Longitudinal Shear of Anisotropic Elastic Half-space
with an Elastic Flat Inclusion*

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The present paper deals with the longitudinal shear of an anisotropic elastic half-space with an elastic flat inclusion. The method of the replacement of flat inclusions by continuous distributions of concentrated forces is expanded to apply to the longitudinal shear problem of an anisotropic medium. The advantage of this method is that the singular integral equations can be derived systematically and the strength of the stress singularity at the tips of the elastic flat inclusion can be easily obtained. In order to clarify the influence of anisotropy, free surface, inclination of the inclusion and rigidity of the inclusion, on the strength of stress singularity at the tips of the inclusion, numerical calculations are performed and the results are shown in figures.

Key Words : Elasticity, Longitudinal Shear, Anisotropic Body, Flat Inclusion, Fracture Mechanics, Strength of Stress Singularity, Singular Point Method

1. Introduction

A great number of longitudinal shear problems of elastic solids have been studied because of their practical importance and the mathematical tractability. In particular, the estimation of strength of structural components weakened by cracks is one of the most fundamental problems of fracture mechanics and many investigations⁽¹⁾ have been made about the crack problems, and a few handbooks⁽²⁾ have already been published. As the other cause of the degradation of strength of structural components, stress concentration due to inclusions which exist unavoidably in engineering materials is mentioned. Especially, in the case of an inclusion with a sharp corner, a remarkable stress concentration arises and a crack initiates from the sharp corner^{(3), (4)}. Therefore, the knowledge of character of stress singularity in the neighborhood of the sharp corner is very important to estimation of the strength of structural components. The flat inclusion is the most typical inclusion with a sharp corner, and several investigations have been made up to date. For instance, the plane problems of a rigid flat inclusion parallel to the inter-

face of two bonded elastic bodies⁽⁵⁾ and of collinear rigid flat inclusion⁽⁶⁾ have been analyzed. On the related subject, a bending problem⁽⁷⁾ based on Reissner's theory and a thermoelastic problem⁽⁸⁾ have been studied.

In all of those investigations, however, the matrix is considered to be isotropic and only a few investigations dealt with the flat inclusion problems concerning the anisotropy of matrix. The longitudinal shear problem of a single rigid flat inclusion in an anisotropic infinite body, was treated by Berezhnitskii et al.⁽⁹⁾ by reducing it to a Hilbert problem. Sumi⁽¹⁰⁾ has analyzed the thermoelastic problem of an orthotropic rectangular plate with a rigid flat inclusion by means of the analytic continuation procedure and the collocation method.

In the present paper, we deal with the problem of a flat inclusion in an anisotropic half-space which is subjected to longitudinal shear at infinity. The

Fig. 1 A flat inclusion in an anisotropic half-space and coordinate system

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flat inclusion is assumed to be isotropic. By means of the method of the replacement of flat inclusions by continuous distributions of concentrated forces, we can readily set up a singular integral equation for density function of the concentrated forces. Transforming (11) the singular integral equation into linear algebraic equation by using the orthogonality relations of Chebyshev polynomials, we can evaluate the strength of stress singularity at the tips of the flat inclusion. Following the analysis, some numerical calculations are performed and the results about the strength of stress singularity of longitudinal shear mode are shown against the geometrical parameters of arrangement of the flat inclusion and the physical properties of the matrix and the inclusion.

2. Statement of Problem and Basic Equations

The x_1 axis of a rectangular coordinate system x_k ($k=1,2$) lies in the bounding surface L' of an anisotropic half-space and the y_1 axis runs perpendicular to it through the center of a flat inclusion L , as shown in Fig.1. The flat inclusion, which is of thickness h and of length $2l$, makes an angle β with x_1 axis. The distances from the center and the tip of the flat inclusion to the bounding surface are denoted by d_0 and d respectively. In the following analysis, we also use a supplementary rectangular coordinate system (x, y) whose origin is located at the center of the flat inclusion. The anisotropic half-space is subjected to a longitudinal shear stress τ_0 at infinity.

When the anisotropy is characterized by the elastic symmetry with respect to the plane (x_1, y_1) , the generalized Hooke's law (12) in the condition of a longitudinal shear is

$$\left. \begin{aligned} \tau_{x_1 y_1} &= C_{44} \frac{\partial u}{\partial y_1} + C_{45} \frac{\partial u}{\partial x_1} \\ \tau_{x_1 x_1} &= C_{45} \frac{\partial u}{\partial y_1} + C_{55} \frac{\partial u}{\partial x_1} \end{aligned} \right\} \quad (1)$$

where u is the displacement normal to the plane (x_1, y_1) and C_{ij} ($i, j=4,5$) are elastic constants. Substituting Eq. (1) into the equilibrium equation and using $z=x_1+iy_1$ and $\bar{z}=x_1-iy_1$, we obtain the following equation:

$$k_{11} \frac{\partial^2 u}{\partial z^2} + 2k_{12} \frac{\partial^2 u}{\partial z \partial \bar{z}} + k_{22} \frac{\partial^2 u}{\partial \bar{z}^2} = 0 \quad (2)$$

where

$$\left. \begin{aligned} k_{11} &= C_{55} - C_{44} + 2iC_{45}, & k_{12} &= C_{55} + C_{44} \\ k_{22} &= C_{55} - C_{44} - 2iC_{45} \end{aligned} \right\} \quad (3)$$

The general solution of Eq. (2) is given by the analytic function $\phi(Z_1)$, namely

$$u(x_1, y_1) = \phi(Z_1) + \bar{\phi}(\bar{Z}_1) \quad (4)$$

where $Z_1 = z + \gamma_1 \bar{z}$, the constant γ_1 is a characteristic value, which satisfies the condition $|\gamma_1| < 1$, of the following algebraic equation:

$$k_{22}\gamma_1^2 + 2k_{12}\gamma_1 + k_{11} = 0 \quad (5)$$

Substituting Eq. (4) into Eq. (1), the stress components are written as

$$\tau_{x_1 x_1} + i\tau_{x_1 y_1} = k_1 \phi'(Z_1) + \bar{k}_2 \bar{\phi}'(\bar{Z}_1) \quad (6)$$

where a prime represents differentiation with respect to a complex variable, and

$$k_1 = k_{11} + \gamma_1 k_{12}, \quad \bar{k}_2 = k_{12} + \bar{\gamma}_1 k_{11} \quad (7)$$

To handle the problem of the anisotropic semi-infinite medium, it is convenient to introduce the complex variables ζ_1 , t_1 defined by

$$\left. \begin{aligned} \zeta_1 &= \frac{z + \gamma_1 \bar{z}}{1 + \gamma_1}, & t_1 &= \frac{l + \Gamma_1 \bar{l}}{1 + \Gamma_1} \\ d_1 &= \frac{1 - \gamma_1}{1 + \gamma_1} (-id_0) \end{aligned} \right\} \quad (8)$$

where $t = x + iy$. The constant Γ_1 is a characteristic value referred to the coordinate system (x, y) . By using the transformation formula (13) of the elastic constants under rotation of the rectangular coordinate system, it is easily seen that the following relation holds

$$\gamma_1 = \Gamma_1 e^{2i\beta} \quad (9)$$

The relation between ζ_1 and t_1 is given by

$$\zeta_1 - d_1 = \left(\frac{1 + \Gamma_1}{1 + \gamma_1} \right) t_1 e^{i\beta} \quad (10)$$

By the use of the complex variable ζ_1 , the displacement and the stress components are written as

$$u(x_1, y_1) = \phi(\zeta_1) + \bar{\phi}(\bar{\zeta}_1) \quad (11)$$

$$\tau_{x_1 x_1} + i\tau_{x_1 y_1} = \frac{k_1}{1 + \gamma_1} \phi'(\zeta_1) + \frac{\bar{k}_2}{1 + \bar{\gamma}_1} \bar{\phi}'(\bar{\zeta}_1) \quad (12)$$

Similarly, by the use of the complex variable t_1 , we obtain

$$u(x, y) = \Phi(t_1) + \bar{\Phi}(\bar{t}_1) \quad (13)$$

$$\tau_{xx} + i\tau_{xy} = \frac{K_1}{1 + \Gamma_1} \Phi'(t_1) + \frac{\bar{K}_2}{1 + \bar{\Gamma}_1} \bar{\Phi}'(\bar{t}_1) \quad (14)$$

where

$$\Phi'(t_1) = e^{i\beta} \left(\frac{1 + \Gamma_1}{1 + \gamma_1} \right) \phi'(\zeta_1) \quad (15)$$

$$K_1 = k_1 e^{-2i\beta}, \quad K_2 = k_2 \quad (16)$$

Eqs. (11)-(14) are the basic equations of the present problem. The expressions differ from those given by Sih and Liebowitz (12), but it is easily seen that both basic equations are equivalent. It is convenient to use Eqs. (11)-(14) in the case of construction of a complex displacement function referred to the rectangular coordinate system (x, y) besides the coordinate system (x_1, y_1) .

By using Eqs. (11)-(14), the boundary conditions of the present problem are written as follows:

(a) On the bounding surface of the half-space, $\tau_{z_1 y_1} = 0$:

$$\text{Im} \left\{ \frac{k_1}{1 + \gamma_1} \phi'(x_1) + \frac{\bar{k}_2}{1 + \bar{\gamma}_1} \bar{\phi}'(x_1) \right\} = 0 \quad (17)$$

(b) From the continuity condition of displacement between the flat inclusion and the matrix, it follows that

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial u''(x, y)}{\partial x} \quad (18)$$

where

$$\frac{\partial u(x, y)}{\partial x} = \phi'(x) + \overline{\phi'(x)} \quad (19)$$

In what follows, the superscript (i) will refer to quantities associated with the flat inclusion.

(c) At infinity, $\tau_{z_1 y_1} = 0$, $\tau_{z_1 x_1} = \tau_0$:

$$\phi'(\zeta_1) \rightarrow \frac{\tau_0(1+\gamma_1)(\bar{k}_1 - \bar{k}_2)}{(k_1 k_1 - k_2 k_2)} \quad (20)$$

3. Derivation of Singular Integral Equation and Its Solution

The complex displacement function of a concentrated force acting at the point

$z=z_0$ in an anisotropic half-space is easily obtained as a solution of the Hilbert problem in the form

$$\phi'(\zeta_1) = \frac{1}{2\pi(k+\bar{k})} \frac{f}{\zeta_1 - \zeta_{10}} + \frac{\bar{k}}{2\pi k(k+\bar{k})} \frac{f}{\zeta_1 - \bar{\zeta}_{10}} \quad (21)$$

where f is the magnitude of the concentrated force, and

$$k = -\frac{(k_1 - k_2)}{2(1+\gamma_1)}, \quad \zeta_{10} = \frac{z_0 + \gamma_1 \bar{z}_0}{1+\gamma_1} \quad (22)$$

When the concentrated force acts at the point $t=s$ ($x=s, y=0$), the complex displacement function $\phi'(t_1)$ is given by

$$\phi'(t_1) = \frac{1}{2\pi(k+\bar{k})} \frac{f}{t_1 - s} + \frac{\bar{k}(1+\Gamma_1)e^{i\theta}}{2\pi k(k+\bar{k})(1+\gamma_1)} \left\{ \lambda e^{i\theta} t_1 - \lambda e^{-i\theta} s + d_1 - \bar{d}_1 \right\} \quad (23)$$

where

$$\lambda = \frac{1+\Gamma_1}{1+\gamma_1}$$

To find the solution, the concentrated forces mentioned above will be distributed continuously along the position of the flat inclusion. By considering the behaviour of the complex displacement function at infinity given by Eq. (20), we can get the required complex displacement function as

$$\phi'(t_1) = \frac{\tau_0 e^{i\theta}(1+\Gamma_1)(\bar{k}_1 - \bar{k}_2)}{k_1 k_1 - k_2 k_2} + \frac{1}{2\pi(k+\bar{k})} \int_{-l}^l \frac{f(s)}{t_1 - s} ds + \frac{\bar{k}(1+\Gamma_1)e^{i\theta}}{2\pi k(k+\bar{k})(1+\gamma_1)} \int_{-l}^l \frac{f(s) ds}{\lambda e^{i\theta} t_1 - \lambda e^{-i\theta} s + d_1 - \bar{d}_1} \quad (24)$$

It is found that Eq. (24) automatically satisfies the boundary conditions Eqs. (17) and (20). When the flat inclusion supposed to be in static self-equilibrium, the resultant force acting on the flat inclusion must vanish as follows:

$$\int_{-l}^l f(s) ds = 0 \quad (25)$$

Equation (25) is a subsidiary condition to find the density function. The equilibrium of the stress τ_{zx} and the force acting on the surface of the flat inclusion gives

$$\frac{\partial u^{(i)}}{\partial x} = \frac{1}{\mu^{(i)} h} \int_{-l}^l f(s) ds \quad (26)$$

where $\mu^{(i)}$ is the shear modulus of the isotropic flat inclusion. In view of Eq. (26), substitution of Eq. (24) into the boundary condition (18) yields a Volterra type singular integral equation:

$$\frac{1}{\pi} \int_{-l}^l \frac{f(s)}{x-s} ds + \frac{1}{\pi} \int_{-l}^l m(x, s) f(s) ds - \frac{k+\bar{k}}{\mu^{(i)} h} \int_{-l}^l f(s) ds = C_0 \quad (27)$$

where

$$m(x, s) = \text{Re} \left\{ \frac{(\bar{k}_1 - \bar{k}_2)(1+\Gamma_1)e^{i\theta}}{(k_1 - k_2)(1+\gamma_1)(\lambda e^{i\theta} x - \lambda e^{-i\theta} s + d_1 - \bar{d}_1)} \right\} \quad (28)$$

$$C_0 = -\frac{2\tau_0(k+\bar{k})}{k_1 k_1 - k_2 k_2} \text{Re} \{ e^{i\theta}(1+\Gamma_1)(\bar{k}_1 - \bar{k}_2) \} \quad (29)$$

Before solving the singular integral equation (27), we introduce the following non-dimensional notations:

$$X = x/l, \quad S = s/l, \quad F(S) = f(s)/(k+\bar{k}), \quad D_0 = d_0/l \quad (30)$$

By the use of Eq. (30), the normalized singular integral equation is

$$\frac{1}{\pi} \int_{-1}^1 \frac{F(S)}{X-S} dS + \frac{1}{\pi} \int_{-1}^1 M(X, S) F(S) dS - \frac{k^*}{\rho^*} \int_{-1}^1 F(S) dS = C_1 \quad (31)$$

where

$$\left. \begin{aligned} M(X, S) &= \text{Re} \left\{ \frac{(\bar{k}_1 - \bar{k}_2)(1+\Gamma_1)e^{i\theta}}{(k_1 - k_2)(1+\gamma_1)(\lambda e^{i\theta} X - \lambda e^{-i\theta} S + D_1 - \bar{D}_1)} \right\} \\ C_1 &= -\frac{2\tau_0}{k_1 k_1 - k_2 k_2} \text{Re} \{ e^{i\theta}(1+\Gamma_1)(\bar{k}_1 - \bar{k}_2) \} \\ D_1 &= \frac{(1-\gamma_1)}{(1+\gamma_1)} (-iD_0), \quad \rho^* = \frac{h\mu^{(i)}}{lC_{55}}, \quad k^* = \frac{2\text{Re}(k)}{C_{55}} \end{aligned} \right\} \quad (32)$$

The subsidiary condition (25) becomes

$$\int_{-1}^1 F(S) dS = 0 \quad (33)$$

The problem is reduced to a solution of the singular integral equation (31) subject to the subsidiary condition (33). Once the density function has been found, the complex displacement

ment function $\Phi'(t_1)$ is obtained from Eq. (24), and then the stress fields are determined. For the particular case of a rigid flat inclusion in an anisotropic infinite medium, a closed form solution is obtained. If the inclusion is rigid, it follows that $\rho^* \rightarrow \infty$ in Eq. (32). Furthermore, we obtain $M(X, S) \rightarrow 0$ for an infinite medium. Therefore, the singular integral equation (31) becomes

$$\frac{1}{\pi} \int_{-1}^1 \frac{F(S)}{X-S} dS = C_1 \quad (34)$$

By using the Plemelj formula of Cauchy integral, the solution of Eq. (34) is

$$F(S) = -C_1 S / \sqrt{1-S^2} \quad (35)$$

In order to find a solution to the singular integral equation (31), we adopt the procedure proposed by Erdogan⁽¹¹⁾. Since the solution of the dominant equation of the singular integral equation (31) corresponds

to the weight function of Chebyshev polynomials of the first kind, we expand the non-dimensional density function $F(S)$ in the following series

$$F(S) = \sum_{n=0}^{\infty} \frac{A_n T_n(S)}{\sqrt{1-S^2}} \quad (36)$$

where A_n are unknown constants, and $T_n(S)$ are Chebyshev polynomials of the first kind. Using the integral formula associated with $T_n(S)$, we get from Eq. (33)

$$A_0 = 0 \quad (37)$$

Substituting Eq. (36) into Eq. (31) and taking account of the orthogonality relation of Chebyshev polynomials, we obtain the following simultaneous linear equation:

$$\sum_{n=1}^{\infty} (-\delta_{mn} + C_{mn} + G_{mn}) A_n = C_1 \delta_{1m} \quad (m=1, 2, \dots) \quad (38)$$

where δ_{ij} is the Kronecker delta, and G_{mn} and C_{mn} are given by

$$G_{mn} = -\frac{2}{\pi} \frac{k^*}{\rho^*} \int_{-1}^1 U_{m-1}(X) \sqrt{1-X^2} \int_{-1}^1 \frac{T_n(S) dS dX}{\sqrt{1-S^2}} = \begin{cases} \frac{2}{\pi} \frac{k^*}{\rho^*} \left[\frac{1}{n^2 - (m+1)^2} - \frac{1}{n^2 - (m-1)^2} \right] & (m+n: \text{even}) \\ 0 & (m+n: \text{odd}) \end{cases} \quad (39)$$

$$C_{mn} = \frac{2}{\pi^2} \int_{-1}^1 U_{m-1}(X) \sqrt{1-X^2} \int_{-1}^1 \frac{M(X, S) T_n(S) dS dX}{\sqrt{1-S^2}} \quad (40)$$

where $U_{m-1}(X)$ are Chebyshev polynomials of the second kind. The unknown constant A_n are found from Eq. (38) and the density function $F(S)$ is determined in the form of Eq. (36). The integrals in expression (40) are of Gauss-Chebyshev type so that the integrals can be easily evaluated by using a quadrature formula given by

$$C_{mn} \approx \frac{2}{(I+1)J} \sum_{\eta=1}^I \sum_{\xi=1}^J \sin v_{\eta} \sin mv_{\eta} \cos w_{\xi} M(X_{\eta}, S_{\xi}) \quad (41)$$

where

$$\left. \begin{aligned} v_{\eta} &= \eta\pi/(I+1), \quad X_{\eta} = \cos v_{\eta}, \quad w_{\xi} = (2\xi-1)\pi/2J, \quad S_{\xi} = \cos w_{\xi} \\ (\eta &= 1, 2, \dots, I; \xi = 1, 2, \dots, J) \end{aligned} \right\} \quad (42)$$

For the analysis, it has been assumed that the flat inclusion is imbedded in the half-space completely. Then the kernel $M(X, S)$ is a Fredholm kernel, that is, there is no singularity in $|X, S| \leq 1$. Now, some considerations will be made about the case that the tip of the flat inclusion reaches the bounding surface, the kernel $M(X, S)$ being given in the form

$$M(X, S) = \text{Re} \left[\frac{(\bar{k}_1 - \bar{k}_2)(1 + \bar{\gamma}_1) e^{i\beta}}{(k_1 - k_2)(1 + \bar{\gamma}_1)} \frac{1}{[(X-S) \cos \beta + i \sin \beta \left\{ (X-1) \frac{1-\gamma_1}{1+\gamma_1} + (S-1) \frac{1-\gamma_1}{1+\gamma_1} \right\}]} \right] \quad (43)$$

The kernel $M(X, S)$ has a singularity at $X=S=1$. In the case of $\beta=0$, the kernel $M(X, S)$ becomes a Cauchy kernel. Thus the singular integral equation (31) is simplified as

$$\frac{1}{\pi} \int_{-1}^1 \frac{F(S)}{X-S} dS - \left[k^* / \rho^* \left\{ 1 + \text{Re} \left\{ \frac{(\bar{k}_1 - \bar{k}_2)(1 + \bar{\gamma}_1)}{(k_1 - k_2)(1 + \bar{\gamma}_1)} \right\} \right\} \right] \int_{-1}^1 F(S) dS = C_1 / \left[1 + \text{Re} \left\{ \frac{(\bar{k}_1 - \bar{k}_2)(1 + \bar{\gamma}_1)}{(k_1 - k_2)(1 + \bar{\gamma}_1)} \right\} \right] \quad (44)$$

Therefore, the density function $F(S)$ has a singularity in the form of $1/\sqrt{1-S^2}$, and the shearing stress has a singularity $r^{1/2}$ at the tips of the flat inclusion where r is a distance from the tip of the flat inclusion. This result shows that although the tip reaches the bounding surface of the half-space, the strength of the stress singularity at the tip of the flat inclusion is finite.

4. Strength of Stress Singularity at the Tips of the Flat Inclusion

For the stress singularity in the vicinity of the tips of the flat inclusion, we have from Eq. (24)

$$\tau_{xx} + i\tau_{xy} = \frac{k_1 l}{1 + \bar{\gamma}_1} \sum_{n=1}^{\infty} A_n \Psi_n(t_1) + \frac{\bar{k}_2 l}{1 + \bar{\gamma}_1} \sum_{n=1}^{\infty} A_n \Psi_n(\bar{t}_1) \quad (45)$$

where $\Psi_n(t_1)$ is the following Cauchy integral:

$$\Psi_n(t_1) = \frac{1}{2\pi i} \int_{L_1} \frac{T_n(s/l) ds}{\sqrt{s^2 - l^2} (s - t_1)} \quad (46)$$

The path of integration L_1 coincides with the rim of the flat inclusion. The function $\sqrt{l^2 - t^2}$ is understood to be a branch, single-valued in the t_1 -plane cut along L_1 , such as $\sqrt{l^2 - t^2} \rightarrow t_1$ for $|t_1| \rightarrow \infty$. Considering the behaviour of the Cauchy integral near the tips of the flat inclusion and introducing the local coordinates $re^{i\theta} = t_1 - l$ we have the behavior of the stress singularity near the tip of the flat inclusion $t_1 = 1$, as follows:

$$\left. \begin{aligned} \tau_{xx} &= \frac{k_3(l)}{\sqrt{2r}} \operatorname{Im} \left[\frac{\nu_1}{\sqrt{\cos \theta + \nu_1 \sin \theta}} \right] \\ \tau_{xy} &= -\frac{k_3(l)}{\sqrt{2r}} \operatorname{Im} \left[\frac{1}{\sqrt{\cos \theta + \nu_1 \sin \theta}} \right] \end{aligned} \right\} \dots (47)$$

where

$$\begin{aligned} \nu_1 &= i(1-\gamma_1)/(1+\gamma_1) \\ k_3(l) &= \frac{\sqrt{T(1+\Gamma_1)(1+\bar{\Gamma}_1)}}{2(1-\Gamma_1\bar{\Gamma}_1)} \\ &\times \operatorname{Re} \left[\frac{K_1+K_2}{1+\Gamma_1} \right] \sum_{n=1}^{\infty} A_n \dots (48) \end{aligned}$$

Similarly, the behavior of the stress singularity near another tip of the inclusion $t_1 = -1$ is obtained by replacing $k_3(l)$ in Eq. (47) with $k_3(-l)$ which is expressed as

$$\begin{aligned} k_3(-l) &= -\frac{\sqrt{T(1+\Gamma_1)(1+\bar{\Gamma}_1)}}{2(1-\Gamma_1\bar{\Gamma}_1)} \\ &\times \operatorname{Re} \left[\frac{K_1+K_2}{1+\Gamma_1} \right] \sum_{n=1}^{\infty} (-1)^n A_n \dots (49) \end{aligned}$$

It is seen from Eq. (47) that the stresses have the singularity $r^{1/2}$ and the variation of the singularity with the argument θ is affected by the characteristic value γ_1 . This result is similar to that of the crack problem of an anisotropic medium under longitudinal shear, but Eqs. (47) are different from the expressions for the crack problem. The quantities $k_3(\pm l)$ are the strengths of stress singularity of longitudinal shear type at the tips $t_1 = \pm 1$ of the flat inclusion. In the case of a rigid flat inclusion in an anisotropic medium, the strength of stress singularity is given by

$$\begin{aligned} k_3(l) &= \tau_0 \sqrt{T} \frac{(1+\Gamma_1)(1+\bar{\Gamma}_1)}{(1-\Gamma_1\bar{\Gamma}_1)(k_1\bar{k}_1 - k_2\bar{k}_2)} \\ &\times \operatorname{Re} \left[\frac{K_1+K_2}{1+\Gamma_1} \right] \operatorname{Re} [e^{i\theta}(1+\Gamma_1)(\bar{k}_1 - \bar{k}_2)] \dots (50) \end{aligned}$$

When the matrix is isotropic, the elastic constants are reduced to $C_{55} = C_{44} = \mu_1$, $C_{45} = 0$, so that $\gamma_1 = 0$ from Eq. (5). Consequently, by using Eq. (7), (9) and (16), the expression (50) becomes $\tau_0 \sqrt{l} \cos \beta$.

5. Numerical Results and Discussion

In order to clarify the influence of anisotropy, free surface, inclinations of the flat inclusion and rigidity of the flat

Table 1
Ratios of elastic constants

	A	B	C	D	E
C_{45}/C_{55}	0	0	0	0.5	1
C_{44}/C_{55}	1	2	0.5	2	2

Table 2
Dependence of the speed of convergence on ρ^*

$C_{45}/C_{55} = 0, C_{44}/C_{55} = 2, \beta = 0, d/l = 0.1$				
$\rho^* \backslash N$	10	20	30	40
0.1	0.168020	0.198604	0.204991	0.207113
1.0	0.574836	0.581800	0.583417	0.583493
10	0.955684	0.955919	0.956293	0.956371

inclusion on the strength of stress singularity at the tips of the flat inclusion, numerical calculations were performed by the analysis method mentioned above.

The ratios of the elastic constants of the matrix are taken from Table 1. By using the values in Table 1, the influence of anisotropy could be clarified. In Table 1, the symbol A denotes an isotropic medium and symbols B and C correspond to an orthotropic medium. For the medium B, we examined the speed of convergence of the non-dimensional strength of the stress singularity $k_3(l)/\tau_0 \sqrt{l}$. The series for $F(S)$ given in Eq. (36) is truncated after the first finite N terms. The accuracy of the calculated result is dependent on N, I and J in Eq. (41). In the numerical calculations, we set $I=J=N+10$, since it has been already found⁽¹⁾ that the result for $I=J=N+10$ gives good accuracy.

The convergence of Eq. (48) and (49) is strongly dependent on ρ^* and d/l . An example of the convergence in the case of $d/l = 0.1$ is shown in Table 2. It may be said from the table that the result for $N=20$ is sufficiently accurate from a practical point of view. Consequently, the numerical calculations were performed for all d/l with $N=20$.

Fig. 2 shows the variation of the non-dimensional strength of stress singularity $k_3(l)/\tau_0 \sqrt{l} (=F_3(l))$ with d/l for $\rho^*=10$ and $\beta=0$. It is seen from Fig. 2 that for all cases of A-E, the strength of stress singularity $F_3(l)$ decreases as the distance d/l decreases, and $F_3(l)$ decreases remarkably in the vicinity of $d/l=0$. This result can be easily understood from the fact that the line of force is sparser, as the flat inclusion is closer to the bounding surface of the half-space. The larger C_{44}/C_{55} and C_{45}/C_{55} are, the larger $F_3(l)$ is, and it is seen that the influence of anisotropy on $F_3(l)$ is remarkable in comparison with the isotropic case. In Fig. 2, the arrows denote

Fig. 2 Variation of the strength of stress singularity of longitudinal shear mode $k_3(l)/\tau_0 \sqrt{l}$ with d/l

the limiting case of $d/l \rightarrow 0$ and the broken lines correspond to $d/l \rightarrow \infty$. Fig.3(a) shows the variation of $F_3(1)$ with d/l and β for the cases of A and B. It is a common tendency that as d/l becomes smaller, $F_3(1)$ becomes smaller. $F_3(1)$ becomes also smaller with an increasing $|\beta|$. Fig.3(b) corresponds to the cases of D and E. The tendency is similar to that shown in Fig.3(a), but the variation of $F_3(1)$ with d/l is very different from that in Fig.3(a) for the case of negative β . Fig.4(a) and 4(b) show the variation of $F_3(-1)$ with d/l and β for $\rho^*=5$. As can be seen, the variation of $F_3(-1)$ with d/l and β is similar to that shown in Fig. 3(a) and 3(b). Fig.5(a) and 5(b) show the

variation of $F_3(1)$ with d/l and ρ^* for $\beta=0$. It can be seen from Fig.5(a) that $F_3(1)$ increases as ρ^* becomes larger. This result is easily understood from ρ^* being a parameter of the rigidity of the flat inclusion. As shown already in Fig.2, the larger C_{44}/C_{55} is, the larger $F_3(1)$ becomes. Fig.5(b) corresponds to the cases of D and E. The variation of $F_3(1)$ with an increasing of ρ^* is similar to that shown in Fig.5(a). Fig.6 shows the influence of anisotropy on $F_3(1)$ for all of ρ^* in the case of $\beta=0$ and $d/l=0.6$. It is seen from the figure that $F_3(1)$ decreases monotonically as ρ^* decreases and it tends to zero as $\rho^*=0$.

(a)

(a)

(b)

(b)

Fig.3 Variation of the strength of stress singularity of longitudinal shear mode $k_3(1)/\tau_0\sqrt{l}$ with d/l and β

Fig.4 Variation of the strength of stress singularity of longitudinal shear mode $k_3(-1)/\tau_0\sqrt{l}$ with d/l and β

6. Conclusions

Applying the method of replacing the flat inclusion with continuous distribution of concentrated forces, we analyzed the longitudinal shear problem of an anisotropic half-space with an elastic flat inclusion. Numerical results clarified the influence of anisotropy, free surface, inclination of the flat inclusion and the rigidity of the inclusion on the strength of the stress singularity at the tips of the flat inclusion. The conclusions were as follows:

(1) The larger C_{44}/C_{55} and C_{45}/C_{55} are, the larger the strength of stress singularity

of the flat inclusion becomes (Fig.2, Fig.6).

(2) The strength of stress singularity decreases as the tips of the inclusion come closer to the free surface (Fig.2-Fig.5).

(3) the strength of stress singularity becomes larger with an increasing ρ^* which is the parameter of the rigidity of the inclusion (Fig.5, Fig.6).

(4) The strength of stress singularity changes with the inclination of the flat inclusion, and the tendency of the variation is more pronounced stronger anisotropy (Fig.3, Fig.4).

(5) In the case of the isotropic matrix, the results are easily obtained by putting $C_{55} = C_{44} = \mu_1$, $C_{45} = 0$ and $\gamma_1 = 0$.

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(a)

(b)

Fig.5 Variation of the strength of stress singularity of longitudinal shear mode $k_3(l)/\tau_0\sqrt{l}$ with d/l and ρ^*

Fig.6 Variation of the strength of stress singularity of longitudinal shear mode $k_3(l)/\tau_0\sqrt{l}$ with ρ^*

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