Vibration Analysis by Substructure Synthesis Method*

(Part 4, Calculation of Residual Compliance Matrix)

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The authors propose a method for calculating accurately the residual compliance matrix of structures with free-free boundary condition in this paper.

A total structure must be divided into several substructures when it is analyzed by substructure synthesis methods. In doing this, the majority of the substructures will be under free-free boundary condition. For such substructures the residual compliance matrix cannot be obtained accurately by the usual method. So it is important to propose a new method for calculating the accurate residual compliance matrix of the structures with free-free boundary condition.

Algorithms of the usual method and the method proposed by the authors as well as Hansteen are given. Numerical examples about a simple case are also shown.

Key Words : Vibration, Finite Element Method, Substructures Synthesis Method, Residual Compliance Matrix, Modal Analysis, Natural Frequency

1. Introduction

The procedure of substructure synthesis methods is as follows. A total structure is divided into several substructures. Next, the dynamic characteristics of the substructures are analyzed either by the experimental modal analysis or by calculations (for example, FEM). Then, the vibration of the total structure is analyzed by synthesizing the dynamic characteristics of the substructures. There are two kinds of major characteristics used in these methods. One is "transfer function", and the other is "natural mode".

In the vibration analysis by substructure synthesis methods, the influences of the natural modes in lower and higher frequency sections than in the analyzed section are considered in terms of "interior restraint" and "residual compliance" respectively. In the experimental modal analysis, both influences can be estimated by using the methods named "curve fitting". But, in the calculation, it is hard to obtain the residual compliance efficiently. Especially the residual compliance of the substructures with free-free boundary condition cannot be calculated.

Since many substructures with free-free boundary condition appear in the analysis by the substructure synthesis methods, it is important to propose an efficient method which enables calculating an accurate residual compliance of such substructures.

In this paper, the procedures of the usual method, the method proposed by Hansteen and the method proposed by the authors are explained.

2. Usual Method

In frequency domain, the equation of motion of an N degrees of freedom model is written as

$$([K] - \omega^2[M])\{\delta\} = \{F\} \quad (1)$$

where $[K]$: stiffness matrix
 $[M]$: mass matrix
 $\{\delta\}$: displacement vector
 $\{F\}$: force vector

By calculating the equation of the eigenvalue problem,

$$([K] - \lambda[M])\{x\} = \{0\} \quad (2)$$

the natural frequencies and the natural modes are obtained. Here, it is assumed that the natural modes are normalized with the mass matrix $[M]$. The modal matrix with the natural modes from the first order to the p -th ($p < N$) is expressed as $[\phi_p]$. The matrix of the natural modes with order higher than the p -th is expressed as $[\phi_h]$. The diagonal matrix with the natural frequencies Ω from the first order to the p -th is expressed as $[Q_p^*]$. The matrix with the natural frequencies with order higher than the p -th is expressed as $[Q_h^*]$.

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As the displacement is expressed by a linear combination of these natural modes, Eq. (3) is given.

$$\{\delta\} = [\phi_i]\{\xi_i\} + [\phi_n]\{\xi_n\} \quad (3)$$

Substituting Eq. (3) into Eq. (1) yields

$$([K][\phi_i] - \omega^2[M][\phi_i])\{\xi_i\} + ([K][\phi_n] - \omega^2[M][\phi_n])\{\xi_n\} = \{F\} \quad (4)$$

Multiplying the matrix $[\phi_n]^T$ by each term of Eq. (4) and considering the orthogonality of the natural modes, yields

$$(\tau^2 \mathcal{Q}_{ii} - \omega^2 \tau^2 I_i)\{\xi_i\} = [\phi_n]^T \{F\} \quad (5)$$

Therefore,

$$\{\xi_n\} = (\tau^2 \mathcal{Q}_{ii} - \omega^2 \tau^2 I_i)^{-1} [\phi_n]^T \{F\} \quad (6)$$

The displacement $\{\delta_n\}$ under the influence of the higher natural modes is written as

$$\{\delta_n\} = [\phi_n](\tau^2 \mathcal{Q}_{ii} - \omega^2 \tau^2 I_i)^{-1} [\phi_n]^T \{F\} \quad (7)$$

Here, it is assumed that the analyzed frequency range is much lower than the $(p+1)th$ natural frequency. Then, Eq. (7) can be expressed approximately as

$$\{\delta_n\} = [\phi_n] \tau^2 \mathcal{Q}_{ii}^{-1} [\phi_n]^T \{F\} \quad (8)$$

From Eq. (8), the residual compliance matrix $[Z]$ is obtained as

$$[Z] = [\phi_n] \tau^2 \mathcal{Q}_{ii}^{-1} [\phi_n]^T \quad (9)$$

All natural modes with order higher than the $p-th$ must be used to calculate an exact residual compliance matrix (Eq. (9)). But, it is difficult to use all natural modes because the degree of freedom, N , is very large in general. Then, only natural modes from the $(p+1)th$ order to the $q-th$ ($p < q < N$) are used practically. The natural modes with order higher than the $q-th$ are neglected. So, it is difficult to obtain an accurate residual compliance matrix by the usual method. The usual method is not efficient.

3. Hansteen's Method

Hansteen proposed a method by which the influence of all natural modes with order higher than adopted ones can be estimated using only the natural modes.

Number of adopted natural modes is assumed to be p here. The displacement is written as

$$\{\delta\} = [\phi_i \phi_n] \begin{Bmatrix} \xi_i \\ \xi_n \end{Bmatrix} \quad (10)$$

like Eq. (2). The modal matrix $[\phi_n]$ and the diagonal matrix $\tau^2 \mathcal{Q}_{ii}$ are used just for convenience here though they are unknown. Substituting Eq. (10) into Eq. (1), and multiplying $[\phi_i \phi_n]^T$ by each term of Eq. (1), yields

$$\begin{pmatrix} \tau^2 \mathcal{Q}_{ii} & \\ & -\omega^2 \tau^2 I_i \end{pmatrix} \begin{Bmatrix} \xi_i \\ \xi_n \end{Bmatrix} = [\phi_i \phi_n]^T \{F\} \quad (11)$$

The right part of Eq. (11) is rewritten as

$$[\phi_i \phi_n]^T \{F\} = \begin{Bmatrix} f_i \\ f_n \end{Bmatrix} \quad (12)$$

Since the natural modes are normalized with respect to the mass matrix, we have

$$[\phi_i \phi_n]^T [M] [\phi_i \phi_n] = \tau^2 I_i \quad (13)$$

Equation (12) can be written as

$$[\phi_i \phi_n]^T \{F\} = \tau^2 I_i \begin{Bmatrix} f_i \\ f_n \end{Bmatrix} \quad (14)$$

Substituting Eq. (13) into Eq. (14) yields

$$[\phi_i \phi_n]^T \{F\} = [\phi_i \phi_n]^T [M] [\phi_i \phi_n] \begin{Bmatrix} f_i \\ f_n \end{Bmatrix} \quad (15)$$

Then, the force vector $\{F\}$ can be expressed as

$$\begin{aligned} \{F\} &= [M] [\phi_i \phi_n] \begin{Bmatrix} f_i \\ f_n \end{Bmatrix} \\ &= [M] [\phi_i] \{f_i\} + [M] [\phi_n] \{f_n\} \\ &= \{F_i\} + \{F_n\} \end{aligned} \quad (16)$$

Equation (16) means that the force vector is divided into two vectors, a force vector $\{F_i\}$ exciting the natural modes from the first order to the $p-th$ and a force vector $\{F_n\}$ exciting ones from the $(p+1)th$ to the $N-th$.

The force vector $\{F_n\}$ is written as

$$\begin{aligned} \{F_n\} &= \{F\} - \{F_i\} \\ &= \{F\} - [M] [\phi_i] \{f_i\} \end{aligned} \quad (17)$$

From Eq. (12), we get

$$\{f_i\} = [\phi_i]^T \{F\} \quad (18)$$

Substituting Eq. (18) into Eq. (17), yields

$$\begin{aligned} \{F_n\} &= \{F\} - [M] [\phi_i] [\phi_i]^T \{F\} \\ &= (\tau^2 I_i - [M] [\phi_i] [\phi_i]^T) \{F\} \end{aligned} \quad (19)$$

Then, the displacement $\{\delta_n\}$ under the influence of the force vector $\{F_n\}$ is expressed as

$$\{\delta_n\} = ([K] - \omega^2 [M])^{-1} \{F_n\} \quad (20)$$

Equation (20) means that the displacement $\{\delta_n\}$ can be calculated using only the natural modes from the first order to the $p-th$.

Since the analyzed frequency section is assumed to be much lower than the $(p+1)th$ natural frequency here, Eq. (20) can be rewritten approximately as

$$\{\delta_n\} = [K]^{-1} \{F_n\} \quad (21)$$

Substituting Eq. (19) into Eq. (21), we get

$$\{\delta_n\} = [K]^{-1} (\tau^2 I_i - [M] [\phi_i] [\phi_i]^T) \{F\} \quad (22)$$

This is an expression proposed by Hansteen in order to estimate the influence of the natural modes with order higher than the adopted ones. Then, the residual compliance matrix is represented as

$$[Z] = [K]^{-1} (\tau^2 I_i - [M] [\phi_i] [\phi_i]^T) \quad (23)$$

As this method does not need the natural modes with order higher than the adopted ones, it is more efficient than the usual method explained in Chapter 2.

But this method cannot be applied to structures with free-free boundary condition directly because the inverse matrix $[K]^{-1}$ cannot be obtained. The reason is that the stiffness matrix $[K]$ with free-free boundary condition is not positive definite.

Then, the authors try to eliminate this disadvantage in the next chapter.

4. Residual Compliance Matrix of the Structure with Free-free Boundary Condition

It is assumed here the number of nodal points of a structure is n and that the degree of freedom of a nodal point is δ and that the structure is under free-free boundary condition. The stiffness matrix and the mass matrix are expressed as $[K]$ and $[M]$ respectively. The equation of the motion of the structure is expressed as

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{F\} \quad (24)$$

As the structure is under free-free boundary condition, the natural frequencies from the first order to the sixth are all 0 Hz. And the natural modes of the same orders are rigid natural modes.

The residual compliance matrix $[Z]$ can be obtained by calculating

$$[K]\{\delta_n\} = \{F_n\} \quad (25)$$

where $\{F_n\}$ is the force vector which is exciting the natural modes with order higher than the p -th ($p \geq 7$). The displacement $\{\delta_n\}$ is expressed by a linear combination of the natural modes from the $(p+1)$ th order to the δn -th.

The mass center does not move when the structure with free-free boundary condition is vibrating freely. Therefore, the mass center of the structure does not move when it displaces $\{\delta_n\}$, either. Though the degree of freedom of Eq.(25) looks like δn , the real rank is $(\delta n - \delta)$. This is the reason why Eq.(25) cannot be solved directly. But the degree of freedom can be reduced to $(\delta n - \delta)$ according to the law of motion of the mass center.

The δ degrees of freedom of a point n is reduced here. Using a matrix $(\delta \times (\delta n - \delta))$ represented as $[S]$, the displacement $\{\delta_n\}$ is expressed as

$$\{\delta_n\} = \begin{bmatrix} I \\ S \end{bmatrix} \{\psi\} \quad (26)$$

where I is a unit matrix $((\delta n - \delta) \times (\delta n - \delta))$, and the vector $\{\psi\}$ is a displacement vector corresponding with the degrees of freedom from the nodal point No. $(n-1)$. The procedure of calculating the matrix $[S]$ is explained later. Substituting Eq.(26) into Eq.(25), and multiplying $[I, S^T]$ by all terms, yields

$$[I \ S^T][K] \begin{bmatrix} I \\ S \end{bmatrix} \{\psi\} = [I \ S^T]\{F_n\} \quad (27)$$

Equation (27) is rewritten as

$$[K']\{\psi\} = [I \ S^T]\{F_n\} \quad (28)$$

where

$$[K'] = [I \ S^T][K] \begin{bmatrix} I \\ S \end{bmatrix} \quad (29)$$

The degree of freedom of the stiffness matrix $[K']$ is $(\delta n - \delta)$. As the matrix $[K']$ does not express the displacement of the rigid modes, the matrix $[K']$ is positive definite. Therefore, Eq.(28) can be solved as

$$\{\psi\} = [K']^{-1} [I \ S^T]\{F_n\} \quad (30)$$

Substituting Eq.(30) into Eq.(26) gives

$$\{\delta_n\} = \begin{bmatrix} I \\ S \end{bmatrix} [K']^{-1} [I \ S^T]\{F_n\} \quad (31)$$

Substituting Eq.(31) into Eq.(19) yields

$$\{\delta_n\} = \begin{bmatrix} I \\ S \end{bmatrix} [K']^{-1} [I \ S^T] \times ({}^T I_I - [M][\phi_i][\phi_i]^T)\{F\} \quad (32)$$

Then, the residual compliance matrix is obtained as

$$[Z] = \begin{bmatrix} I \\ S \end{bmatrix} [K']^{-1} [I \ S^T] \times ({}^T I_I - [M][\phi_i][\phi_i]^T) \quad (33)$$

Equation (33) can be used for calculating generally the residual compliance matrix of structures with free-free boundary condition.

By the way, the matrix $[S]$ is calculated as follows. The matrix $[S]$ is obtained by restraining the movement of the mass center.

The displacement Δx_c of the mass center along x-axis is expressed as

$$\Delta x_c = \frac{\{D_x\}^T [M] \{x\}}{\{D_x\}^T [M] \{D_x\}} \quad (34)$$

where the vector $\{x\}$ is a displacement vector of all nodal points, and $\{D_x\}$ is a vector in which only x-coordinate elements are 1 and the others are all 0. Since the mass center either does not move as the vector $\{\delta_n\}$, we have

$$\{D_x\}^T [M] \{\delta_n\} = 0 \quad (35)$$

As the mass center does not move in the direction of y-axis and z-axis, the following are valid

$$\{D_y\}^T [M] \{\delta_n\} = 0 \quad (36)$$

$$\{D_z\}^T [M] \{\delta_n\} = 0 \quad (37)$$

In the vector $\{D_y\}$, only y-coordinate elements are 1 and the other elements are all 0. Only z-coordinate elements are 1 and the other elements are 0 in the vector $\{D_z\}$.

The angular displacement $\Delta \theta_{xc}$ of the

mass center is expressed as

$$\Delta\theta_{xc} = \frac{\{R_x\}^T[M]\{x\}}{\{R_x\}^T[M]\{R_x\}} \quad (38)$$

The vector $\{R_x\}$ is expressed as

$$\{R_x\}^T = (0, -r_{1z}, r_{1y}, 1, 0, 0, \dots, \dots, 0, -r_{nz}, r_{ny}, 1, 0, 0) \quad (39)$$

where r_{iz} is the relative z-coordinate of a point i to z-coordinate of the mass center, and r_{iy} is the relative y-coordinate of a point i .

When only the stiffness matrix and the mass matrix are given as input data, the vector $\{R_x\}$ can be obtained as follows. It is clear that the vector $\{R_x\}$ does not express the parallel displacement with x-axis and that the mass center does not move. Then, three conditions should be considered with respect to the vector $\{R_x\}$. The first one is that the vector does not cause a strain in the structure.

$$\{K\}\{R_x\} = \{0\} \quad (40)$$

The second is that the mass center does not move in the direction of y-axis.

$$\{D_y\}^T[M]\{R_x\} = 0 \quad (41)$$

The third one is that the mass center does not move in the direction of z-axis, either.

$$\{D_z\}^T[M]\{R_x\} = 0 \quad (42)$$

Substituting Eq.(39) into these three equations, the unknown elements in the vector $\{R_x\}$ can be obtained.

When the structure displaces by $\{\delta_n\}$, the angular displacement $\Delta\theta_{xc}$ of the mass center equals 0. Then we have

$$\{R_x\}^T[M]\{\delta_n\} = 0 \quad (43)$$

The angular displacement $\Delta\theta_{xc}$ of the mass center equals 0, too.

$$\{R_y\}^T[M]\{\delta_n\} = 0 \quad (44)$$

where, the vector $\{R_y\}$ is written as

$$\{R_y\}^T = (r_{1z}, 0, -r_{1z}, 0, 1, 0, \dots, \dots, r_{nz}, 0, -r_{nz}, 0, 1, 0) \quad (45)$$

The unknown elements in Eq.(45) can be obtained by solving the following three equation.

$$\{K\}\{R_y\} = \{0\} \quad (46)$$

$$\{D_x\}^T[M]\{R_y\} = 0 \quad (47)$$

$$\{D_z\}^T[M]\{R_y\} = 0 \quad (48)$$

The elements r_{iz} have already been obtained in Eqs.(40), (41) and (42). As the angular displacement $\Delta\theta_{xc}$ of the mass center is equal to 0, too, we have

$$\{R_z\}^T[M]\{\delta_n\} = 0 \quad (49)$$

where the vector $\{R_z\}$ is

$$\{R_z\}^T = (-r_{1y}, r_{1x}, 0, 0, 0, 1, \dots, \dots, r_{ny}, r_{nx}, 0, 0, 0, 1) \quad (50)$$

The unknown elements in Eq.(50) have already been obtained. Anyway, if they are solved at this step of the process, these elements can be obtained by the following three equations.

$$\{K\}\{R_z\} = \{0\} \quad (51)$$

$$\{D_x\}^T[M]\{R_z\} = 0 \quad (52)$$

$$\{D_z\}^T[M]\{R_z\} = 0 \quad (53)$$

On the other hand, if the coordinates of all nodal points can be used, they can be substituted for r_{iz} , r_{iy} , and r_{iz} . The process of calculating the vectors $\{R_x\}$, $\{R_y\}$, $\{R_z\}$, $\{D_x\}$, $\{D_y\}$, and $\{D_z\}$ is not needed. The rigid natural modes can be used for the matrix $[S]$, too.

With the matrix $[Y]$,

$$[Y]^T = [\{D_x\}, \{D_y\}, \{D_z\}, \{R_x\}, \{R_y\}, \{R_z\}] \quad (54)$$

equations (36), (37), (43), (44) and (49) are put together as

$$[Y][M]\{\delta_n\} = \{0\} \quad (55)$$

The sub-matrix whose freedom is from that of the nodal point 1 and $(n-1)$ is expressed as $[Y_b]$. $[Y_n]$ is the sub-matrix whose freedom corresponds with that of the nodal point n . The displacement vector is divided into two sub-vectors corresponding with the matrix $[Y]$. Eq.(55) is changed as

$$[Y_b \ Y_n] \begin{Bmatrix} M_{bb} & M_{bn} \\ M_{nb} & M_{nn} \end{Bmatrix} \begin{Bmatrix} \delta_b \\ \delta_n \end{Bmatrix} = \{0\} \quad (56)$$

The sub-vector $\{\delta_n\}$ is

$$\{\delta_n\} = -([Y_b][M_{bn}] + [Y_n][M_{nn}])^{-1} \times ([Y_b][M_{bb}] + [Y_n][M_{nb}])\{\delta_b\} \quad (57)$$

Equation (57) means that δ degrees of freedom of the nodal point n can be expressed with the freedom of the nodal points 1 to $(n-1)$. Therefore, the matrix $[S]$ can be obtained as

$$[S] = -([Y_b][M_{bn}] + [Y_n][M_{nn}])^{-1} \times ([Y_b][M_{bb}] + [Y_n][M_{nb}]) \quad (58)$$

5. Application to Eigenvalue Problem of the Structure on Free-free Boundary Condition

Any eigenvalue solver which can not be used to analyze the structures whose stiffness matrices are not positive definite can be applied to such structures directly by using the proposed method. The elastic natural modes and the natural frequencies can be obtained with $[K']$, and $[M']$ in Eq.(59) and (60).

$$[K'] = [I \ S^T][K] \begin{Bmatrix} I \\ S \end{Bmatrix} \quad (59)$$

$$[M'] = [I \ S^T][M] \begin{bmatrix} I \\ S \end{bmatrix} \dots\dots\dots (60)$$

6. Simple Numerical Example

As a simple numerical example, the residual compliance matrix of the model with four masses and four springs shown in Fig.1. is analyzed.

Fig.1 Model with masses and springs

6.1 Exact result

The stiffness matrix and the mass matrix of the model are given as

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \dots\dots\dots (61)$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dots\dots\dots (62)$$

The natural frequencies and the natural modes are obtained as follows.

$$\begin{aligned} \phi_1 &= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \\ \phi_2 &= \begin{bmatrix} -0.25\sqrt{4+2\sqrt{2}} \\ -0.25(\sqrt{2}-1)\sqrt{4+2\sqrt{2}} \\ 0.25(\sqrt{2}-1)\sqrt{4+2\sqrt{2}} \\ 0.25\sqrt{4+2\sqrt{2}} \end{bmatrix} \\ \phi_3 &= \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix} \\ \phi_4 &= \begin{bmatrix} -0.25\sqrt{4-2\sqrt{2}} \\ 0.25(\sqrt{2}+1)\sqrt{4-2\sqrt{2}} \\ -0.25(\sqrt{2}+1)\sqrt{4-2\sqrt{2}} \\ 0.25\sqrt{4-2\sqrt{2}} \end{bmatrix} \end{aligned} \dots\dots\dots (63)$$

Now, it is assumed that the number of adopted natural modes is two (the first and the second modes). There are two natural modes in higher frequency section than in the second natural frequency.

Then, substituting the third and the fourth values in Eq.(63) into Eq.(9), the residual compliance matrix is obtained as

$$[Z] = \begin{bmatrix} \frac{2-\sqrt{2}}{4} & \frac{\sqrt{2}}{8} & \frac{-2+\sqrt{2}}{8} & \frac{-1+\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{8} & \frac{1}{4} & 0 & \frac{-2+\sqrt{2}}{8} \\ \frac{-2+\sqrt{2}}{8} & 0 & \frac{1}{4} & -\frac{\sqrt{2}}{8} \\ \frac{-1+\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{8} & -\frac{\sqrt{2}}{8} & \frac{2-\sqrt{2}}{4} \end{bmatrix} \dots\dots\dots (64)$$

This is an exact residual compliance matrix because all natural modes with order higher than the adopted ones are used in this calculation.

6.2 Usual method

As the degrees of freedom of the stiffness matrix and the mass matrix are very large in general, it is difficult to use all natural modes. Then, the residual compliance matrix is calculated using only the third natural mode by the usual method in this case.

Substituting the values of only the third natural mode into Eq.(9), the residual compliance matrix is obtained as

$$[Z] = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix} \dots\dots\dots (65)$$

The values in Eq.(65) have errors as compared with the values in Eq.(64). These errors occur because the fourth natural mode is omitted.

6.3 Proposed method

Hansteen's method cannot be applied to this model because the stiffness matrix is not positive definite. Then, the method proposed by the authors is applied.

It is assumed that third and the fourth natural modes and the natural frequencies are all unknown. As the mass center does not move when the model displaces like the second, the third and the fourth natural mode shapes, equation (66) holds.

$$[1, 1, 1, 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \dots\dots\dots (66)$$

An unknown value x_4 is expressed as

$$\{x_4\} = [-1, -1, -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \dots\dots\dots (67)$$

Then, the matrix $[S]$ shown Eq.(58) is

$$[S] = [-1, -1, -1] \dots\dots\dots (68)$$

in this case.

Substituting Eq.(61) and (68) into Eq.(28), yields

$$[K'] = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix} \dots\dots\dots (69)$$

The stiffness matrix $[K']$ in Eq.(69) is positive definite. The inverse matrix of $[K']$ is obtained as

$$[K']^{-1} = \frac{1}{8} \begin{bmatrix} 7 & 1 & -3 \\ 1 & 3 & -1 \\ -3 & -1 & 3 \end{bmatrix} \dots\dots\dots (70)$$

Substituting Eqs.(62), (70) and the values of the first and the second natural modes into Eq.(63), the residual compliance matrix can be obtained. And this is the same matrix as Eq.(64).

The response at the mass (m_3) is shown in Table 1 when a unit harmonic force excites the model at the mass (m_3). The result with the residual compliance matrix calculated by the proposed method is more accurate than that by the usual method and the result without the residual compliance matrix in available frequency section.

Table 1 Transfer function

Driving point: m_3
Measured point: m_3

Frequency (Hz)	Exact result	Modal Analysis with the first and the second natural modes		
		A	B	C
0.01	-62.95	-62.95	-63.07	-63.20
0.02	-15.45	-15.45	-15.58	-15.70
0.03	-6.650	-6.653	-6.778	-6.903
0.04	-3.561	-3.568	-3.693	-3.818
0.05	-2.122	-2.133	-2.258	-2.383
0.06	-1.329	-1.344	-1.469	-1.594
0.07	-0.8348	-0.8557	-0.9807	-1.106
0.08	-0.4916	-0.5197	-0.6447	-0.7697
0.09	-0.2198	-0.2565	-0.3815	-0.5065
0.10	0.04719	0.0001062	-0.1249	-0.2499
0.11	0.4636	0.4040	0.2790	0.1540
0.12	4.118	4.043	3.918	3.793
0.13	-0.9313	-1.024	-1.149	-1.274

A: with the residual compliance matrix (Eq.(65)) calculated by the proposed method.

B: with the residual compliance matrix (Eq.(66)) calculated with the third natural mode by the former method.

C: without the residual compliance.

7. Conclusions

In this paper, the authors propose a method for calculating accurately the residual compliance matrix of the structure with free-free boundary condition. The authors' proposed method are explained. In addition to that, a simple numerical example is shown. Then, it is found that the proposed method is useful to calculate the residual compliance matrix of the structure with free-free boundary condition and that the residual compliance matrix of the structure with any boundary condition can be obtained accurately by either the authors' method or Hansteen's.

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