CONTACT STRESS OF AN ELASTIC LAYER PRESSED ON A RIGID BASE
WITH A SMOOTH PROTRUSION (Second Report)\textsuperscript{\textcopyright}

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In an elastic contact between two uneven surfaces, elastic bodies come in contact with each other smoothly along the inner and outer contact edges, and both edges vary with the applied force or displacement. In the case of a small protrusion, we can approximate it as a parabolic one. In the previous paper, we analyzed the three-dimensional elastic contact problem of an elastic layer pressed by uniform pressure on a rigid base with a smooth parabolic protrusion. In the present paper, we consider a similar problem of an elastic layer being pressed by uniform displacement on the upper surface of the layer.

KEY WORDS: Elasticity, Contact Stress, Thick Plate, Axisymmetric Stress, Mixed Boundary Value Problem

1. INTRODUCTION

In the previous paper\textsuperscript{**}, the authors analyzed the three-dimensional elastic contact problem of an elastic layer pressed by uniform pressure on a rigid base with a smooth parabolic protrusion. In the present paper, we consider a similar problem of it being pressed by uniform displacement. Numerical results are shown for the distributions of contact stresses and surface displacements and for the effect of the protrusion and layer thickness on the stress states.

2. STRESS ANALYSIS

2.1 Stress functions

In a cylindrical coordinate system \((r, \theta, z)\), the axisymmetrical general solutions of the elastic basic equations without torsion are given by Boussinesq's stress functions \(\phi(r, \theta)\) and \(\psi(r, z)\). The displacement \(u_r, \theta, z\) and stresses \(\sigma_r, \theta, z\) are

\[
\begin{align*}
\sigma_r &= 2G\varepsilon_r + \frac{3}{2}G\varepsilon_\theta + (3-4\nu)\varepsilon_z, \\
\sigma_\theta &= 2G\varepsilon_\theta + \frac{3}{2}G\varepsilon_r - 2(1-\nu)\varepsilon_z, \\
r_\theta &= \frac{3G\varepsilon_r}{r} + \frac{3G\varepsilon_\theta}{r} - (1-2\nu)\frac{\partial \varepsilon_z}{\partial r},
\end{align*}
\]

where \(G\) is the shear modulus, \(\nu\) Poisson's ratio, \(r, \theta, z\) cylindrical harmonic functions.

2.2 The method of the stress analysis

A general protrusion may be approximated by a parabolic one if its height is small. Then as shown in Figure 1(a), we consider the three-part mixed boundary value problem of pressing an elastic layer (thickness \(h\)) onto a rigid base with a parabolic protrusion by displacement \(\delta\). The height of the protrusion is \(\varepsilon_c\) (small). The surface of the layer makes contact with the rigid base in the regions \(0 \leq z \leq h\), \(r = r_c\), and \(z = 0\). We assume frictionless contact. At \(r = r_0\), and \(r = r_c\), the layer separates smoothly from the base. The boundary conditions of the problem are

\[
\begin{align*}
(\nu) & (\sigma_r)_{r=r_0} = 0 & (0 \leq \theta \leq \pi) \\
(\mu) & (\sigma_r)_{r=r_c} = 0 & (0 < \theta < \pi) \\
(\tau) & \left(\frac{\partial \sigma_r}{\partial r}\right)_{r=r_0} = 0 & (0 < \theta < \pi) \\
\end{align*}
\]

where \(k\) is a parameter for the parabolic protrusion. The condition (\(\mu\)) means that the layer separates smoothly from the base at \(r_0\) and \(r_0\). If condition (\(\nu\)) is satisfied, \(\left(\frac{\partial \sigma_r}{\partial r}\right)_{r=r_0} = 0\) at \(r = r_0\) and \(\left(\frac{\partial \sigma_r}{\partial r}\right)_{r=r_c} = 0\) at \(r = r_c\), automatically. Then the condition (\(\tau\)) becomes \(\left(\frac{\partial \sigma_r}{\partial r}\right)_{r=r_0} = 0\) at \(r = r_0\) and \(\left(\frac{\partial \sigma_r}{\partial r}\right)_{r=r_c} = 0\) at \(r = r_c\), under condition (\(\nu\)). In this paper, we divide the problem into two problems (Problems I and II) to facilitate the analysis, as shown in Figures 1(b) and 1(c).

Problem I is a case of the layer being pressed onto a rigid base with a cylindrical protrusion (height \(\varepsilon_c\) and radius \(r_c\)) by uniform displacement \(\delta'(=0)\). The boundary conditions are

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Fig. 1 The geometry of the problem

2.3 Stress analysis of Problem I

This problem is the same as Problem I in Ref. (2). The details of the stress analysis are omitted here and only the results used in the present paper are given.

The surface displacement \( (w)_{\infty} \) is

\[
0 \leq r \leq r_1:
\nu
r_1 < r < r_2:
\nu
r_2 < r < \infty:\n
(\nu)_{\infty} = 0
\]

The stresses \((\sigma_x)_{\infty}\) and \((\sigma_z)_{\infty}\) are

\[
\frac{E}{\nu}(\sigma_x)_{\infty} = -1 - \frac{2\pi}{\lambda \nu} \omega \int_{r_2}^{r_1} \left[ \left( \frac{r}{r_2} \right)^{2n+1} - \left( \frac{r}{r_1} \right)^{2n+1} \right] j_n(\lambda r) \, dr
\]

\[
\frac{E}{\nu}(\sigma_z)_{\infty} = -\frac{2\pi}{\lambda \nu} \omega \int_{r_2}^{r_1} \left( \frac{r}{r_2} \right)^{2n} \sinh \lambda h \, j_n(\lambda r) \, dr
\]

where \(E = 2G(1 + \nu)\) is Young’s modulus, \(H(x)\) is Heaviside’s step function, \(j_n(\lambda r)\) is a Bessel function of the first kind in order \(n\), and
\[ r_{x} = r_{y} + b, \quad r_{z} = r_{y} - b, \quad r = \sqrt{r_{x}^{2} + r_{y}^{2} + 2r_{x}r_{y}b \cos \varphi}, \quad (r_{x} \leq r \leq r_{y}) \]

\[ Z(\lambda) = f(\lambda) \sinh(\lambda h) \]

\[ f(\lambda) = \cosh(\lambda h), \quad g(\lambda) = \frac{\lambda h}{\sinh(\lambda h)} \]

\[ t(\lambda) = f(\lambda) - g(\lambda) = \frac{\lambda h}{\sinh(\lambda h)} \]

From the conditions (i) and (ii) in equation (3), we obtain

\[ \alpha_{0} = \frac{-2(1 - \nu)\lambda h}{\lambda h}, \quad \sum_{n=1}^{\infty} \lambda_{n} \alpha_{n} = 0 \]

Equation (10) implies that the stress singularity of \( (\sigma_{x} \lambda)^{m} \) vanishes at \( r = r_{y} = 0 \). The coefficients \( \alpha_{0} + \alpha_{1} \) are determined by two infinite systems of simultaneous equations

\[ \sum_{n=1}^{\infty} \lambda_{n}^{2} (a_{n} \alpha_{0} + a_{2} \alpha_{1}) \int_{\lambda} A(\lambda) \frac{dZ(\lambda)}{d\lambda} d\lambda = \left( \frac{1}{2} \lambda_{n}^{2} + \frac{1}{2} \lambda_{n} \right) \delta_{m}, \quad (m = 0, 1, 2, \ldots) \]

\[ \sum_{n=1}^{\infty} \lambda_{n}^{2} (a_{n} \alpha_{0} + a_{2} \alpha_{1}) \int_{\lambda} C(\lambda) \frac{dZ(\lambda)}{d\lambda} d\lambda = \left( \frac{1}{2} \lambda_{n}^{2} + \frac{1}{2} \lambda_{n} \right) \delta_{m}, \quad (m = 0, 1, 2, \ldots) \]

where \( \delta_{m} \) is Kronecker's delta.

2.4 Stress analysis of Problem II

We choose the stress functions \( \sigma_{x} \) and \( \sigma_{y} \) as follows:

\[ \sigma_{x} = \frac{-2G^{2}(2 \lambda^{2} - r^{2}) + \int_{\lambda} A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z f(\lambda r) d\lambda}{\int_{\lambda} A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z f(\lambda r) d\lambda} \]

\[ \sigma_{y} = \frac{-2G^{2} \lambda^{2} + \int_{\lambda} C(\lambda) \cosh \lambda z + D(\lambda) \sinh \lambda z f(\lambda r) d\lambda}{\int_{\lambda} C(\lambda) \cosh \lambda z + D(\lambda) \sinh \lambda z f(\lambda r) d\lambda} \]

where \( A(\lambda) \) to \( D(\lambda) \) are arbitrary functions with respect to \( \lambda \). Using the boundary conditions (iv) and (v) of equation (4), we can express \( A(\lambda) \), \( B(\lambda) \) and \( D(\lambda) \) in terms of \( C(\lambda) \) as follows:

\[ \lambda A(\lambda) = -[\{1 - 2\nu\}f(\lambda r) + \rho(\lambda)c(\lambda)], \quad \lambda B(\lambda) = [\{1 - 2\nu\}C(\lambda)], \quad \lambda D(\lambda) = -f(\lambda)c(\lambda) \]

Then, we obtain \( \sigma_{x} \) and \( \sigma_{y} \) as follows:

\[ \sigma_{x} = \frac{-2G^{2} \lambda^{2} + \int_{\lambda} C(\lambda) \{ f(\lambda r) - \rho(\lambda)c(\lambda) \} \sinh \lambda z - \{2(1 - \nu) + \rho(\lambda)c(\lambda)\} \cosh \lambda z f(\lambda r) d\lambda}{\int_{\lambda} C(\lambda) \{ f(\lambda r) - \rho(\lambda)c(\lambda) \} \sinh \lambda z - \{2(1 - \nu) + \rho(\lambda)c(\lambda)\} \cosh \lambda z f(\lambda r) d\lambda} \]

\[ \sigma_{y} = \frac{-2G^{2} \lambda^{2} + \int_{\lambda} C(\lambda) \{ f(\lambda r) - \rho(\lambda)c(\lambda) \} \cosh \lambda z f(\lambda r) d\lambda}{\int_{\lambda} C(\lambda) \{ f(\lambda r) - \rho(\lambda)c(\lambda) \} \cosh \lambda z f(\lambda r) d\lambda} \]

The boundary conditions (i) and (ii) of equation (4) give

\[ (w_{x})_{x} = \frac{-1}{G} \int_{\lambda} C(\lambda) f(\lambda r) d\lambda = \begin{cases} \frac{G}{r} & (0 < r \leq r_{y}) \\ 0 & (r_{y} < r < \infty) \end{cases} \]

\[ (w_{y})_{x} = \begin{cases} \frac{-G}{r} & (0 < r \leq r_{y}) \\ 0 & (r_{y} < r < \infty) \end{cases} \]

and the problem is reduced to determining \( C(\lambda) \) such as to satisfy equation (17) under the complementary condition \( (w_{x})_{x} = 0, (r_{y} = 0) \).

We assume \( C(\lambda) \) as follows:

\[ C(\lambda) = \int_{\lambda} F(\phi) f(\lambda r) d\phi, \quad R = \sqrt{r^{2} + b^{2} + 2r_{x} b \cos \phi} \]

where \( F(\phi) \) is a continuous function with respect to \( \phi \). Substituting equation (17) into the first one of equations (16), changing the order of integration and using integral formula (3), we obtain

\[ (w_{x})_{x} = \frac{-1}{G} \int_{\lambda} F(\phi) d\phi \int_{\lambda} f(\lambda r) f(\lambda r) d\lambda = \frac{-2(1 - \nu) b}{G} \int_{\lambda} F(\phi) H(\lambda r) d\phi \]

Because of \( r_{y} \leq r \leq r_{x} \), we obtain \( H(\lambda r) = 1 \) and \((w_{x})_{x} = (r_{y} r_{y}) \) for \( 0 < r \leq r_{y} \), and \( H(\lambda r) = 0 \) and \((w_{x})_{x} = 0 \) for \( r_{y} < r < \infty \). Moreover, since a step function in \( r_{y} < r \leq r_{x} \) is \( H(\lambda r) = 0 \) if \( 0 < \phi < \pi \), \( (r_{y} r_{y}) \) for \( r_{y} < r < \infty \), \( (w_{x})_{x} = 0, (r_{y} < r < \infty) \), \( (w_{x})_{x} = \frac{G}{r} \) in \( r_{y} < r \leq r_{x} \) is expressed by the integration in interval of \( \phi = \frac{x}{r} \), and becomes a function of \( r_{y} \). Then, \( C(\lambda) \) given by equation (18) satisfies quantitatively the first one of equations (16) for an arbitrary function of \( F(\phi) \). By assuming the continuous function \( F(\phi) \) in Fourier cosine series with a loss of generality as follows:

\[ \frac{F(\phi)}{R^{2}} = \frac{E}{2\pi b} \frac{1}{\cos \theta} \]

\[ \int_{\lambda} F(\phi) H(\lambda r) d\phi = \int_{\lambda} \frac{E}{2\pi b} \frac{1}{\cos \theta} H(\lambda r) d\phi \]
the displacement \((w/r)\) is from equation (19)

\[
0 \leq r < r_r: (w/r) = -\frac{2(1-\nu^2)\mu}{h} \frac{\partial^2 w}{\partial r^2} - \frac{2(1-\nu^2)\mu}{h} \left( r - \frac{\phi}{h} \right) \frac{\partial w}{\partial r} - \frac{\phi}{h} \sin \frac{\phi}{h} \frac{\partial w}{\partial \theta} - \frac{\phi}{h} \cos \frac{\phi}{h} \frac{\partial w}{\partial \theta} + \frac{2}{r} (w/r) = \text{constant}
\]

(20)

\[
0 < r < \infty: (w/r) = \text{constant}
\]

(21)

where \(b_0\) are arbitrary coefficients. From the condition (i) of equation (4), we obtain

\[
h = \frac{2(1-\nu^2)\mu}{h} b_0
\]

(21)

The condition (iii) of equation (4) at \(r = r_r\), \((\phi = \pi)\) is equivalent to the following condition

\[
\sum b_0 = 0
\]

(22)

Equation (20) satisfies always the conditions (i) and (iii) of equation (14) for arbitrary values of \(A\) under the conditions (21) and (22). To determine the coefficients \(b_0\), we use the condition for \(w/r\). Substituting equation (19) into equation (17) and using the integral formula \(n\), we find that the function \(C(\lambda)\) is

\[
C(\lambda) = \frac{E^2}{h} \sum b_0 \frac{\partial H(\lambda)}{\partial \lambda}
\]

(23)

where

\[
H(\lambda) = \frac{1}{\lambda} \frac{\partial Z(\lambda)}{\partial \lambda}
\]

Substituting equation (23) into the second equation of (16), we obtain

\[
\sum b_0 \int_{r_0}^{r} \frac{\partial H(\lambda)}{\partial \lambda} \lambda d\lambda = 0 \quad (r_i < r < r_r)
\]

(24)

or

\[
\sum b_0 \int_{r_0}^{r} \frac{\partial H(\lambda)}{\partial \lambda} \lambda d\lambda = 1 \quad (r_i < r < r_r)
\]

(25)

In consideration of the formulae

\[
\lambda r f_i(\lambda) = (\phi/\lambda)(r f_i(\lambda)), \quad \lambda r f_i(\lambda) = (\phi/\lambda)(r f_i(\lambda))
\]

(26)

multiplying both sides of equation (25) by \(r\), and integrating them with respect to \(r\), we get

\[
\sum b_0 \int_{r_0}^{r} \frac{\partial H(\lambda)}{\partial \lambda} \lambda r f_i(\lambda) d\lambda = 1 + 2 c_i \quad (r_i < r < r_r)
\]

(27)

Equation (27) holds in interval \(r_i < r < r_r\), because of integrating stress with respect to \(r\). Moreover, equation (27) by \(r\) and integrating with respect to \(r\), we obtain

\[
\sum b_0 \int_{r_0}^{r} \frac{1}{\lambda} \frac{\partial H(\lambda)}{\partial \lambda} \lambda r f_i(\lambda) d\lambda = 1 + 2 c_i \quad (r_i < r < r_r)
\]

(28)

Dividing equation (28) by \(r^2\) and differentiating it with respect to \(r\) in consideration of the formula

\[
\frac{\partial}{\partial \phi} (r f_i(\lambda)) = \frac{1}{r} \frac{\partial}{\partial \phi} (r f_i(\lambda))
\]

(29)

we get

\[
\sum b_0 \int_{r_0}^{r} \frac{\partial H(\lambda)}{\partial \lambda} \frac{\partial (r f_i(\lambda))}{\partial \phi} d\lambda = 1 + 2 c_i \quad (r_i < r < r_n)
\]

(30)

Equation (30) holds for arbitrary values of \(r\) in \(r_i < r < r_n\). Expanding \(r f_i(\lambda)\) in \(\sin \phi \) we get

\[
r f_i(\lambda) = \frac{2}{\lambda} 
\]

(31)

and substituting equation (31) into (30) we find

\[
\sum b_0 \int_{r_0}^{r} \frac{\partial H(\lambda)}{\partial \lambda} \frac{\partial (r f_i(\lambda))}{\partial \phi} d\lambda = 1 + 2 c_i \quad (r_i < r < r_n)
\]

(32)

We assume that the coefficients \(b_0\) can be expressed by

\[
b_0 = b_0 + c_0
\]

(33)
As equations (32) hold for arbitrary value of $\phi$, we equate the coefficients of $\cos m\phi$ in both sides of (32) and obtain two infinite systems of simultaneous equations with respect to $a_i$ and $h$,

\[
\frac{\partial H_i}{\partial \alpha} + (\beta + 1) \int [r(\alpha)\frac{\partial H_i}{\partial \beta} + 2r\frac{H_i}{\partial \beta}] + \frac{r^2}{2} \beta_i(r) + \frac{r^2}{4} \beta_i(r) + \frac{r^2}{4} \delta_i(r) - 2\delta_i(r) = 0
\]

(m = 0, 1, 2, ...)

(34)

The constant $c_1$ is determined by the condition (22) as follows:

\[
c_1 = \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

(35)

After determining $h$, by equations (34) and (35), we obtain the displacement $(\alpha, \beta)$, from equation (20). To obtain $(\alpha, \beta)$, we use the asymptotic expansion of $\lambda \delta H_i(\lambda)/\delta \alpha$ and $\lambda(\lambda)$ for large values of $\lambda$ as follows:

\[
\frac{H_i}{\partial \alpha} = \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

(36)

Then, we can express the stress $(\sigma, \beta)$ by separating the singular terms

\[
\frac{h}{E_{c}}(\alpha, \beta) = -1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n \lambda(\lambda)
\]

\[
+ \frac{r}{\sqrt{r} \lambda} \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

\[
\sinh \lambda h + \lambda \cosh \lambda h \Delta H_i(\lambda) \Delta f_i(\lambda)
\]

(37)

The last term of equation (37) becomes zero from equation (22). Then, the singularity of $(\sigma, \beta)$ vanishes at $r = r_0$. The stress of the upper surface $(\alpha, \beta)$ is

\[
\frac{h}{E_{c}}(\alpha, \beta) = -1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n \lambda(\lambda)
\]

\[
+ \frac{r}{\sqrt{r} \lambda} \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

\[
\sinh \lambda h + \lambda \cosh \lambda h \Delta H_i(\lambda) \Delta f_i(\lambda)
\]

(38)

2.5 Stress analysis of the present problem

Putting $\beta = \alpha$ and $\beta = -\beta$ and substituting the results of Problems I and II into equation (5), we obtain

\[
\left( \frac{\partial \sigma_\alpha}{\partial \alpha} + \frac{\partial \sigma_\beta}{\partial \beta} \right) \| \delta \|_{\infty} = \frac{z(h-r)}{2\pi r h} \left[ r \sin \lambda h - (\beta^2 + 1) \lambda r \sin \lambda h \right] f_i(\lambda)
\]

(39)

then

\[
\sigma - \beta = 1, \sum_{m=0}^{\infty} (\alpha + \beta) b_m = 0
\]

(40)

We can determine the constants $\alpha$ and $\beta$ from equation (40). The displacement $(\alpha, \beta) = (\alpha, \beta) - (\alpha, \beta)$ is

\[
0 < \lambda < r : (\alpha, \beta) = \frac{-r^2}{\beta} \left[ 1 - (\beta^2 + 1) \lambda r \sin \lambda h \right]
\]

\[
\sinh \lambda h + \lambda \cosh \lambda h \Delta H_i(\lambda) \Delta f_i(\lambda) = 0
\]

(41)

\[
r \lambda \lambda < \delta \| \delta \|_{\infty} = 0
\]

(42)

The stress $(\alpha, \beta) = (\alpha, \beta) - (\alpha, \beta)$ is

\[
\frac{h}{E_{c}}(\alpha, \beta) = -1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n \lambda(\lambda)
\]

\[
+ \frac{r}{\sqrt{r} \lambda} \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

\[
\sinh \lambda h + \lambda \cosh \lambda h \Delta H_i(\lambda) \Delta f_i(\lambda) = 0
\]

(43)

and the stress singular term at $r = r_0$ vanishes because of equation (40). The stress on the upper surface, $(\alpha, \beta) = (\alpha, \beta) - (\alpha, \beta)$, is given by

\[
\frac{h}{E_{c}}(\alpha, \beta) = -1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n \lambda(\lambda)
\]

\[
+ \frac{r}{\sqrt{r} \lambda} \sum_{m=0}^{\infty} (\beta + 1)^m b_m \sum_{n=0}^{\infty} (\beta + 1)^n c_n
\]

\[
\sinh \lambda h + \lambda \cosh \lambda h \Delta H_i(\lambda) \Delta f_i(\lambda) = 0
\]

(44)

Table 1 The relationship among $h/r_c$, $r/r_c$, $h/\delta_i$, and $\alpha/\beta_i$.

<table>
<thead>
<tr>
<th>$h/r_c$</th>
<th>$r/r_c$</th>
<th>$h/\delta_i$</th>
<th>$\alpha/\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
<td>1.4869 (-6.1028)</td>
</tr>
<tr>
<td>0.7</td>
<td>2.7959</td>
<td>(-13.2555)</td>
<td>1.4588 (-3.2566)</td>
</tr>
<tr>
<td>1.0</td>
<td>8.9762</td>
<td>(-18.6448)</td>
<td>1.4460 (-2.1514)</td>
</tr>
<tr>
<td>2.0</td>
<td>8.8627</td>
<td>(-9.2966)</td>
<td>1.4365 (-3.5866)</td>
</tr>
<tr>
<td>5.0</td>
<td>8.8562</td>
<td>(-8.2908)</td>
<td>1.4359 (-3.2864)</td>
</tr>
</tbody>
</table>
3. NUMERICAL RESULTS

There are six parameters, \( r_1, r_2, r_3, k, \delta, h \) and two relationships among them. Sometimes, we must determine the inner and outer radii \( r_1 \) and \( r_2 \) of the contact regions under the protrusion profile, when the layer thickness and the displacement are given. However, it is difficult to determine the radii \( r_1 \) and \( r_2 \) in such a process because the coefficients of the simultaneous equations involve unknown values. Therefore, we solve the simultaneous equations (11) and (34) by giving the values \( r_1/r_2 \) and \( h \), and then obtain the relationship among \( r_1, r_2, k, \delta, h/r_2 \), and \( h \).

Table 1 shows the values of \( k r_1/\delta \) and \( \Delta \) for \( r_1/r_2 = 0.25, 0.5, 0.75 \) and \( h/r_2 = 0.5, 0.7, 1.0, 2.0, 3.0 \).

Figure 2 shows the radial distributions of \((w_1)_\infty \) (dashed curves), and \((\sigma_\infty)_\infty \) (solid curves), in the case of \( r_1/r_2 = 0.25, 0.5, 0.75 \), and \( h/r_2 = 1.0 \). The chained curves denote the profiles of parabolic punches. The displacement \((w_1)_\infty \) in each case coincides with the protrusion \( 0 \leq r < r_1 \), and separates smoothly from the protrusion \( r_1 \leq r < r_2 \), and decreases with an increasing \( r \) for \( r_2 \leq r < \infty \), and makes smooth contact with the rigid base for \( r_1 < r_\infty \) again. The contact stress \((\sigma_\infty)_\infty \) in each case for \( 0 \leq r < r_1 \), and \( r_1 < r_\infty \) is always compressive and has no singularities. The stress \((\sigma_\infty)_\infty \) for \( 0 \leq r < r_1 \) decreases with an increasing \( r \) and becomes zero at \( r = r_1 \). But the slope of \((\sigma_\infty)_\infty \) at \( r = r_1 = 0 \) is infinity. The maximum contact stress at \( r = 0 \) decreases with an increasing \( r_1/r_2 \). The stress \((\sigma_\infty)_\infty \) for \( r_1 < r_\infty \) rises from zero with an infinite slope and tends to \(-E\delta/h \) with an increasing \( r \).

We can obtain several results for the other parameter \( k \) constant by converting scales of one of Figure 2 only. Multiplying the stresses and the displacements of Figure 2 by \(-r_1/\delta \), we get

\[
\frac{h(\sigma_\infty)_\infty}{E \delta} \left( \frac{r_1}{\delta} \right) = \frac{1 - \nu}{E \delta} \left( \frac{r_1}{\delta} \right) \frac{h(\sigma_\infty)_\infty}{E \delta} \frac{h(\sigma_\infty)_\infty}{E \delta} \left( \frac{r_1}{\delta} \right) = \frac{1 - \nu}{E \delta} \left( \frac{r_1}{\delta} \right)
\]

Then, we obtain the results for \( \xi = \text{constant} \) and show them in Figure 3. When \( r_1/r_2 \) increases, \( k \) decreases and the parabolic protrusion becomes flatter. Then the maximum contact stress at \( r = 0 \) decreases.

Multiplying equation (44) by \( E \delta/k r_1 \), we get

\[
\frac{1 - \nu}{E \delta} \left( \frac{r_1}{\delta} \right) \frac{h(\sigma_\infty)_\infty}{E \delta} \left( \frac{r_1}{\delta} \right) = \frac{1 - \nu}{E \delta} \left( \frac{r_1}{\delta} \right) \frac{h(\sigma_\infty)_\infty}{E \delta} \left( \frac{r_1}{\delta} \right) = \frac{1 - \nu}{E \delta} \left( \frac{r_1}{\delta} \right)
\]

Fig. 2 The radial distributions of \((w_1)_\infty \) and \((\sigma_\infty)_\infty \) \( (\delta = \text{constant}) \)

Fig. 3 The radial distributions of \((w_1)_\infty \) and \((\sigma_\infty)_\infty \) \( (\delta \text{ = constant}) \)

Fig. 4 The radial distributions of \((w_1)_\infty \) and \((\sigma_\infty)_\infty \) \( (k \text{ = constant}) \)
Then we obtain the results for \( k = \text{constant} \) and show them in Figure 4. When the value of \( r/r_{0} \) increases, the height of the protrusion increases and the contact stress \( (a_{3})_{0} \) increases also. Using the characteristic length \( r_{0} \) defined by

\[
\frac{r_{0}}{r_{i}} = \frac{\sqrt{\frac{h}{h_{0}+h_{0}}} - \frac{1}{2}}{2} \tag{46}
\]

we get

\[
\frac{1}{r_{0}} \frac{r_{0}}{r_{i}} \frac{r_{0}}{r_{i}} = \frac{(1-\nu) E_{0}^{2}}{E_{0}^{2}} \]

\[
\frac{r_{0}}{r_{i}} \frac{r_{0}}{r_{i}} = \frac{r_{0}}{r_{i}} \frac{r_{0}}{r_{i}} \tag{47}
\]

Then, we obtain the results for the parabolic protrusion = constant \( (\varepsilon_{0} = \text{constant} \) and \( k = \text{constant} \) \) and show them in Figure 5. In this case, the displacement \( \delta \) is given by \( 2(1-\nu) \delta / \varepsilon_{0} = -h / \varepsilon_{0} \), and its value for \( h/r_{0} = 1.0 \) is

\[
\frac{2(1-\nu) \delta}{\varepsilon_{0}} = 0.053 \frac{h}{r_{0}} = 0.25 \]

\[
0.173 \frac{h}{r_{0}} = 0.5 \]

\[
0.464 \frac{h}{r_{0}} = 0.75 \]

\[
\tag{48}
\]

With an increasing \( \delta \), the inner radius \( r_{i} \) increases, the outer radius \( r_{o} \) decreases and the maximum contact stress at \( r_{0} \) increases.

Figure 6 shows the radial distributions of \( (a_{3})_{0} \) and \( (a_{3})_{or} \) for various values of \( h/r_{0} \) and \( r_{0}/r_{i} = 0.5 \). With an increasing \( h/r_{0} \), the contact stress decreases and tends to those in the case of an elastic half-space.

Figure 7 shows the radial distributions of \( (a_{3})_{or} \) for \( r_{0}/r_{i} = 0.5 \). The stress in each case is always compression. With an increasing \( h/r_{0} \), the stress tends to a uniform distribution of simple compression, \(-E\sigma/4\).

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