Railway Wheel Squeal*

(3rd Report, Squeal of a disk simulating
a wheel in internal resonances)

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We made an apparatus with a steel rod and a thin steel disk. A disk serving as a railway wheel was clamped at inner radius and was free at its end. When a certain relationship exists among the natural frequencies in the axial direction of the disk, various internal resonances occur. We analyzed a squeal of the disk in internal resonances theoretically. Consequently, in internal resonances between two i -th and j -th vibration modes, such as $\omega_j \cong 3\omega_i$ or $\omega_j \cong 5\omega_i$ (ω : natural circular frequency of the disk), squeals with only a frequency of j -th mode and with frequencies of two modes (i -th and j -th modes) occur, but a squeal of a single i -th mode does not occur. If ω_j does not coincide with $3\omega_i$ or $5\omega_i$ completely, a frequency component of i -th mode is predominant in the squeal of two modes. When the internal resonances exist among three modes, a squeal with frequencies of triple modes occurs. However, its generating region is very narrow.

Key Words : Friction, Sound, Vibration of Continuous System, Non-linear Vibration, Squeal, Internal Resonance

1. Introduction

We have analyzed a wheel squeal without taking account of the internal resonances among the natural frequencies in the axial direction of the disk in our previous reports.⁽¹⁾⁽²⁾ It has been made clear that a squeal of a single mode occurs in steady state. However, detailed observations of frequency spectra of squeals revealed that there was a slight vibration of another mode in squeal. And different ratios between outer and inner radii of a disk simulating a wheel are likely to cause several kinds of internal resonances among natural frequencies in the axial direction.

It is very important for reducing the squeal that we investigate what kinds of squeals occur in internal resonances. A study on the non-linear vibration of the disk for internal resonances has been made by Yasuda et al.⁽³⁾ In their study, axisymmetric forced subharmonic vibration of a disk was analyzed, when a uniform initial tension was applied to a disk in radial direction. However, there are few analyses on an autonomous multi-degrees of freedom non-linear vibration in internal resonances except an investigation on multi-degree modes in non-linear electric circuit.⁽⁴⁾

In this report, we analyze the self-excited vibration of a disk which is clamped at inner radius and is free at outer radius and investigate to see if a squeal other than one of only a single mode can be generated.

2. Experimental Results

The experiment was carried out by an apparatus composed of a thin steel disk and a steel rod.⁽¹⁾ One side of a squared rod is pressed against the circumference of the disk with a constant load and a squeal noise is generated when the rod is moved in the axial direction of the disk.

Table 1 The specifications of the disk and the rods tested.

Disk	Outer radius	Inner radius	Thickness
	mm	mm	mm
	150	25	7.5
Steel rod	Length of square section		Length
	mm		mm
	10		600
	40		600
	10		300

Table 2 Axial natural frequencies of the disk.

n	s	
	0	1
2	530 (502)	3320 (3278)
3	1030 (1029)	4440 (4459)
4	1780 (1794)	5920 (6062)
5	2750 (2752)	7660 (7951)
6	3840 (3892)	9590 (10069)
7	5120 (5213)	
8	6530 (6712)	
9	8070 (8388)	
10	9770 (10240)	

() Calculated values Hz

* Received 1st October 1982.

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Table 1 shows the specifications of the disk and the rods tested. Table 2 presents the main axial natural frequencies of the disk obtained by experimentally exciting it. Also, in this table, theoretical natural frequencies calculated by neglecting the effects of shear and rotatory inertia are shown. From Table 2, it is found that internal resonance relations such as $\omega_j \approx 3\omega_i$ ($j > i$, ω : natural circular frequency of a disk) and $\omega_j \approx 5\omega_i$ exist among the axial natural frequencies of the disk. (For example, $\omega_{1-0} \approx 3\omega_{4-0}$, $\omega_{3-0} \approx 3\omega_{5-0}$, $\omega_{7-0} \approx 5\omega_{3-0}$) The centers of different rods were made to rub against the disk and squeals of i -th mode, i.e. ω_{3-0} (1030 Hz), ω_{4-0} (1780 Hz), and ω_{5-0} (2750 Hz) are generated when an internal resonance relation exists between i -th and j -th modes. Then, axial accelerations of the disk were measured by a light piezotype accelerometer.

Figures 1(a), (b) and (c) show these frequency spectra. (80dB = 10 m/s²) In all frequency spectra, a frequency component near j -th mode which internally resonates to i -th mode occurs slightly with a frequency of i -th mode. In Fig. 1(a), a frequency of $5\omega_{3-0}$ near ω_{7-0} occurs besides 1030 Hz and $3\omega_{4-0}$ ($\approx \omega_{7-0}$) in Fig. 1(b) and $3\omega_{5-0}$ ($\approx \omega_{7-0}$) in Fig. 1(c) occur respectively. On the other hand, we examine a squeal of j -th mode which is in internal resonance with i -th mode. For the 10x10x30 rod, both squeals of 1780 Hz and 5120 Hz

which have an internal resonance of $\omega_{7-0} \approx 3\omega_{4-0}$ occur.⁽²⁾ Figure 2 shows a frequency spectrum of 5120 Hz (= squeal of j -th mode) and it can be seen that a frequency component of 1780 Hz does not occur and a squeal of only single j -th mode occurs.

3. Theoretical Discussions

Under the following assumptions, we analyzed the frictional vibration in presence of internal resonance.

(1) As described in the previous report,⁽²⁾ the vibration of the rod has an effect on the input level to the axial vibration of the disk. Thus, neglecting the vibration of the rod, we consider only the frictional vibration of a disk.

(2) Coefficients of friction as a non-linear term can be expressed by the polynomial of the relative sliding velocity v_r .

When a frictional force μP_0 is applied in the axial direction to a circumferential point (r_0, θ_0) of the disk, the equation of motion for the transverse displacement $w(r, \theta, t)$ of a thin disk at the equilibrium position is given by⁽¹⁾

$$\ddot{\xi}_{ns}(t) + C_{ns}\dot{\xi}_{ns}(t) + \omega_{ns}^2 \xi_{ns}(t) = \frac{gP_0 \varphi_{ns}(r_0, \theta_0)}{\gamma h N_{ns}} \{\mu(v_r) - \mu(V)\} \dots (1)$$

where we neglect shear and rotatory inertia, $\xi_{ns}(t)$: unknown function of time t to be determined, n : number of the nodal diameter modes, s : number of the nodal circular modes, P_0 : contact load between a disk and a rod, μ : coefficients of friction expressed as a function of relative sliding velocity $v_r = V - \dot{w}$, V : sliding velocity, w : transverse displacement at the contact point (r_0, θ_0)

$$w = \sum_{n,s} \varphi_{ns}(r_0, \theta_0) \xi_{ns}(t),$$

h : thickness of the disk, C_{ns} : damping coefficient, γ : specific weight, g : acceleration of gravity, r_0, r_i : outer and inner radii of the disk,

$$N_{ns} = \int_0^{2\pi} \int_{r_i}^{r_0} \varphi_{ns}^2(r, \theta) r dr d\theta,$$

$\varphi_{ns}(r, \theta)$: eigenfunctions obtained from the boundary conditions under which the disk is clamped at $r=r_i$ and free at $r=r_0$, that is,

$$\varphi_{ns}(r, \theta) = \{J_n(k_{ns}r) + \bar{B}_{ns}Y_n(k_{ns}r) + \bar{C}_{ns}I_n(k_{ns}r) + \bar{D}_{ns}K_n(k_{ns}r)\} \cos n\theta,$$

$k_{ns}, \bar{B}_{ns}, \bar{C}_{ns}, \bar{D}_{ns}$: real constants,⁽¹⁾ $J_n(k_{ns}r)$, $Y_n(k_{ns}r)$: the first and second kind Bessel functions of n -order, $I_n(k_{ns}r)$, $K_n(k_{ns}r)$: the modified first and second kind Bessel functions of n -order, $\theta_0=0$, since the contact point is an anti-node of vibrational mode at squeal occurrence, $[\cdot]$: derivative with respect to time t . Now, we express coefficient of friction $\mu(v_r)$ in the third order term of v_r as

$$\mu(v_r) = \mu_0 + \mu_1 v_r + \mu_2 v_r^2 \dots (2)$$

Fig.1 Frequency spectra of acceleration in squeals of i -th mode.

Fig.2 Frequency spectrum of acceleration in squeal of j -th mode.

where μ_0 , μ_1 and μ_2 are constants.

Considering n from 0 to n_0 , s from 1 to s_0 , equation (1) leads to the equations with $N=(n_0+1)s_0$ degrees of freedom. Replacing the variables in Eq.(1) as follows:

$$\begin{aligned}\xi_i^* &= \xi_{ns}^*(t), \quad \varphi_i = \varphi_{ns}, \quad \bar{\omega}_i = \omega_{ns}, \\ D_i &= gP_0\varphi_{ns}/\gamma h N_{ns}, \quad C_i = C_{ns} \dots \dots \dots (3)\end{aligned}$$

Equation (1) can be rewritten as

$$\begin{aligned}\ddot{\xi}_i^* + \bar{\omega}_i^2 \xi_i^* &= D_i \left\{ \mu \left(V - \sum_{m=1}^N \varphi_m \xi_m^* \right) \right. \\ &\quad \left. - \mu(V) \right\} - C_i \xi_i^* \quad (i=1, \dots, N) \dots \dots (4)\end{aligned}$$

Furthermore, by putting

$$\tau = \omega_0 t, \quad v = V/\omega_0, \quad x_i = \varphi_i \omega_0 \sqrt{-\frac{\mu_2}{(\mu_1 + 3\mu_2 \omega_0^2 v^2)}} \xi_i^*$$

in Eq.(4), we obtain

$$\begin{aligned}x_i'' + \omega_i^2 x_i &= \epsilon B_i \left\{ \sum_{m=1}^N x_m' - A^* \left(\sum_{m=1}^N x_m' \right)^2 \right. \\ &\quad \left. - \left(\sum_{m=1}^N x_m' \right)^3 + C_i^* x_i' \right\} \dots \dots \dots (5)\end{aligned}$$

where $[\cdot]$: derivative with respect to τ , ω_0 : standard natural circular frequency, $\omega_i = \bar{\omega}_i/\omega_0$, $B_i = D_i \varphi_i / D_0 \varphi_0$, D_0, φ_0 : values of D_i and φ_i at ω_0 ,

$$\begin{aligned}\epsilon &= -\frac{D_0 \varphi_0 (\mu_1 + 3\mu_2 \omega_0^2 v^2)}{\omega_0}, \\ C_i^* &= \frac{C_i}{D_i \varphi_i (\mu_1 + 3\mu_2 \omega_0^2 v^2)}, \\ A^* &= 3\omega_0 v \sqrt{-\frac{\mu_2}{(\mu_1 + 3\mu_2 \omega_0^2 v^2)}}\end{aligned}$$

In Eq.(5), the condition $\epsilon > 0$ must hold since $B_i > 0$ in order that the self-excited vibration occurs. By using the averaging method in Eq.(5), the following four kinds of internal resonances are generated when coefficient of friction is expressed as Eq.(2).

- (i) $\omega_j \approx 2\omega_i$, (ii) $\omega_j \approx 3\omega_i$,
- (iii) $\omega_k - \omega_j \approx 2\omega_i$, $\omega_k - \omega_i \approx 2\omega_j$, $\omega_k + \omega_i \approx 2\omega_j$,
- (iv) $\omega_i + \omega_j \approx \omega_k$,

Fig.3 Axial natural frequencies of the disk for various values of r_i/r_0 .

where $i < j < k$.

We examine if these internal resonances exist according to r_i/r_0 . Figures 3 and 4 present the calculated results of non-dimensional quantities $(\lambda r_0)^2$ ($\lambda^2 = 12\gamma(1-\nu^2)\omega_{ns}^2/g h^2 E$, ν : Poisson's ratio, E : Young's modulus) versus r_i/r_0 about natural frequencies of nodal diameter modes of which squeals occur easily. Furthermore, to investigate the internal resonance, these figures show $2\omega_i, 3\omega_i, (\omega_k - \omega_j)/2$ and so on. The values of r_i/r_0 are 0.15-0.35 in actual railway wheel. It is evident that there are possibilities that internal resonances of (ii)-(iv) except for case (i) occur for r_i/r_0 less than 0.4.

3.1 Case of $\omega_j \approx 3\omega_i$

Neglecting high order terms of ϵ , ω_i and ω_j can be written as follows:⁽⁴⁾

$$\begin{aligned}\omega_i &= p_{i0} + \epsilon p_{i1} \\ \omega_j &= p_{j0} + \epsilon p_{j1}\end{aligned} \dots \dots \dots (6)$$

where p_{i0} and p_{j0} are related by $p_{j0} = 3p_{i0}$. Substituting Eq.(6) into Eq.(5) for i -th and j -th modes related to internal resonance and neglecting high order terms, we obtain

$$\begin{aligned}x_i'' + p_{i0}^2 x_i &= \epsilon B_i \left\{ \sum_{m=1}^N x_m' - A^* \left(\sum_{m=1}^N x_m' \right)^2 \right. \\ &\quad \left. - \left(\sum_{m=1}^N x_m' \right)^3 + C_i^* x_i' \right\} - 2\epsilon p_{i0} p_{j1} x_i \\ x_j'' + p_{j0}^2 x_j &= \epsilon B_j \left\{ \sum_{m=1}^N x_m' - A^* \left(\sum_{m=1}^N x_m' \right)^2 \right. \\ &\quad \left. - \left(\sum_{m=1}^N x_m' \right)^3 + C_j^* x_j' \right\} - 2\epsilon p_{j0} p_{i1} x_j \\ x_l'' + p_{l0}^2 x_l &= \epsilon B_l \left\{ \sum_{m=1}^N x_m' - A^* \left(\sum_{m=1}^N x_m' \right)^2 \right. \\ &\quad \left. - \left(\sum_{m=1}^N x_m' \right)^3 + C_l^* x_l' \right\} \\ &\quad (l=1, \dots, N, l \neq i \neq j)\end{aligned} \dots \dots \dots (7)$$

Now, assuming that the solutions of Eq.(7) are of the form

$$\begin{aligned}x_i &= a_i \sin(p_{i0}\tau + \theta_i), \\ x_j &= a_j \sin(p_{j0}\tau + \theta_j), \\ x_l &= a_l \sin(p_{l0}\tau + \theta_l)\end{aligned} \dots \dots \dots (8)$$

where a_i, a_j, a_l : amplitudes, $\theta_i, \theta_j, \theta_l$: phase angles, and using the averaging method for Eq.(7),

Fig.4 Axial natural frequencies of the disk for various values of r_i/r_0 .

we have

$$\begin{aligned}\frac{da_i}{d\tau} &= \frac{\epsilon B_i}{2p_{i0}} \left\{ a_i p_{i0} D_i^* - \frac{3}{4} a_i^3 p_{i0}^3 \right. \\ &\quad \left. - \frac{3}{2} a_i p_{i0} \sum_{m=1}^N a_m^2 p_{m0}^2 - \frac{3}{4} a_i^3 p_{i0}^3 \cos \Theta_i \right\} \\ \frac{da_j}{d\tau} &= \frac{\epsilon B_j}{2p_{j0}} \left\{ a_j p_{j0} D_j^* - \frac{3}{4} a_j^3 p_{j0}^3 \right. \\ &\quad \left. - \frac{3}{2} a_j p_{j0} \sum_{m=1}^N a_m^2 p_{m0}^2 - \frac{1}{4} a_j^3 p_{j0}^3 \cos \Theta_j \right\} \\ \frac{da_i}{d\tau} &= \frac{\epsilon B_i}{2p_{i0}} \left\{ a_i p_{i0} D_i^* - \frac{3}{4} a_i^3 p_{i0}^3 \right. \\ &\quad \left. - \frac{3}{2} a_i p_{i0} \sum_{m=1}^N a_m^2 p_{m0}^2 \right\} \dots \dots \dots (9)\end{aligned}$$

where $D_i^* = 1 + C_i^*$, $\Theta_i = 3\theta_i - \theta_j$, $p_{m0} = \omega_m$ ($m=1, \dots, N, m \neq i, j$)

$$\begin{aligned}a_i \frac{d\theta_i}{d\tau} &= \frac{\epsilon}{2p_{i0}} \left\{ 2a_i p_{i0} p_{i1} + \frac{3}{4} B_i a_i^3 p_{i0}^3 \sin \Theta_i \right\} \\ a_j \frac{d\theta_j}{d\tau} &= \frac{\epsilon}{2p_{j0}} \left\{ 2a_j p_{j0} p_{j1} - \frac{1}{4} B_j a_j^3 p_{j0}^3 \sin \Theta_j \right\} \\ a_i \frac{d\theta_i}{d\tau} &= 0\end{aligned}$$

.....(10)

The steady-state solutions of Eqs.(9) and (10) are obtained from equations which put the right-hand sides of Eqs.(9) and (10) as equal to zero, (these equations are expressed as Eq.(9') and Eq.(10')) and are classified, i.e.,

- (a) (0, 0, 0, ..., 0)
 - (b) (0, a_i , 0, ..., 0)
 - (c) (0, 0, 0, 0, ..., 0, a_j , 0, 0) under $p_{j1}=0$
 - (d) (0, 0, 0, a_i , 0, ..., 0, a_j , 0, 0)
 - (e) (i) $\overbrace{(a_1, a_2, 0, \dots, a_i, 0, \dots, a_j, a_{j+1}, a_n)}^n$
 n ($3 \leq n \leq N$) solutions including a_i and a_j
 - (ii) $\overbrace{(a_1, a_2, 0, \dots, 0, 0, \dots, 0, a_{j+1}, a_n)}^n$
 n ($2 \leq n \leq N$) solutions in the absence of a_i and a_j
-(11)

Solutions (a), (b), (c) and (e)(ii) out of these solutions have the same stabilities and stable regions of solutions as those in the absence of the internal resonance.⁽¹¹⁾ That is, solutions (a), (b) and (c) are stable under a certain condition and the solutions (e)(ii) are always unstable. The solutions (e)(i) are unstable according to numerical calculations. Furthermore, it is obvious from Eqs.(9') and (10') that the solution (0, 0, ..., a_i , 0, ..., 0) can not exist. This is correlated to the experimental result; a squeal of single j -th mode occurs, but a squeal of single i -th mode does not occur.

Lastly, we investigate the solution (d) which is double-mode vibration in the

internal resonance. Now, introducing $\beta_i = a_i^2 p_{i0}^3$, $\beta_j = a_j^2 p_{j0}^3$ and eliminating $\cos \Theta_i$ from Eq.(9)', we get

$$\beta_i^2 - 4(D_i^*/3 + \beta_j)\beta_i - 3\beta_j^2 + 4\beta_j D_j^* = 0 \dots \dots (12)$$

From Eqs.(9') and (10'), we have

$$\begin{aligned}&\left\{ 3B_i \left(\frac{3}{4}\beta_i + \frac{3}{2}\beta_j - D_i^* \right) + B_j \left(\frac{3}{4}\beta_j + \frac{3}{2}\beta_i - D_j^* \right) \right\}^2 \\ &+ (2\eta\omega_i)^2 - \frac{1}{16} \frac{\beta_i}{\beta_j} (9B_i\beta_j + B_j\beta_i)^2 = 0 \dots (13)\end{aligned}$$

$$\text{where } \eta = \frac{3p_{i1} - p_{j1}}{\omega_i} = \frac{3\omega_i - \omega_j}{\epsilon\omega_i} \dots \dots \dots (14)$$

and η is consistence ratio between ω_i and $3\omega_j$.

From Eqs.(6), (9'), (10'), (12) and (13), Θ_{i0} (steady-state value of Θ_i), p_{i1} , p_{j1} , p_{i0} and p_{j0} are obtained. $\Theta_{i0} = \pi$ is derived from Eq.(10') when $\eta=0$. Substitution of this value into Eqs.(12) and (13) gives β_i and β_j , when $\eta=0$. For the solution (d) to be stable, stability conditions (omitted for simplicity) derived from the variational equations for i -th and j -th modes and the following condition must be satisfied.

$$D_i^* < \frac{3}{2}(\beta_i + \beta_j) \quad (i=1, \dots, N, i \neq j) \dots \dots \dots (15)$$

The numerical calculations for $\omega_{s-0} \approx 3\omega_{i-0}$ were carried out using the following values. $h=0.75$ cm, $P_0=19.6$ N, $\mu_0=1.69$, $\mu_1=-5.5 \times 10^{-3}$ s/cm, $\mu_2=1 \times 10^{-7}$ s²/cm³, $V=70$ cm/s, ω_0 : the natural circular frequency of (3-0) mode of which squeal occurs most frequently at $r_i/r_0=2.5/15.0$, C_i : values in the previous report.⁽¹¹⁾

Figure 5 shows calculated results of amplitudes a_i , a_j and accelerations $a_i p_{i0}^3$, $a_j p_{j0}^3$ (taken as dB) at the contact point of the disk for different values of r_i/r_0 (r_i is constant). In this figure, $\phi = \epsilon\eta$ is also shown. These solutions are all stable. There is a slight vibration of j -th mode near $r_i/r_0=0.2$ where $3\omega_i$ does not coincide with ω_j . However, if $3\omega_i$ and ω_j are sufficiently close to each other ($\phi=0$), acceleration level of j -th mode

Fig.5 Amplitudes and accelerations for r_i/r_0 when $\omega_i=3\omega_j$.

increases and double-mode vibration with (5-0) and (9-0) modes occurs. The region of r_i/r_0 where vibrational level of j -th mode becomes high is narrow and the condition under which the double-mode vibration occurs is restricted.

We shall discuss the effects of the damping coefficients on the amplitudes and the stabilities of solutions. Figure 6 shows the amplitudes for two cases at $r_i/r_0 = 0.4065$ where double-mode vibration is of highest level. The first case is when damping coefficient C_j of (9-0) mode changes under constant C_i of (5-0) mode ($= 10\text{s}^{-1}$). The second case is when C_i changes under constant C_j ($= 40\text{s}^{-1}$). As C_j increases for the constant C_i , amplitudes of two modes do hardly change. However, with an increase of C_i for the constant C_j , amplitudes of two modes decrease and these solutions become unstable. From these results, it is obvious that the generating condition for the double-mode vibration depends mainly on the vibration of i -th mode and that a squeal with almost a single i -th mode occurs, provided $3\omega_i$ does not coincide completely with ω_j .

3.2 Case of $\omega_k - \omega_j \cong 2\omega_i$

The following equations for amplitudes and phase angles are obtained by the same procedure as Eqs.(6),(7) and (8):

$$\left. \begin{aligned} \frac{da_i}{d\tau} &= \frac{\epsilon B_i}{2p_{i0}} \left\{ a_i p_{i0} D_i^* - \frac{3}{4} a_i^3 p_{i0}^3 \right. \\ &\quad - \frac{3}{2} a_i p_{i0} \sum_{m=1}^N a_m^2 p_{m0}^2 \\ &\quad \left. - \frac{3}{2} a_i p_{i0} a_j p_{j0} a_k p_{k0} \cos \Theta_2 \right\} \\ \frac{da_j}{d\tau} &= \frac{\epsilon B_j}{2p_{j0}} \left\{ a_j p_{j0} D_j^* - \frac{3}{4} a_j^3 p_{j0}^3 \right. \\ &\quad - \frac{3}{2} a_j p_{j0} \sum_{m=1}^N a_m^2 p_{m0}^2 \\ &\quad \left. - \frac{3}{4} a_j^3 p_{i0}^3 a_k p_{k0} \cos \Theta_2 \right\} \\ \frac{da_k}{d\tau} &= \frac{\epsilon B_k}{2p_{k0}} \left\{ a_k p_{k0} D_k^* - \frac{3}{4} a_k^3 p_{k0}^3 \right. \\ &\quad - \frac{3}{2} a_k p_{k0} \sum_{m=1}^N a_m^2 p_{m0}^2 \\ &\quad \left. - \frac{3}{4} a_k^3 p_{i0}^3 a_j p_{j0} \cos \Theta_2 \right\} \end{aligned} \right\} \dots (16)$$

Fig.6 Influence of damping coefficients on double-mode vibration.

$$\left. \begin{aligned} a_i \frac{d\theta_i}{d\tau} &= \frac{\epsilon}{2p_{i0}} \left\{ 2a_i p_{i0} p_{i1} \right. \\ &\quad \left. - \frac{3}{2} B_i a_i p_{i0} a_j p_{j0} a_k p_{k0} \sin \Theta_2 \right\} \\ a_j \frac{d\theta_j}{d\tau} &= \frac{\epsilon}{2p_{j0}} \left\{ 2a_j p_{j0} p_{j1} \right. \\ &\quad \left. - \frac{3}{4} B_j a_j^3 p_{i0}^3 a_k p_{k0} \sin \Theta_2 \right\} \\ a_k \frac{d\theta_k}{d\tau} &= \frac{\epsilon}{2p_{k0}} \left\{ 2a_k p_{k0} p_{k1} \right. \\ &\quad \left. + \frac{3}{4} B_k a_k^3 p_{i0}^3 a_j p_{j0} \sin \Theta_2 \right\} \end{aligned} \right\} \dots (17)$$

where $p_{m0} = \omega_m$ ($m=1, \dots, N, m \neq i \neq j \neq k$) and $\Theta_2 = \theta_k - \theta_j - 2\theta_i$ and equations for modes except i -th, j -th and k -th modes are omitted, because they are the same equations as the third equation of Eqs.(9) and (10) for l -th mode. Besides solutions (a) and (b) of Eq.(11), the steady-state solutions of Eqs.(16) and (17) are of the form

$$\left. \begin{aligned} (c) & (0, 0, \dots, a_i, 0, \dots, 0) \\ & p_{i1} = 0 \\ (d) & (0, 0, \dots, 0, 0, a_j, 0, \dots, 0) \\ & p_{j1} = 0 \\ (e) & (0, 0, \dots, 0, 0, 0, \dots, 0, a_k, \dots, 0) \\ & p_{k1} = 0 \\ (f) & (0, 0, \dots, a_i, 0, a_j, \dots, 0, a_k, \dots, 0) \end{aligned} \right\} \dots (18)$$

We do not discuss various other solutions described above. The solutions (c), (d) and (e) are the same solutions as those in the absence of the internal resonance. However, the stability of the solution (c) is different from that in the absence of the internal resonance as described later. We now study the solution (f) with triple-mode vibration. Equating the right-hand side of Eq.(16) to zero, we get

$$\left. \begin{aligned} \beta_i \left(\frac{2}{3} D_i^* - \frac{\beta_i}{2} - \beta_k \right) \\ - \beta_j \left(\frac{4}{3} D_j^* - \beta_j - \beta_i - 2\beta_k \right) = 0 \\ \beta_k \left(\frac{4}{3} D_k^* - \beta_k - 2\beta_i \right) \\ - \beta_j \left(\frac{4}{3} D_j^* - \beta_j - 2\beta_i \right) = 0 \end{aligned} \right\} \dots (19)$$

where $\beta_k = a_k^3 p_{k0}^3$.

Eliminating Θ_2 from the equations obtained by equating the right-hand sides of Eqs.(16) and (17) to zero yields

$$\begin{aligned} (2\bar{\eta}\omega_i)^2 + \left[B_i \left\{ \frac{3}{2} \beta_i + 3(\beta_j + \beta_k) - 2D_i^* \right\} \right. \\ + B_j \left\{ \frac{3}{4} \beta_j + \frac{3}{2} (\beta_i + \beta_k) - D_j^* \right\} \\ + B_k \left\{ \frac{3}{4} \beta_k + \frac{3}{2} (\beta_i + \beta_j) - D_k^* \right\} \left. \right]^2 \\ - \frac{9}{16} \frac{1}{\beta_i \beta_k} (4B_i \beta_j \beta_k + B_j \beta_i \beta_k + B_k \beta_i \beta_j)^2 = 0 \end{aligned} \dots (20)$$

where $\bar{\eta} = (p_{k1} - p_{j1} - 2p_{i1})/\omega_i$ (21)

From Eqs.(19) and (20), $\beta_i, \beta_j, \beta_k$ and then Θ_{20} (the steady-state value of Θ_2), $p_{i1}, p_{j1}, p_{k1}, p_{i0}, p_{j0}$ and p_{k0} can be obtained. The stability of the solution (f) is studied by solving the variational equation (omitted) derived from Eqs.(16) and (17) and by the following condition for all modes except for i -th, j -th and k -th modes.

$$D_i^2 < 3(\beta_i + \beta_j + \beta_k)/2 \quad (22)$$

The stability conditions for the solution (c) with a single i -th mode are influenced by j -th and k -th modes because of the internal resonance and these conditions are given by

$$D_i^2 > 0 \quad (23)$$

$$B_j(2D_i^2 - D_j^2) + B_k(2D_i^2 - D_k^2) > 0 \quad (24)$$

$$\begin{aligned} & [(2\bar{\eta}\omega_i)^2 + \{B_j(2D_i^2 - D_j^2) \\ & + B_k(2D_i^2 - D_k^2)\}^2] (2D_i^2 - D_j^2)(2D_i^2 - D_k^2) \\ & - D_i^2 \{B_j(2D_i^2 - D_j^2) + B_k(2D_i^2 - D_k^2)\}^2 > 0 \end{aligned} \quad (25)$$

$$2D_i^2 > D_i^2 \quad (l \neq i \neq j \neq k) \quad (26)$$

Calculated amplitudes of i -th (3-0), j -th (4-0) and k -th (6-0) modes are shown in Fig.7 under $P_0 = 58.8$ N and $r_i/r_0 = 0.22-0.24$. These modes satisfy a relation $\omega_k - \omega_j \cong 2\omega_i$. Stable and unstable solutions for triple-mode vibration are shown in Fig.7 by thin solid and chain lines respectively. When $r_i/r_0 = 0.231$ ($\phi \cong \epsilon\bar{\eta} = 0$), three kinds of solutions (A, B and C) and a stable solution (D) exist. However,

Fig.7 Amplitudes when $\omega_k - \omega_j \cong 2\omega_i$.

when $|\phi|$ increases, unstable solutions (A and C) occur. In the above Fig.7, thick solid and chain lines show stable and unstable solutions for a single i -th mode vibration respectively. From this figure, we can see that a stable triple-mode vibration occurs if a single i -th mode vibration becomes unstable.

We shall discuss the generating region for the triple-mode vibration.⁽⁴⁾ When both β_j and β_k which correspond to the amplitudes of (4-0) and (6-0) modes approach zero, from Eq.(19), we get

$$\beta_i \rightarrow 4D_i^2/3 \quad (27)$$

Furthermore,

$$\frac{\beta_j}{\beta_k} = \frac{2D_i^2 - D_j^2}{2D_i^2 - D_k^2} \quad (28)$$

In view of $\beta_j \rightarrow 0, \beta_k \rightarrow 0$ in Eq.(20), the following equation is derived from Eqs.(27) and (28).

$$\begin{aligned} & [(2\bar{\eta}\omega_i)^2 + \{B_j(2D_i^2 - D_j^2) \\ & + B_k(2D_i^2 - D_k^2)\}^2] (2D_i^2 - D_j^2)(2D_i^2 - D_k^2) \\ & - D_i^2 \{B_j(2D_i^2 - D_j^2) + B_k(2D_i^2 - D_k^2)\}^2 = 0 \end{aligned} \quad (29)$$

Equation (29) is equal to the limiting condition for the stability of a single i -th mode (Eq.(25)). This implies that a stable i -th mode vibration becomes unstable due to the exciting vibrations of j -th and k -th modes.

Figure 8 shows the regions of damping coefficient C_{3-0} of (3-0)-mode in which stable single (3-0)-mode or triple-mode vibrations occur under the constant damping coefficients of (4-0) and (6-0)-modes. Curves a, b and d in this figure are obtained from Eqs.(29), (24) and (26) respectively and curve c is obtained from the stability conditions for the

Fig.8 Generating region for triple-mode vibration.

58.8 N {
 $\square$ generating region for stable single (3-0) mode vibration.
 $\square$ generating region for stable triple-mode vibration.
 19.6 N $\blacksquare$ generating region for stable triple-mode vibration.

triple-mode vibration. Generating region for the triple-mode vibration is very narrow, especially under a small load. In practice, a triple-mode vibration may occur rarely in the narrow generating region and under more difficult initial condition than a single mode vibration.

From the experimental results of natural frequencies in Table 2, it is clear that natural frequencies of the tested disk satisfy perfectly the relation $\omega_k - \omega_j \approx 2\omega_i$. Thus, there are some possibilities of occurrence of stable triple-mode vibration. Figure 9 presents a waveform and frequency spectrum of acceleration when triple-mode vibrations occur several times in hundreds of squeal occurrences. The tested disk rubbed against 22 cm position from the end of the 10x10x300 rod under 58.8 N in load.

3.3 Case of $\omega_k - \omega_j \approx 2\omega_i$, $\omega_k + \omega_j \approx 2\omega_i$ and $\omega_i + \omega_j \approx \omega_k$

Fig.9 A waveform and frequency spectrum of acceleration for squeal with triple-mode vibration.

$P_0 = 58.8 \text{ N}$, $C_{1-0} = 19 \text{ s}^{-1}$,
 $C_{2-0} = 19 \text{ s}^{-1}$, $C_{3-0} = 24 \text{ s}^{-1}$

Fig.10 Amplitudes when $\omega_k - \omega_j \approx 2\omega_i$. (unstable solutions)

Figure 10 shows calculated amplitudes for i -th (3-0), j -th (5-0) and k -th (8-0) modes which have a relation $\omega_k - \omega_j \approx 2\omega_i$ of the internal resonance. All solutions similar to those in Fig.7 are unstable. Also, Fig.11 shows calculated amplitudes for i -th (3-0), j -th (4-0) and k -th (5-0) modes ($\omega_k + \omega_j \approx 2\omega_i$). There are two kinds of unstable solutions which are different from those in Figs.7 and 10, because natural frequencies of those modes do not satisfy $\omega_{k-0} + \omega_{j-0} = 2\omega_{i-0}$ as shown in Fig.4. Figure 12 shows calculated amplitudes for i -th (3-0), j -th (4-0) and k -th (5-0) modes ($\omega_i + \omega_j \approx \omega_k$). Three

$P_0 = 58.8 \text{ N}$, $C_{1-0} = 19 \text{ s}^{-1}$,
 $C_{2-0} = 19 \text{ s}^{-1}$, $C_{3-0} = 19 \text{ s}^{-1}$

Fig.11 Amplitudes when $\omega_k + \omega_j \approx 2\omega_i$. (unstable solutions)

$P_0 = 58.8 \text{ N}$, $C_{1-0} = 19 \text{ s}^{-1}$,
 $C_{2-0} = 19 \text{ s}^{-1}$, $C_{3-0} = 19 \text{ s}^{-1}$

Fig.12 Amplitudes when $\omega_i + \omega_j \approx \omega_k$. (unstable solutions)

kinds of unstable solutions have the same tendency as those in Fig.11.

According to the calculations, stable steady-state solutions do not exist in all cases. Therefore, these triple-mode squeals do not occur in the experiment, since natural frequencies of the tested disk do not satisfy the relationships of the internal resonances for $r_i/r_o < 0.5$ as described above.

3.4 Case of $\omega_j \approx 5\omega_i$

The internal resonance of $\omega_j \approx 5\omega_i$ as shown in Fig.1(a) must be studied using the coefficients of friction considered up to at least fifth-order of v_r . Thus, the coefficients of friction are of the form

$$\mu(v_r) = \mu_0 + \mu_1 v_r + \mu_2 v_r^3 + \mu_3 v_r^5 + \dots \quad (30)$$

Using Eq.(30), the following equations for amplitudes and phase angles can be derived.

$$\left. \begin{aligned} \frac{da_i}{d\tau} &= \frac{\epsilon B_i}{2p_{i0}} \left[a_i p_{i0} D_i^* - E_i \left\{ \frac{3}{4} a_i^3 p_{i0}^3 \right. \right. \\ &\quad + \frac{3}{2} a_i p_{i0} \sum_{m=1}^N a_m^2 p_{m0}^2 \left. \right\} \\ &\quad + \frac{5}{8} E_i \left\{ a_i^3 p_{i0}^3 + 6 a_i^3 p_{i0}^3 a_j^3 p_{j0}^3 \right. \\ &\quad + 3 a_i p_{i0} a_j^3 p_{j0}^3 + \frac{1}{2} a_i^3 p_{i0}^3 a_j p_{j0} \cos \Theta_3 \left. \right\} \left. \right] \\ \frac{da_j}{d\tau} &= \frac{\epsilon B_j}{2p_{j0}} \left[a_j p_{j0} D_j^* \right. \\ &\quad - E_j \left\{ \frac{3}{4} a_j^3 p_{j0}^3 + \frac{3}{2} a_j p_{j0} \sum_{m=1}^N a_m^2 p_{m0}^2 \right\} \\ &\quad + \frac{5}{8} E_j \left\{ a_j^3 p_{j0}^3 + 6 a_j^3 p_{j0}^3 a_i^3 p_{i0}^3 \right. \\ &\quad + 3 a_j p_{j0} a_i^3 p_{i0}^3 + \frac{1}{10} a_i^3 p_{i0}^3 \cos \Theta_3 \left. \right\} \left. \right] \end{aligned} \right\} \quad (31)$$

where

$$\begin{aligned} D_i^* &= D_i^* + \frac{5\mu_3 \omega_i^3 v^4}{\mu_1 + 3\mu_2 \omega_i^3 v^2}, \\ E_i &= \frac{\mu_2 + 10\mu_3 \omega_i^3 v^2}{\mu_2}, \\ E_j &= \frac{\mu_3 (\mu_1 + 3\mu_2 \omega_j^3 v^2)}{\mu_2^2}, \quad \Theta_3 = 5\theta_i - \theta_j \end{aligned}$$

$$\left. \begin{aligned} a_i \frac{d\theta_i}{d\tau} &= \frac{\epsilon}{2p_{i0}} \left\{ 2a_i p_{i0} p_{i1} \right. \\ &\quad \left. - \frac{5}{16} E_i B_i a_i^3 p_{i0}^3 a_j p_{j0} \sin \Theta_3 \right\} \\ a_j \frac{d\theta_j}{d\tau} &= \frac{\epsilon}{2p_{j0}} \left\{ 2a_j p_{j0} p_{j1} \right. \\ &\quad \left. + \frac{5}{16} E_j B_j a_j^3 p_{j0}^3 \sin \Theta_3 \right\} \end{aligned} \right\} \quad (32)$$

Using these equations, steady-state solutions and their stabilities can be examined. Calculated accelerations are shown in Fig.13 for i -th (3-0) and j -th (7-0) modes related to the internal resonance of $\omega_j \approx 5\omega_i$. Constants in Eq.(30) were determined such that curve of the coefficients of friction might be almost

$$P_0 = 58.8 \text{ N}, C_{i0} = 19 \text{ s}^{-1}, C_{r0} = 24 \text{ s}^{-1}$$

Fig.13 Accelerations for various values of r_i/r_o .

the same curve as Eq.(2); $\mu_0 = 1.69$, $\mu_1 = -6.2 \times 10^{-3} \text{ s/cm}$, $\mu_2 = 8.0 \times 10^{-8} \text{ s}^3/\text{cm}^3$, $\mu_3 = 1.0 \times 10^{-12} \text{ s}^5/\text{cm}^5$, $V = 70 \text{ cm/s}$. These solutions are all stable, and this figure shows that a double-mode vibration occurs if ω_j approaches $5\omega_i$ ($\phi = 0$).

4. Conclusions

From theoretical and experimental results about the self-excited vibration of a disk which is clamped at inner radius and is free at outer radius in the internal resonances among axial natural frequencies of the disk, the following conclusions are obtained.

(1) For internal resonances between two i -th and j -th modes such as $\omega_j \approx 3\omega_i$ or $\omega_j \approx 5\omega_i$, squeals with only a frequency of j -th mode and with frequencies of two modes (i -th and j -th modes) occur, but squeal of a single i -th mode does not occur. If ω_j does not coincide with $3\omega_i$ or $5\omega_i$ completely, a frequency component of i -th mode is predominant in the squeal with two modes.

(2) For the internal resonance of $\omega_k - \omega_j \approx 2\omega_i$ among the three modes, a triple-mode vibration occurs. However, its generating region is very narrow. Squeal with a frequency of a single i -th, j -th or k -th mode occurs even in the presence of the internal resonances. However, a single i -th mode vibration becomes unstable if triple-mode vibration is excited.

(3) For other internal resonances such as $\omega_k - \omega_i \approx 2\omega_j$, $\omega_k + \omega_i \approx 2\omega_j$ and $\omega_i + \omega_j \approx \omega_k$, squeals with stable triple-mode vibration do not occur under our calculating condition.

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