DYNAMIC SHEAR STRESS ANALYSIS OF DISCS SUBJECTED TO VARIABLE ROTATIONS*

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Dynamic circumferential displacements and shear stresses are analyzed for a hollow disc which rotates at variable speeds, and the inner face of which is fixed on a rigid shaft. The problem is solved by using the Laplace transform and Cauchy's integral theorem. For a disc accelerating exponentially with respect to time,

$$\omega(t) = \omega_0 [1 - \exp(-ct)],$$

the shear stress changes cyclically as a sine function of time. The relation between a parameter c in the above equation, which specifies an increasing rate of rotations, and the amplitude $|\Delta \bar{\tau}_{r\theta}|$ of the cyclic variations of the shear stress becomes linear, that is,

$$|\Delta \bar{\tau}_{r\theta}| = 0.1126 \times 10^{-3} c.$$

The ratio of the maximum dynamic and quasi-static stresses, and the period of the cyclically changing stresses are obtained.

Key Words : Elasticity, Structural Analysis, Rotating Discs, Dynamic Shear Stress, Variable Rotations

1. Introduction

When rotating machines start up, slow down or abruptly stop in some accidents, their rotating machine-elements may have a large stress in them as compared with the state of rotating at a constant speed. Stern⁽¹⁾, Tang⁽²⁾, Deresiewicz⁽³⁾, Reddy et al.⁽⁴⁾ and Gurushankar et al.⁽⁵⁾ obtained the circumferential displacement and shear stress in discs subjected to variable rotations based on the quasi-static analysis. However, it may be insufficient to apply the quasi-static analysis to discs which rotate at rapidly changing speeds.

A dynamic problem of discs rotating exponentially with respect to time was solved by Samanta⁽⁶⁾ and Kikuchi et al.⁽⁷⁾ who used the method of separation of variables to derive a solution. However, they have not discussed on the time-response of stress and displacement in detail. The dynamic analyses of the rotating disc by Johnson⁽⁸⁾ and the author⁽⁹⁾ revealed that the dynamic behavior of the rotating discs becomes similar to the oscillating one of a spring-mass system. From this fact we may expect that the dynamic behavior in the circumferential direction will be the same as that of a spring-mass system.

This report presents a dynamic analysis of the circumferential displacement and shear stress in a hollow disc subjected to arbitrarily changing rotations, the inner face of which is glued to a rigid shaft.

The Laplace transforms, Cauchy's and convolution theorems are employed to solve the problem. The numerical computations are carried out for a disc whose rotation increases exponentially with respect to time until a given speed from rest. The dynamic circumferential displacement and shear stress are computed, and their variations with respect to time are discussed. Finally, the obtained results are compared with the solutions based on the quasi-static analysis.

2. Fundamental Equations

A hollow disc with the inner radius a and outer radius b is rotating at the variable angular speed of $\omega(t)$ as shown in Fig.1. The inner face of the disc is glued to the rigid shaft and the disc is driven by this shaft. Suppose the polar coordinate system (r, θ) is fixed on the rigid shaft. The circumferential and radial displacements u, v of the disc are measured from

Fig.1 Rotating disc

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(r, θ)-system. The equation⁽⁹⁾ of motion in the circumferential direction is given by

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} = \alpha^2 \left[\frac{\partial^2 v}{\partial t^2} + 2\omega \frac{\partial u}{\partial t} + (r+u) \frac{d\omega}{dt} - v\omega^2 \right] \quad (1)$$

The constant α is defined by

$$\alpha = (\rho/G)^{1/2} \quad (2)$$

where ρ is the density and G the shear modulus of the disc materials. Assuming that the value of $d\omega/dt$ is larger than the other acceleration components, the following inequality may hold among the acceleration components in the right-hand side of Eq. (1);

$$\left| \frac{\partial^2 v}{\partial t^2} \right|, \left| r \frac{d\omega}{dt} \right| \gg \left| 2\omega \frac{\partial u}{\partial t} \right|, \left| u \frac{d\omega}{dt} \right|, |v\omega^2| \quad (3)$$

Hence, the equation of motion (1) is simplified to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} = \alpha^2 \left[\frac{\partial^2 v}{\partial t^2} + r\Omega_2 \right] \quad (4)$$

where

$$\Omega_2 = d\omega/dt \quad (5)$$

The shear stress is expressed in terms of the circumferential displacement v ,

$$\tau_{r\theta} = G \left[\frac{\partial v}{\partial r} - \frac{v}{r} \right] \quad (6)$$

Since no external force acts on the outer periphery of the disc and the inner face is glued to the rigid shaft, the boundary conditions are given by

$$\tau_{r\theta} = G \left[\frac{\partial v}{\partial r} - \frac{v}{r} \right] = 0, \text{ at } r=b \quad (7)$$

$$v=0, \text{ at } r=a \quad (8)$$

The initial condition of the disc is

$$v = \partial v / \partial t = 0, \text{ at } t=0 \quad (9)$$

which stands for the state of rest.

Non-dimensional quantities are introduced as follows:

$$\bar{v} = v/b, \quad \bar{r} = r/b, \quad \bar{a} = a/b$$

$$\bar{\tau}_{r\theta} = \tau_{r\theta}/G, \quad \bar{\omega} = \omega(b\beta)$$

$$\bar{t} = t/(b\beta), \quad \bar{\alpha} = \alpha/\beta, \quad \bar{\Omega}_2 = d\bar{\omega}/d\bar{t}$$

$$\quad (10)$$

where, by using Young's modulus E and Poisson's ratio ν , the constant β is defined by

$$\beta = [\rho(1-\nu^2)/E]^{1/2} \quad (11)$$

The equation of motion (4) is represented in terms of the non-dimensional quantities (10) by

$$\frac{\partial^2 \bar{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}^2} = \bar{\alpha}^2 \left[\frac{\partial^2 \bar{v}}{\partial \bar{t}^2} + \bar{r}\bar{\Omega}_2 \right] \quad (12)$$

The stress-displacement relation (6) is given by

$$\bar{\tau}_{r\theta} = \frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}} \quad (13)$$

The boundary conditions are

$$\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}} = 0, \text{ at } \bar{r}=1, \text{ for } \bar{t} > 0 \quad (14)$$

$$\bar{v}=0, \text{ at } \bar{r}=\bar{a}, \text{ for } \bar{t} > 0 \quad (15)$$

The initial condition is

$$\bar{v} = \partial \bar{v} / \partial \bar{t}, \text{ at } \bar{t}=0 \quad (16)$$

3. Derivation of Solution

Applying the Laplace transform

$$\tilde{H}(\bar{r}, s) = \int_0^\infty H(\bar{r}, \bar{t}) e^{-s\bar{t}} d\bar{t} \quad (17)$$

to Eq. (12) with the initial condition (16), the following equation is obtained,

$$\frac{\partial^2 \tilde{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \tilde{v}}{\partial \bar{r}} - \left(s^2 \bar{\alpha}^2 + \frac{1}{\bar{r}^2} \right) \tilde{v} = \bar{\alpha}^2 \bar{r} \bar{\Omega}_2 \quad (18)$$

which is a modified Bessel equation of the 1st order. The Laplace transforms of the boundary conditions are

$$\frac{\partial \tilde{v}}{\partial \bar{r}} - \frac{\tilde{v}}{\bar{r}} = 0, \text{ at } \bar{r}=1 \quad (19)$$

$$\tilde{v}=0, \text{ at } \bar{r}=\bar{a} \quad (20)$$

A solution of Eq. (18) is represented by the modified Bessel functions of the 1st and 2nd kinds, I_1 and K_1 , as follows:

$$\tilde{v} = A I_1(s\bar{\alpha}\bar{r}) + B K_1(s\bar{\alpha}\bar{r}) - \bar{r} \bar{\Omega}_2 / s^2 \quad (21)$$

The constants A and B in Eq. (21) are determined by the boundary conditions (19) and (20). Then, the displacement v becomes

$$\begin{aligned} \tilde{v} = & \{ [s\bar{\alpha} K_0(s\bar{\alpha}) + 2K_1(s\bar{\alpha}) + s\bar{\alpha} K_2(s\bar{\alpha})] I_1(s\bar{\alpha}\bar{r}) \\ & + [s\bar{\alpha} I_0(s\bar{\alpha}) - 2I_1(s\bar{\alpha}) + s\bar{\alpha} I_2(s\bar{\alpha})] K_1(s\bar{\alpha}\bar{r}) \} \\ & \times \bar{\alpha} \bar{\Omega}_2 / s^2 \Phi(s, \bar{\alpha}, \bar{a}) - \bar{r} \bar{\Omega}_2 / s^2 \quad (22) \end{aligned}$$

where the function Φ is defined by

$$\begin{aligned} \Phi(s, \bar{\alpha}, \bar{a}) = & I_1(s\bar{\alpha}\bar{a}) [s\bar{\alpha} K_0(s\bar{\alpha}) + 2K_1(s\bar{\alpha}) + \\ & s\bar{\alpha} K_2(s\bar{\alpha})] + K_1(s\bar{\alpha}\bar{a}) [s\bar{\alpha} I_0(s\bar{\alpha}) \\ & - 2I_1(s\bar{\alpha}) + s\bar{\alpha} I_2(s\bar{\alpha})] \quad (23) \end{aligned}$$

The inversion formula⁽¹⁰⁾ of Eq. (22) is given by

$$\bar{v}(\bar{r}, \bar{t}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{v}(\bar{r}, s) e^{s\bar{t}} ds \quad (24)$$

Applying the convolution theorem⁽¹¹⁾ to Eq. (22), the displacement $\bar{v}(\bar{r}, \bar{t})$ is obtained in the integral form,

$$\begin{aligned} \bar{v}(\bar{r}, \bar{t}) = & \bar{a} \int_0^{\bar{t}} \bar{\Omega}_2(\bar{t}-\xi) W(\bar{r}, \xi) d\xi \\ & - \bar{r} \int_0^{\bar{t}} \bar{\Omega}_2(\bar{t}-\xi) \xi d\xi \quad (25) \end{aligned}$$

where the function W is defined by

$$W(\bar{r}, \bar{t}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Psi(s) ds \quad (26)$$

The integrand $\Psi(s)$ in the integral (26) is given by

$$\begin{aligned} \Psi(s) = & \{ [s\bar{\alpha} K_0(s\bar{\alpha}) + 2K_1(s\bar{\alpha}) \\ & + s\bar{\alpha} K_2(s\bar{\alpha})] I_1(s\bar{\alpha}\bar{r}) + [s\bar{\alpha} I_0(s\bar{\alpha}) - 2I_1(s\bar{\alpha}) \end{aligned}$$

$$+s\bar{a}I_2(s\bar{a})]K_1(s\bar{a}\bar{r})\}e^{s\bar{t}}/s^2\Phi(s,\bar{a},\bar{a})\dots(27)$$

The Cauchy integral theorem is employed to obtain the integral (26). Consider a closed loop C on the complex s -plane as shown in Fig.2. Integrating the integrand

Fig.2 Closed contour and poles

given by (27) along the path C , we get

$$\oint_C \Psi(s)ds = 2\pi i \sum_{n=1}^N \text{Res}\{\Psi(s)\} \dots(28)$$

where N is the number of poles inside C . Dividing the closed loop C into two parts, a line segment C_L parallel to the imaginary axis and a circular arc C_R with radius R , Eq.(28) becomes

$$\oint_C \Psi(s)ds = \int_{C_R} \Psi(s)ds + \int_{C_L} \Psi(s)ds \dots(29)$$

After the operation of making the radius of the circular arc C_R infinite, Eq.(29) is rewritten as

$$\lim_{R \rightarrow \infty} \int_{C_R} \Psi(s)ds + \int_{\gamma-i\infty}^{\gamma+i\infty} \Psi(s)ds = 2\pi i \sum_{n=1}^{\infty} \text{Res}\{\Psi(s)\} \dots(30)$$

Since the 1st term of the left-hand side of Eq.(30) is zero, the function W is expressed in terms of residues by

$$W(\bar{r}, \bar{t}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Psi(s)ds = \sum_{n=1}^{\infty} \text{Res}\{\Psi(s)\} \dots(31)$$

The poles of the function $\Psi(s)$ are derived from the value of s satisfying

$$s^2\Phi(s,\bar{a},\bar{a})=0 \dots(32)$$

Since the function Φ has distinct simple poles (12) except for $s=0$, the roots of Eq.(32) are represented by

$$s = \pm i\zeta_n, \quad n=1, 2, \dots \dots(33)$$

where ζ_n is a positive and real number. The characteristic equation becomes

$$\zeta_n \bar{a} [J_1(\zeta_n \bar{a} \bar{a}) Y_0(\zeta_n \bar{a}) - Y_1(\zeta_n \bar{a} \bar{a}) J_0(\zeta_n \bar{a})] - 2[J_1(\zeta_n \bar{a} \bar{a}) Y_1(\zeta_n \bar{a}) - Y_1(\zeta_n \bar{a} \bar{a}) J_1(\zeta_n \bar{a})]$$

$$- \zeta_n \bar{a} [J_1(\zeta_n \bar{a} \bar{a}) Y_2(\zeta_n \bar{a}) - Y_1(\zeta_n \bar{a} \bar{a}) J_2(\zeta_n \bar{a})] = 0, \quad n=1, 2, \dots \dots(34)$$

The residues at $s = \pm i\zeta_n$ are given by

$$\text{Res}\{\Psi(s)\}|_{s=\pm i\zeta_n} = -\frac{4}{\zeta_n^2 \bar{a}} \frac{\Gamma_1}{\Gamma_0} \sin(\zeta_n \bar{t}) \dots(35)$$

where the functions Γ_0 and Γ_1 are defined by

$$\begin{aligned} \Gamma_0 = & \bar{a} [Y_0(\zeta_n \bar{a} \bar{a}) - Y_2(\zeta_n \bar{a} \bar{a})] [\zeta_n \bar{a} J_0(\zeta_n \bar{a}) - 2J_1(\zeta_n \bar{a}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a})] - \bar{a} [J_0(\zeta_n \bar{a} \bar{a}) - J_2(\zeta_n \bar{a} \bar{a})] [\zeta_n \bar{a} Y_0(\zeta_n \bar{a}) - 2Y_1(\zeta_n \bar{a}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a})] + \zeta_n \bar{a} Y_1(\zeta_n \bar{a} \bar{a}) [-3J_1(\zeta_n \bar{a}) + J_3(\zeta_n \bar{a})] - \zeta_n \bar{a} J_1(\zeta_n \bar{a} \bar{a}) [-3Y_1(\zeta_n \bar{a}) + Y_3(\zeta_n \bar{a})] \dots(36) \end{aligned}$$

$$\begin{aligned} \Gamma_1 = & J_1(\zeta_n \bar{a} \bar{r}) [\zeta_n \bar{a} Y_0(\zeta_n \bar{a}) - 2Y_1(\zeta_n \bar{a}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a})] - Y_1(\zeta_n \bar{a} \bar{r}) [\zeta_n \bar{a} J_0(\zeta_n \bar{a}) - 2J_1(\zeta_n \bar{a}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a})] \dots(37) \end{aligned}$$

In order to obtain the residue at $s=0$, the function $\Psi(s)$ must be expanded into a power series of s ,

$$\begin{aligned} \Psi(s) = & \{ 2\bar{r} + (2\bar{r}\bar{t})s + [\bar{a}^2(1/\bar{r} - 2\gamma\bar{r} + \bar{r}^3)/4 + \bar{t}^2\bar{r}]s^2 + 0(s^3) \} / \{ s^2 [2\bar{a} + \bar{a}^2[(1/\bar{a} - 2\gamma\bar{a} + \bar{a}^3)/4]s^2 + 0(s^4)] \}, \\ & \gamma \approx 0.5772 \dots(38) \end{aligned}$$

It can be understood from Eq.(38) that the function $\Psi(s)$ has a double pole at $s=0$. Hence, the residue (13) at $s=0$ is given by

$$\text{Res}\{\Psi(s)\}|_{s=0} = \frac{d}{ds} [s^2 \Psi(s)]|_{s=0} = \frac{\bar{r}\bar{t}}{\bar{a}} \dots(39)$$

Finally, the function W is obtained as

$$W(\bar{r}, \bar{t}) = -\frac{4}{\bar{a}} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2} \frac{\Gamma_1}{\Gamma_0} \sin(\zeta_n \bar{t}) + \frac{\bar{r}\bar{t}}{\bar{a}} \dots(40)$$

The circumferential displacement $\bar{v}$ and shear stress $\bar{\tau}_{r\theta}$ are represented by

$$\begin{aligned} \bar{v}(\bar{r}, \bar{t}) = & -\frac{4\bar{a}}{\bar{a}} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2} \frac{\Gamma_1}{\Gamma_0} \int_0^{\bar{t}} \bar{Q}_2(\bar{t}-\xi) \sin(\zeta_n \xi) d\xi \\ \bar{\tau}_{r\theta}(\bar{r}, \bar{t}) = & -\frac{2\bar{a}}{\bar{a}} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2} \frac{\Gamma_1}{\Gamma_0} \{ [\zeta_n \bar{a} J_0(\zeta_n \bar{a} \bar{r}) - (2/\bar{r}) J_1(\zeta_n \bar{a} \bar{r}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a} \bar{r})] [\zeta_n \bar{a} Y_0(\zeta_n \bar{a}) - 2Y_1(\zeta_n \bar{a}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a})] - [\zeta_n \bar{a} Y_0(\zeta_n \bar{a} \bar{r}) - (2/\bar{r}) Y_1(\zeta_n \bar{a} \bar{r}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a} \bar{r})] [\zeta_n \bar{a} J_0(\zeta_n \bar{a}) - 2J_1(\zeta_n \bar{a}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a})] \} \int_0^{\bar{t}} \bar{Q}_2(\bar{t}-\xi) \sin(\zeta_n \xi) d\xi \dots(42) \end{aligned}$$

4. Disc Rotating at Exponentially Increasing Speeds

A rotating process in which the rotation of the disc increases exponentially with respect to time until a constant value, is adopted as a numerical example as shown in Fig.3

$$\omega = \omega_0 [1 - \exp(-ct)] \dots(43)$$

Expressing Eq.(43) in terms of the non-dimensional quantities, yields

Fig.3 Variable rotating process

$$\bar{\omega} = \bar{\omega}_0 [1 - \exp(-\bar{c}\bar{t})] \dots (44)$$

where the constant $\bar{\omega}_0$ and $\bar{c}$ are defined by

$$\bar{\omega}_0 = (b\beta)\omega_0, \quad \bar{c} = (b\beta)c \dots (45)$$

The angular acceleration $\bar{\Omega}_2$ is

$$\bar{\Omega}_2 = \bar{c}\bar{\omega}_0 \exp(-\bar{c}\bar{t}) \dots (46)$$

Let C_n be the convolution integral included in Eqs. (41) and (42). Then C_n is computed from Eq. (44) as follows:

$$C_n = \int_0^{\bar{t}} \bar{\Omega}_2(\bar{t} - \xi) \sin(\zeta_n \xi) d\xi \\ = \bar{c}\bar{\omega}_0 \left[\frac{\bar{c} \sin(\zeta_n \bar{t}) - \zeta_n \cos(\zeta_n \bar{t})}{\bar{c}^2 + \zeta_n^2} + \frac{\zeta_n e^{-\bar{c}\bar{t}}}{\bar{c}^2 + \zeta_n^2} \right] \dots (47)$$

(i) For large increasing rate, $\bar{c} \gg 1$,

$$C_n = \bar{\omega}_0 \sin(\zeta_n \bar{t}) \dots (48)$$

(ii) For small increasing rate, $\bar{c} \ll 1$,

$$C_n = \bar{\omega}_0 \left[-\frac{\bar{c}}{\zeta_n} \cos(\zeta_n \bar{t}) + \frac{\bar{c}}{\zeta_n} e^{-\bar{c}\bar{t}} \right] \dots (49)$$

Hence, the circumferential displacement $\bar{v}$ and shear stress $\bar{\tau}_{r\theta}$ are given by

$$\bar{v} = -\frac{4\bar{a}}{\bar{\alpha}} \sum_{n=1}^{\infty} \frac{C_n}{\zeta_n^2} \frac{\Gamma_1}{\Gamma_0} \dots (50)$$

$$\bar{\tau}_{rs} = -\frac{2\bar{a}}{\bar{\alpha}} \sum_{n=1}^{\infty} \frac{C_n}{\zeta_n^2} \frac{\Gamma_1}{\Gamma_0} \{ [\zeta_n \bar{a} J_0(\zeta_n \bar{a} \bar{r}) - (2/\bar{r}) J_1(\zeta_n \bar{a} \bar{r}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a} \bar{r})] [\zeta_n \bar{a} Y_0(\zeta_n \bar{a}) - 2 Y_1(\zeta_n \bar{a}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a})] - [\zeta_n \bar{a} Y_0(\zeta_n \bar{a} \bar{r}) - (2/\bar{r}) Y_1(\zeta_n \bar{a} \bar{r}) - \zeta_n \bar{a} Y_2(\zeta_n \bar{a} \bar{r})] [\zeta_n \bar{a} J_0(\zeta_n \bar{a}) - 2 J_1(\zeta_n \bar{a}) - \zeta_n \bar{a} J_2(\zeta_n \bar{a})] \} \dots (51)$$

5. Computational Results

The numerical computations are carried out for a disc with the outer radius $b=0.15$ m and the inner radius $a=0.015$ m. The disc material is SNCM8 whose properties are $E=205.95$ GPa (2.1×10^5 kg/cm²), $\nu=0.278$ and $\rho=7.702 \times 10^3$ kg/m³. The bounded rotating speed is $N=10,000$ rpm (corresponding to $\bar{\omega}=0.02929$ in Eq. (44)).

Fig.4 shows the time-response of the

Fig.4 Time-response of shear stress at inner boundary

shear stress at the inner boundary $\bar{r}=\bar{a}$ for $c=0.1 \times 10^3$. To grasp the phenomena easily, time t is employed instead of the non-dimensional time $\bar{t}$. The solid line shows the dynamic stress given by Eq. (51) and the dash-double dotted line a quasi-static solution (2) which is given by

$$\bar{v}^q = -\frac{\bar{\alpha}^2}{8} \left[\left(\frac{1}{\bar{\alpha}^2} + \bar{\alpha}^2 \right) \bar{r} - \frac{1}{\bar{r}} - \bar{r}^3 \right] \bar{\Omega}_2 \dots (52)$$

$$\bar{\tau}_{r\theta}^q = -\frac{\bar{\alpha}^2}{4} \left(\frac{1}{\bar{r}^2} - \bar{r}^2 \right) \bar{\Omega}_2 \dots (53)$$

The dynamic stress increases with time from zero and reaches its negative maximum value $\bar{\tau}_{r\theta, \max} = -0.0109$ at $t=0.502 \times 10^{-3}$ s. Then, it decreases and attains the positive maximum value $\bar{\tau}_{r\theta, \max} = 0.521 \times 10^{-3}$ at $t=1.004 \times 10^{-3}$ s. After that the dynamic stress varies sinusoidally with respect to time. The averaged value of the dynamic stress, which corresponds to the quasi-static stress, and the amplitude of the cyclic dynamic stress decreases gradually with time elapsed. Meanwhile, the quasi-static stress decreases monotonically.

Fig.5 Time-response of shear stress at inner boundary

The time-response of the shear stress beyond time $t=1.0 \times 10^{-2}$ s is shown in Fig.5. The averaged value of the dynamic shear stress comes closer to zero. The period of the cyclically varying stress can be estimated from Figs.4 and 5, its value being $T=0.10035 \times 10^{-2}$ s.

Fig.6 Stress and displacement distribution for $c=0.1 \times 10^3$

Fig.6 shows the stress and displacement distributions for $c=0.1 \times 10^3$ at two distinct times when the dynamic solution at the inner boundary passes the crest and trough in the sinusoidal motion as shown in Fig.4. The solid line represents the dynamic stress, the dotted line the quasi-static stress, the dash-dotted line the dynamic displacement and the dash-double dotted line the quasi-static displacement. The dynamic stress at $t=0.5018 \times 10^{-3}$ s has its negative maximum value at the inner face and it decays steeply toward the outer boundary. The dynamic displacement distributes almost linearly and shows a similar distribution to that of the quasi-static displacement.

Fig.7 Stress and displacement distribution for $c=0.1 \times 10^3$

After a sufficiently long time elapsed, the stress and displacement distributions become as shown in Fig.7 for $c=0.1 \times 10^3$. It can be seen that the dynamic stress and

Fig.8 Time-response of shear stress at inner boundary

displacement vary cyclically around the zero-axis.

Fig.8 shows the time-response of the dynamic shear stress at the inner boundary for $c=0.1 \times 10^2$. The amplitude of the sinusoidal dynamic stress becomes smaller than that for $c=0.1 \times 10^3$ as shown in Fig.4.

Fig.9 Stress and displacement distributions for $c=0.1 \times 10^2$

Fig.10 Definition of quantities for time-response of stress

Fig.9 gives the stress and displacement distributions at two distinct times when the dynamic stress at the inner boundary passes the crest and trough in the sinusoidal motion given in Fig.8.

The typical time-response of the dynamic and quasi-static stresses is shown in Fig.10. Let $\bar{T}_{r0, \max}$ be the negative maximum value of the first dynamic stress occurring after the start of the rotation, and $\bar{T}_{r0, \max}^q$ be the maximum value of the quasi-static stress. $|\Delta \bar{T}_{r0}|$ denotes the asymptotic amplitude of the dynamic stress at time when the rotating speed of the disc reaches approximately $N=10,000$ rpm.

Fig.11 gives the relation between the parameter c in Eq. (43) and the amplitude at an approximately constant rotation of the disc. Introducing C_N into Eq. (51) and checking the dependency of $|\Delta \bar{T}_{r0}|$ on c , it is seen that a linear relation holds, which agrees with the results of Fig.11. Hence, we get approximately the following relation,

$$|\Delta \bar{T}_{r0}| = 0.1126 \times 10^{-3} C \quad (54)$$

Fig.11 Dependence of stress amplitude on coefficient c

Fig.12 shows how the amplitude of the sinusoidal dynamic stress changes with time. The dotted line represents the rotating speed. The dynamic shear stress depends on $\bar{\Omega}_2 = d\bar{\omega}/d\bar{t}$ as shown in Eq. (42). As the

Fig.12 Variation of stress amplitude with respect to time

rotating speed increases, $\bar{\Omega}_2$ decreases on the contrary. Hence, $\bar{T}_{r\theta}$ becomes smaller with a decrease of the contribution of the 2nd term in Eq. (47). At the same time, the amplitude $\Delta \bar{T}_{r\theta}$ decreases with time and it approaches an asymptotic value as the rotating speed comes close to a constant speed.

Fig.13 shows the variation of ratio $\bar{T}_{r\theta, \max}^d / \bar{T}_{r\theta, \max}^q$ and $\bar{T}_{r\theta, \max}^q$ for various c . It is almost 2.0 in the range of $c=0.5 \sim 5.0$. Beyond $c=5.0$ it decreases considerably. This feature can be explained by the fact that the contribution of the 2nd term in Eq. (47) to the dynamic stress component decreases in proportion to the decrease of $\bar{\Omega}_2$.

6. Conclusions

Analyzing the dynamic circumferential displacement and shear stress occurring in a hollow disc, the following conclusions

Fig.13 Ratio of dynamic and quasi-static stresses

were obtained:

- (1) A general formula was derived for the displacement and shear stress in the disc subjected to arbitrarily varying rotations. They could be expressed in terms of the Bessel functions and of the convolution integral with an angular acceleration and sine function of time.
- (2) The numerical computations were carried out for a rotating process in which the rotating speed increased exponentially up to $N=10,000$ rpm with respect to time. The increasing ratio c of the rotation in Eq. (43) was adopted as a parameter.
- (3) The shear stress at the inner boundary changes cyclically and its averaged value decreases with time. Finally it approaches zero after sufficiently long time elapsed.
- (4) The period of variation of the sinusoidal shear stress is 0.10035×10^{-2} s.
- (5) When the dynamic stress at the inner boundary passes the crest, the shear stress distributions are obtained. It has the maximum value at the inner boundary and decays steeply until $\bar{r}=0.35$. Then, the displacement distributes almost linearly and increases with radius.
- (6) The relation between the stress amplitude $|\Delta \bar{T}_{r\theta}|$ and the coefficient c is approximately given by Eq. (54).
- (7) The stress amplitude with respect to time decreases with time and approaches a constant value as the rotating speed comes close to $N=10,000$ rpm.
- (8) The ratio of the maximum value of the dynamic and quasi-static stresses become about 2.0 in the range $c=0.5 \sim 5.0$ and then decreases with an increase of c above 5.0.

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