac{3\gamma}{4\eta} (\xi_s \zeta_s + \mu z_{1s} \zeta_s) \right\}, \\
f &= -\frac{\mu}{4\eta} \left\{ 1 + \frac{3\gamma}{4} (3\xi_s^2 + \zeta_s^2) \right\}, \\
g &= \left\{ 3\gamma \xi_s z_{1s} + \frac{\mu}{2} \left(1 + \frac{3\gamma}{4} A_0^2 + \frac{3\gamma}{2} \xi_s^2 \right) \right\} / \left(1 + \frac{3\gamma}{2} A_0^2 \right), \\
h &= 3\gamma \left(\xi_s z_{1s} + \frac{\mu}{4} \xi_s \zeta_s \right) / \left(1 + \frac{3\gamma}{2} A_0^2 \right)
\end{aligned}$$

It is seen that the trivial solution $\xi_s = \zeta_s = z_{1s} = 0$ is always unstable.

5. Solutions in the Neighborhood of Resonance of Order 2

In the neighborhood of this resonance Eq. (1) by R.K.G. method has a vibration with beat character. We shall obtain the maximum and minimum amplitudes of this vibration by approximating another kind of solution to Eq. (11), i.e. a limit cycle as shown in Fig. 14.

For the convenience sake of analysis, we use Eqs. (12) and (13) expressed in the coordinates (A^2, φ) by the relation $d(A^2) = 2A dA$. Now let us put $A^2 = A_0^2 + \Delta A^2$, where a constant component $A_0^2 = 4\beta/\delta$ and ΔA^2 is a variable one as in Ref. (1). Assuming that $A_0^2 \gg |\Delta A^2|$ and $A_0 \gg |z_1|$ and ignoring small terms of higher order, we obtain

$$\frac{d(\Delta A^2)}{d\tau} = -\beta \Delta A^2 - \frac{\mu A_0}{2\eta} \left(1 + \frac{3\gamma}{4} A_0^2 \right) z_1 \sin \varphi, \quad (19)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{4\eta} \left\{ 1 - (2\eta)^2 + \frac{3\gamma}{4} A_0^2 \right\}. \quad (20)$$

Then from Eq. (20) the phase is approximated by

$$\varphi = Q_0 \tau + \varphi_0; \quad Q_0 = -\frac{1}{4\eta} \left\{ 1 - (2\eta)^2 + \frac{3\gamma\beta}{\delta} \right\} \quad (21)$$

where φ_0 is an initial phase angle.

On the other hand, assuming that $A \approx A_0$ and $\xi = A_0 \cos(Q_0 \tau + \varphi_0)$ in Eq. (14), we obtain a typical equation of forced vibration with a damping. Then a steady solution with z_1 is given by

$$z_1 = R_1 \cos(Q_0 \tau + \varphi_0) + R_2 \sin(Q_0 \tau + \varphi_0) \quad (22)$$

$$\text{where } R_1 = \frac{P(W - Q_0^2)}{(W - Q_0^2)^2 + \beta^2 Q_0^2},$$

$$R_2 = \frac{P\beta Q_0}{(W - Q_0^2)^2 + \beta^2 Q_0^2},$$

$$P = -\frac{\mu}{2} \left(1 + \frac{3\gamma}{4} A_0^2 \right) A_0, \quad W = 1 + \frac{3\gamma}{2} A_0^2$$

Substituting Eq. (22) into Eq. (19), a steady solution of ΔA^2 is obtained.

$$\begin{aligned}
\Delta A^2 &= a_0 + a_1 \cos(2Q_0 \tau + \varphi_0) \\
&\quad + b_1 \sin(2Q_0 \tau + \varphi_0); \\
a_0 &= -\frac{c}{\beta} R_2, \quad a_1 = \frac{c(\beta R_2 + 2Q_0 R_1)}{4Q_0^2 + \beta^2}, \\
b_1 &= \frac{c(2Q_0 R_2 - \beta R_1)}{4Q_0^2 + \beta^2}, \quad c = \frac{\mu A_0}{4\eta} \left(1 + \frac{3\gamma}{4} A_0^2 \right)
\end{aligned} \quad (23)$$

From Eqs. (22) and (23) a beat vibration is described as

$$\begin{aligned}
y &= A(\tau) \cos((2\eta - Q_0)\tau - \varphi_0) \\
&\quad + R \cos(Q_0 \tau + \varphi_0 - \psi); \\
A(\tau) &= \sqrt{\frac{4\beta}{\delta} + a_0 + a_1 \cos 2(Q_0 \tau + \varphi_0) + b_1 \sin 2(Q_0 \tau + \varphi_0)}, \\
R &= \frac{P}{\sqrt{(W - Q_0^2)^2 + \beta^2 Q_0^2}}, \quad \tan \psi = \frac{R_2}{R_1} = \frac{\beta Q_0}{W - Q_0^2}
\end{aligned} \quad (24)$$

Accordingly the amplitudes of a beat will be restricted by

$$\begin{aligned}
&A_{\max} + R_1 \quad \text{and} \quad A_{\min} - R; \\
A_{\max} &= \sqrt{\frac{4\beta}{\delta} + a_0 + \sqrt{a_1^2 + b_1^2}}, \\
A_{\min} &= \sqrt{\frac{4\beta}{\delta} + a_0 - \sqrt{a_1^2 + b_1^2}}
\end{aligned} \quad (25)$$

6. Numerical Results and Discussions

Let us compare approximate solutions with numerical solutions of Eq. (1) by R.K.G. method. Numerical calculation is performed for the same parameters as in Ref. (1).

case I: $\beta = 0.05$, $\delta = 0.02$, $\gamma = \mu = 0.2$

Now we designate the extreme values of deflection by a symbol $[y]$, which concretely shows the amplitudes of resonances and beats.

6.1 Behaviors in the region of resonance of order 1/3

We shall investigate resonance phenomena of order 1/3 by differential equations (4) at $\eta = 4.6$. Figure 2(a) shows the trajectories in the plane $\xi - \zeta$ and Figs. (b) and (c) show those in the plane $\bar{E} - Z$ corresponding to the points A and B, respectively. Singular points A, C and E in Fig. (a) have such small components with frequencies η as shown in Fig. (b) and singular points B, D and F have ones as shown in Fig. (c). It is ascertained that a resonance of order 1/3 is described by the non-trivial solution (iv) of Eq. (4). Then a waveform of resonance is shown in Fig. 3(c) together with the components $\eta/3$ [Fig. (a)] and η [Fig. (b)] for point A. Since $\eta/3$ component is much smaller than η one, the waveform of resonance looks like harmonic. In contrast, a waveform of Eq. (1) by R.K.G. method is shown in Fig. 4. It is

recognized that both waveforms have the same appearance. For the sake of comparison with Fig.2, the trajectories of resonance in the stroboscopic phase plane(1)

(a) In the plane $\xi - \zeta$

(b) In the plane $\xi - \zeta$ (Point A)
[$\xi(0) = 2.68$, $\zeta(0) = 0$]

(c) In the plane $\xi - \zeta$ (Point B)
[$\xi(0) = 2.5$, $\zeta(0) = 0$]

Fig.2 Trajectories of Eq. (4) at $\eta = 4.6$
[$\xi(0) = 0.1$, $\zeta(0) = 0$]

Fig.6 Diagram $[y] - \eta$ in the region of resonance of order 1/3

(a) $\eta/3$ component

(b) η component

(c) Waveform of resonance of order 1/3

Fig.3 Waveforms of Eq. (4) at $\eta = 4.6$
(Point A in Fig.2)
[$\xi(0) = 2.6$, $\zeta(0) = 1.4$,
 $\xi(0) = 0.04$, $\zeta(0) = 0.06$]

Fig.4 Waveform of resonance of Eq. (1)
by R.K.G. method at $\eta = 4.6$
[$y(0) = 2.6$, $\dot{y}(0) = 2.6$]

Fig.5 Trajectories in the stroboscopic phase plane at $\eta = 4.6$
(F_s : Stable focus, F_u : Unstable focus, SP : Saddle point)

(a) In the plane $\xi - \zeta$ (b) In the plane $\xi - Z$

Fig.7 Trajectories of Eq. (4) at $\eta = 4.75$ in the neighborhood of resonance of order $1/3$ [$\xi(0) = 3$, $\zeta(0) = 0$, $\Xi(0) = 0.1$, $Z(0) = 0$]

(a) $\eta/3$ component (b) η component

(c) waveform of vibration

Fig.9 Waveforms of Eq. (4) at $\eta = 4.75$ [$\xi(0) = 3$, $\zeta(0) = 0$, $\Xi(0) = 0.1$, $Z(0) = 0$]

are shown in Fig.5. Figure 6 shows a diagram $[y] - \eta$ in the region of resonance of order $1/3$, where a resonance is plotted by the amplitude $A_{1/3} + A_1$ and a beat is by the maximum and minimum amplitudes. Marks $\circ$ and Δ denote amplitudes of Eqs. (1) and (4) by R.K.G. method. Vertical lines connecting two small circles represent frequencies on the boundaries of existence of beat vibrations, so that only a resonance actually occurs within the frequencies bounded by them. Thus it is seen that Eq. (4) itself describes the behaviors in the region of this resonance with high accuracy.

(a) Eq. (4) [$\xi(0) = 3$, $\zeta(0) = 0$, $\Xi(0) = 0.1$, $Z(0) = 0$] (b) Eq. (1) by R.K.G. method [$y(0) = 3.2$, $\dot{y}(0) = 0$]

Fig.8 Trajectories in the plane $\xi - \zeta$ rotating with an angular velocity $\eta/3$ at $\eta = 4.75$

Fig.10 Waveform of Eq. (1) by R.K.G. method at $\eta = 4.75$ [$y(0) = 3.2$, $\dot{y}(0) = 0$]

Next we investigate the trajectories of Eq. (4) at $\eta = 4.75$ in the neighborhood of resonance of order $1/3$. Figure 7 shows that each limit cycle exists in both planes $\xi - \zeta$ and $\Xi - Z$ and suggests that a vibration corresponding to the non-trivial solution occurs. Figure 8 shows trajectories of Eq. (4) and Eq. (1) by R.K.G. method in the coordinate system $\xi - \zeta$ rotating with an angular velocity $\eta/3$. It is recognized that their behaviors are quite similar. Figure 9(c) shows a waveform of vibration together with waveforms of the components $\eta/3$ and η . This pattern of vibration is qualitatively similar to that of Eq. (1) by R.K.G. method in Fig. 10. An apparent character of beat is not recognized in the neighborhood of this resonance.

6.2 Behaviors in the region of resonance of order 2

Firstly a diagram $[y] - \eta$ in this region is shown in Fig. 11, where marks $\circ$ and two vertical lines have the same meaning as in Fig. 6. Marks Δ outside the resonance denote the maximum and minimum amplitudes of Eq. (11). It is seen that the amplitudes of resonance have high accuracy and the actual maximum and minimum amplitudes of a beat are restricted by two curves of Eq. (25). A diagram $|z_1| - \eta$ and a frequency characteristic of resonance are shown in Fig. 12, respectively. Figure 13 shows waveforms of resonance by Eq. (11) and Eq. (1) by R.K.G. method at $\eta = 0.78$. It is ascertained that a resonance of order 2 of system (1) has the same character as that of the system in Ref. (3).

Next we investigate a vibration at $\eta = 0.86$ in the neighborhood of resonance. Figure 14 shows trajectories of Eq. (11) in the planes $\xi - \zeta$ and $z_1 - z_2$, where a trajectory after non-dimensional time $\tau = 150$ is plotted in the latter plane. It is seen that a limit cycle is in existence in each plane. Figure 15(c) shows a waveform by Eq. (24) together with waveforms of the first and second terms. A waveform of

Fig. 11 Diagram $[y] - \eta$ in the region of resonance of order 2

(a) Diagram $|z_1| - \eta$

(b) Frequency characteristic of resonance vector

Fig. 12 Resonance characteristics of order 2

(a) Eq. (11) $[\xi(0) = 3.2, \zeta(0) = 0, z_1(0) = -0.2, z_2(0) = 0]$

(b) Eq. (1) by R.K.G. method $[y(0) = 3, \dot{y}(0) = 0]$

Fig. 13 Waveforms of resonance of order 2 at $\eta = 0.78$

(a) In the plane $\xi - \zeta$ (b) In the plane $z_1 - z_2$
(after $\tau = 150$)

Fig.14 Trajectories of Eq.(11) at $\eta = 0.86$ in the neighborhood of resonance of order 2 [$\xi(0) = 3$, $\zeta(0) = 0$, $z_1(0) = 0.05$, $z_2(0) = 0$]

(a) The first term

(b) The second term $z_1(\tau)$

(c) Waveform of vibration

Fig.15 Waveforms by Eq.(24) at $\eta = 0.86$
($\varphi_0 = 0$)

Eq.(1) by R.K.G. method is shown in Fig.16. A waveform by Eq.(24) has qualitatively high accuracy though the periods of beat are different a little.

6.3 Summary of resonances of orders 2, 1, 1/2 and 1/3

A diagram $[y] - \eta$ in the regions of resonances of orders 2, 1, 1/2 and 1/3 is shown in Fig.17. The results of resonances of orders 1 and 1/2 were reported in Ref. (1). Symbols and lines have the same meaning as in Fig.11. A value $A_0 = \sqrt{48/5}$ is shown there as a common level of amplitudes of beat. It is seen that resonances are analyzed with high accuracy by the averaging method. The beat character near subharmonic resonances of higher orders is not apparently recognized in comparison with that near parametric ones (especially of order 2).

7. Conclusions

In succession to the first report (1), behaviors in the regions of orders 1/3 and 2 are investigated. The summary of the results is as follows:

(1) A subharmonic resonance of order 1/3 is described by the non-trivial steady solution of Eq.(4) with frequencies $\eta/3$ and η . Since η component is much smaller than $\eta/3$ one, the waveform of resonance is regarded as a nearly harmonic one.

(2) A vibration near the resonance of order 1/3, which is described by two limit cycles of Eq.(4), does not have an apparent character of beat. Generally speaking, the beat character is not apparently recognized near subharmonic resonances of higher orders in comparison with parametric ones.

(3) An approximate solution of parametric resonance of order 2 is obtained with high accuracy in the same way as in Ref.(3). A vibration with beat character near the resonance is qualitatively described as Eq.(24).

References

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Fig.16 Waveform of Eq.(1) by R.K.G. method at $\eta = 0.86$
[$y(0) = 3$, $\dot{y}(0) = 0$]

Fig.17 Diagram $[y] - \eta$