On the Subharmonic Vibrations of Order 1/2 of a Nonlinear Vibrating
System with a Duffing Type Restoring Characteristic*
(2nd Report, Case of Soft Spring)

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The subharmonic oscillations of order 1/2 of a system with a Duffing type restoring characteristic are investigated by making use of a harmonic balance method, under the condition that both the periodic exciting force (e.g., an unbalancing force) and the constant force (e.g., gravity) are operative. Unlike in the preceding paper⁽¹⁾, the restoring characteristic of this system is assumed to be a soft spring. As a result, the phenomena of these vibrations are considerably complicated and become extremely different from the same kind of vibrations in the hard spring system. However, the occurrence of the 1/2-th subharmonic vibrations in this system can be similar to the hard spring system, even if the constant stimulating force approaches zero.

Key words: Vibration, Theory of Vibration, Nonlinear Vibration, Subharmonic Vibration of order 1/2, Duffing Equation, Soft Spring Characteristic

1. Introduction

The previous paper⁽¹⁾ dealt with the subharmonic vibrations of order 1/2 of a nonlinear vibrating system with a Duffing type characteristic. As stimulating forces in the system, both a periodic force and a constant force have been taken into account. The restoring characteristics of this system have been restricted using a soft spring.

Now, there are a great number of papers which analyze the nonlinear vibrating phenomena of a system whose restoring characteristic is hard, and there are also a few works⁽²⁾ dealing with such phenomena of a system with a soft spring. It seems, however, that the number of papers about the latter is much less than ones about the former. In addition, it seems that there are very few papers dealing with the subharmonic vibrations of order 1/2 in an oscillating system with a soft nonlinear restoring force, when both a periodic force and a constant force as above mentioned are operative on the system.

Therefore, assuming that the restoring

force of the system in the preceding paper⁽¹⁾ is soft (which a magnetic field may yield), the authors will consider the subharmonic vibrations of order 1/2 of the system by making use of the harmonic balance method and give some numerical examples. Comparing the analytical results with the results obtained by numerical integration from the same system, the existence of these subharmonic oscillations and the validity of the present analysis will be verified.

2. Theoretical Analysis

2.1 Approximate Solution

Let's consider a system with one degree of freedom which is governed by the same differential equation Eq.(1) as that of the preceding paper⁽¹⁾:

$$X'' + hX' + X + qX^3 = (\varphi')^2 \cos \varphi - G \quad \dots\dots\dots (1)$$

where X is the displacement; φ is the angular displacement; τ is the dimensionless time; the prime (') means differentiation with respect to τ ; h , G and q which are constants and positive except for q ; h and $|q|$ are small enough. In the following, it is also assumed that the coefficient of the nonlinear restoring characteristic q is negative, i.e., the spring characteristic is soft. The approximate solution of Eq.(1) is given as⁽¹⁾

$$X = R + C \cos(\psi - \theta) + A \cos(\varphi - \delta) \quad \dots\dots\dots (2)$$

where $\psi = \varphi/n$; $d\varphi/d\tau = v$ (constant); $n=2$; θ and δ are phase angles; R is the unidirectional component⁽⁴⁾; C is the amplitude of the subharmonic component of order 1/2; A is the amplitude of the fundamental compo-

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nent. Substituting Eq.(2) into Eq.(1) and equating the coefficients of the terms containing $\cos(\psi - \theta)$, $\sin(\psi - \theta)$, $\cos(\varphi - \delta)$, $\sin(\varphi - \delta)$ and the non-oscillatory term separately to zero lead to

$$\left. \begin{aligned} C \left[1 - \left(\frac{\nu}{n} \right)^2 + 3q \left(R^2 + \frac{1}{4} C^2 + \frac{1}{2} A^2 \right) \right] + 3qRC A \cos(2\theta - \delta) &= 0, \\ h(\nu/n)C + 3qRC A \sin(2\theta - \delta) &= 0, \\ A \left[1 - \nu^2 + 3q \left(R^2 + \frac{1}{2} C^2 + \frac{1}{4} A^2 \right) \right] + \frac{3}{2} qRC^2 \cos(2\theta - \delta) &= \nu^2 \cos \delta, \\ -h\nu A + \frac{3}{2} qRC^2 \sin(2\theta - \delta) &= -\nu^2 \sin \delta, \\ G + R + qR \left(R^2 + \frac{3}{2} C^2 + \frac{3}{2} A^2 \right) + \frac{3}{4} qC^2 A \cos(2\theta - \delta) &= 0 \end{aligned} \right\} \dots\dots\dots (3)$$

Eliminating the phase angles θ and δ in Eq.(3) yields

$$\left. \begin{aligned} \left[1 - \left(\frac{\nu}{n} \right)^2 + 3q \left(R^2 + \frac{1}{4} C^2 + \frac{1}{2} A^2 \right) \right]^2 + h^2 \left(\frac{\nu}{n} \right)^2 &= (3qRA)^2, \\ A^2 \left[\left\{ 1 - \nu^2 + 3q \left(R^2 + \frac{1}{2} C^2 + \frac{1}{4} A^2 \right) \right\}^2 + (h\nu)^2 \right] + \left(\frac{3}{2} qRC^2 \right)^2 + h^2 \nu \left(\frac{\nu}{n} \right) C^2 & \\ - C^2 \left\{ 1 - \nu^2 + 3q \left(R^2 + \frac{1}{2} C^2 + \frac{1}{4} A^2 \right) \right\} \left[1 - \left(\frac{\nu}{n} \right)^2 + 3q \left(R^2 + \frac{1}{4} C^2 + \frac{1}{2} A^2 \right) \right] &= (\nu^2)^2, \\ R \left[G - R \left\{ 1 + q \left(R^2 + \frac{3}{2} C^2 + \frac{3}{2} A^2 \right) \right\} \right] = \frac{1}{4} C^2 \left[1 - \left(\frac{\nu}{n} \right)^2 + 3q \left(R^2 + \frac{1}{4} C^2 + \frac{1}{2} A^2 \right) \right] & \end{aligned} \right\} \dots\dots\dots (4)$$

Substituting R , C and A obtained from Eq.(4) into Eq.(5), the phase angles θ and δ are given by

$$\left. \begin{aligned} \tan \delta &= \left[h\nu A - \frac{3}{2} qRC^2 \sin(2\theta - \delta) \right] / \left[A \left\{ 1 - \nu^2 + 3q \left(R^2 + \frac{1}{2} C^2 + \frac{1}{4} A^2 \right) \right\} + \frac{3}{2} qRC^2 \cos(2\theta - \delta) \right], \\ \tan(2\theta - \delta) &= h \left(\frac{\nu}{n} \right) / \left[1 - \nu^2 + 3q \left(R^2 + \frac{1}{4} C^2 + \frac{1}{2} A^2 \right) \right] \end{aligned} \right\} \dots\dots\dots (5)$$

If R , C , A , θ and δ are a set of steady solutions for the simultaneous equation Eq.(3), it is clear that the system of R , $-C$, A , θ and δ also is another solution.

2.2 Stability of Steady Solutions

The stability of the steady solutions obtained from Eqs.(4) and (5) is considered. These steady solutions are expressed by C_0 , θ_0 , A_0 , δ_0 and R_0 and sufficiently small variations of them are represented by $\tilde{c}$, $\tilde{\theta}$, $\tilde{a}$, $\tilde{\delta}$ and $\tilde{r}$, respectively:

$$C = C_0 + \tilde{c}(\tau), \quad \theta = \theta_0 + \tilde{\theta}(\tau), \quad A = A_0 + \tilde{a}(\tau), \quad \delta = \delta_0 + \tilde{\delta}(\tau), \quad R = R_0 + \tilde{r}(\tau) \dots\dots\dots (6)$$

If these variations $\tilde{c}$, $\tilde{\theta}$ and so on tend to zero with an increasing τ , the solutions C_0 , θ_0 etc. are stable. In place of Eq.(2), we adopt as an approximate solution the following expression:

$$X = (R_0 + \tilde{r}) + (C_0 + \tilde{c}) \cos\{\psi - (\theta_0 + \tilde{\theta})\} + (A_0 + \tilde{a}) \sin\{\varphi - (\delta_0 + \tilde{\delta})\} \dots\dots\dots (7)$$

Then substituting Eq.(7) into Eq.(1), and equating the coefficients of each harmonic component and of the non-oscillatory term, but neglecting the second and higher powers of the small parameters $\tilde{c}$, $\tilde{\theta}$ and so on yield

$$Bx' = A \cdot x \dots\dots\dots (8)$$

where

$$x^T = [C_0 \tilde{\theta}, \tilde{c}, A_0 \tilde{\delta}, \tilde{a}, \tilde{r}],$$

()^T means transpose of a vector;

$$B = - \begin{bmatrix} 2\frac{\nu}{n} & h & 0 & 0 & 0 \\ h & -2\frac{\nu}{n} & 0 & 0 & 0 \\ 0 & 0 & 2\nu & h & 0 \\ 0 & 0 & h & -2\nu & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix},$$

$$A = A_1 + 3qA_2,$$

$$A_1 = \begin{bmatrix} h\frac{\nu}{n} & 1 - \left(\frac{\nu}{n}\right)^2 & 0 & 0 & 0 \\ 1 - \left(\frac{\nu}{n}\right)^2 & -h\frac{\nu}{n} & 0 & 0 & 0 \\ 0 & 0 & h\nu & 1 - \nu^2 & 0 \\ 0 & 0 & 1 - \nu^2 & -h\nu & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = (a_{ij}), \quad \Theta_0 = 2\theta_0 - \delta_0, \quad a_{11} = -R_0 A_0 \sin \Theta_0, \quad a_{12} = R_0^2 + \frac{3}{4} C_0^2 + \frac{1}{2} A_0^2 + R_0 A_0 \cos \Theta_0,$$

$$a_{13} = R_0 C_0 \sin \Theta_0, \quad a_{14} = C_0 (A_0 + R_0 \cos \Theta_0), \quad a_{15} = C_0 (2R_0 + A_0 \cos \Theta_0),$$

$$a_{21} = R_0^2 + \frac{1}{4} C_0^2 + \frac{1}{2} A_0^2 - R_0 A_0 \cos \Theta_0, \quad a_{22} = -R_0 A_0 \sin \Theta_0, \quad a_{23} = R_0 C_0 \cos \Theta_0, \quad a_{24} = -R_0 C_0 \sin \Theta_0,$$

$$a_{25} = -C_0 A_0 \sin \Theta_0, \quad a_{31} = -R_0 C_0 \sin \Theta_0, \quad a_{32} = C_0 (A_0 + R_0 \cos \Theta_0), \quad a_{33} = 0, \quad a_{34} = R_0^2 + \frac{1}{2} C_0^2 + \frac{3}{4} A_0^2,$$

$$a_{35} = 2R_0 A_0 + \frac{1}{2} C_0^2 \cos \Theta_0, \quad a_{41} = R_0 C_0 \cos \Theta_0, \quad a_{42} = R_0 C_0 \sin \Theta_0, \quad a_{43} = R_0^2 + \frac{1}{2} C_0^2 + \frac{1}{4} A_0^2,$$

$$a_{44} = 0, \quad a_{45} = \frac{1}{2} C_0^2 \sin \Theta_0, \quad a_{51} = -\frac{1}{2} C_0 A_0 \sin \Theta_0, \quad a_{52} = C_0 \left(R_0 + \frac{1}{2} A_0 \cos \Theta_0 \right),$$

$$a_{53} = \frac{1}{4} C_0^2 \sin \Theta_0, \quad a_{54} = R_0 A_0 + \frac{1}{4} C_0^2 \cos \Theta_0, \quad a_{55} = R_0^2 + \frac{1}{2} C_0^2 + \frac{1}{2} A_0^2$$

.....(9)

Substituting $x = x_0 e^{i\tau}$ (x_0 being constant) into Eq.(8) will yield the characteristic equation and applying the Routh-Hurwitz criterion will provide sufficient conditions for the approximate solutions to be stable. In the present paper, however, the authors transform Eq.(8) into ordinary eigenvalue problems and discriminate the stability of the approximate solutions Eq.(2) by means of calculating the real parts of all of eigenvalues.

3. Numerical Examples and Discussion

The steady solutions have been obtained from the nonlinear simultaneous equation (4) [and so Eq.(5)], by making use of the same procedure as in the preceding paper.⁽¹⁾ It is clear from the following numerical results that the characteristic parameters C_0^* , R_0^* and so on, which have been defined and introduced to characterize the phenomena of the subharmonic oscillations of order 1/2 in the preceding paper,⁽¹⁾ are rather ineffective for the soft spring case as comparison with the hard spring case. Therefore, the authors do not use these values, but merely show the characteristic curves of the subharmonic vibrations of order 1/2 for some specific system parameters.

So, in the following figures, the solid lines and the broken lines correspond to the stable state and the unstable state, respectively (as discussed in Section 2.2).

3.1 The case of $G \neq 0$

In Fig.1, the characteristic curves of the subharmonic oscillations of order 1/2 of the system are shown for $G = 0.8$, $h = 0.03$ and $q = -0.0225$. In the figure, the indexes 1 and 2 like A_1 , C_2 and so on denote the combinations of branches A, C and R (i. e., steady solutions) corresponding to each other. So for example the fundamental component and the non-oscillatory term corresponding to branch C_1 of the subharmonic vibrations of order 1/2 are A_1 and R_1 , respectively. This also holds similarly for the phase angles θ and δ .

Figs.1(a) and (b) show the characteristic curves of the amplitude and the phase angle curves, respectively. From these figures, it is clear that stable states exist only in part of the high frequency region on branch 1 and all of the rest of these response curves are unstable. At the lower limit of the stable region, it should be noticed that the gradients of the response curves do not result in a vertical; that is, the limit of stability of the response curves does not necessarily correspond to the vertical tangent. Regardless

of the value of the damping coefficient h , C branch rises from zero. The variations of the phase properties θ_1 and δ_1 corresponding to branch 1 are very small when the frequency ν changes. In Fig.2 the response characteristics of the system are shown for the same parameters as in Fig.1, except for the nonlinearity $q = -0.03$. In general, the tendency of Fig.1 is similar to that of Fig.2, but there is a small difference between these two systems. Namely, if the coefficient of the nonlinearity q of the system decreases (i.e., $q = -0.0225 \rightarrow -0.03$), the response curves are generally depressed (which results in sim-

ilar figures) and they shift in the direction of low frequency. The space between branches C_1 and C_2 expands. However, the frequency at which branch C_1 tends to zero and the values of branches R_1 and A_1 may be nearly constant. The changes connecting the phase properties may also be quite small, although there is a variation with the shift of amplitude curves toward low frequencies (that is, if q decreases, θ_1 comes slightly closer to 90° and δ_1 to 180°).

3.2 The Case of $G = 0$

Figs.3 and 4 show the characteristic curves of the subharmonic oscillations of order $1/2$, when the constant force is equal to zero, i.e., $G = 0$. The difference between Fig.3 and Fig.4 is only in the value of the parameter q ; that is, Fig.3 has $q = -0.0225$ and Fig.4 $q = -0.03$. In these figures, the response curves of the subharmonic vibrations of order $1/2$ in the system are illustrated for five values of the damping coefficient ranging from zero to 0.05. Branches C 's of vibration curves rise from zero only in the system without damping (i.e., $h = 0$). Then the approximate frequency corresponding to the state $C = 0$ may be easily estimated by the following equation⁽¹⁾;

$$(\nu/n) \cong \sqrt{1 + (8/3)q} \quad (10)$$

(a) Amplitude characteristics

(b) Phase characteristics

Fig.1 Response curves of subharmonic vibrations of order $1/2$ (the system with $G \neq 0$)

Present theory { ———, - - - - -

Numerical integration;

	R	C	A
stable	○	△	□
unstable	●	▲	■

Fig.2 Response curves of subharmonic vibrations of order $1/2$ (the system with $G \neq 0$)

By making use of the procedure mentioned in Section 2.2, it may be possible to discriminate the stability of branch 2 in the system without damping. On the other hand, it may be quite difficult to distinguish the stability with regard to branch 1 by that procedure, because the characteristic equation of that system possesses a vanishing root, $\lambda = 0$, as in this branch 1. Here, the calculated curves are merely plotted in figures with chainlines, but their stability will be determined in the numerical integration section later.

As shown in Fig.3(a), the system with for example $h = 0.03$ could be in the stable

(a) Amplitude characteristics

(b) Phase characteristics

Fig.3 Response curves of subharmonic vibrations of order 1/2 (the system with $G = 0$)

frequency region, while it is very limited (i.e., $1.60 < \nu < 1.62$). So branch C_1 with its stable state is connected with the un-

Fig.4 Response curves of subharmonic vibrations of order 1/2 (the system with $G = 0$)

Numerical integration;

	R	C, θ	A, δ
stable	○	△	□
unstable	●	▲	■
stable(neutral)	◇	▽	◇
	R	C	A

(a) the case of $h = 0.03$

(b) the case of $h = 0.0$

Fig.5 Comparison with the results of numerical integration (the system with $G = 0$)

stable branch C_2 at the frequency $\nu \approx 1.62$. When the value of the damping coefficient h increases, the upper limit of the region slowly shifts in the direction of low frequency and the breadth of the region tends to become narrower. If h grows beyond some critical value, it may be easily understood that the subharmonic vibrations of order $1/2$ no longer take place. This tendency is the same as in the hard spring system.⁽¹⁾ In fact, as shown in Fig.3 and Fig.4, the system with $h=0.05$ has no stable states. In addition, it is clear that, if h increases, the variation of the amplitude A of the fundamental harmonics is relatively small and the variation of the non-oscillatory term R large.

Comparing Fig.3 and Fig.4 with Figs.1 and 2, the influence of the nonlinearity q upon the vibrating system with $G=0$ may be understood to be the same as with $G \neq 0$.

3.3 Comparison with Numerical Integration Results

In the present paper, numerically integrated solutions with period 4π of Eq.(1) have been researched by making use of so-called the "shooting method" (or initial value adjustment method).⁽⁶⁾⁽⁷⁾⁽⁸⁾⁽⁹⁾⁽¹⁰⁾⁽¹¹⁾⁽¹²⁾ At the same time the analytical results obtained by the harmonic balance method, as mentioned above, are quantitatively compared with the numerical results, and the stable domains are checked in detail by the numerical results. This shooting method can analyse not only the stable solutions, but also the unstable ones, and the characteristic multiplier obtained by this method can discriminate between the stable and the unstable. Namely, if there is even one of the characteristic multipliers whose absolute value is greater than unity, the solution with these characteristics is an unstable one. The numerical results by this shooting method applied to only branch 1 are plotted in Figs.1 and 5 together with the analytical results.

From Fig.1, concerning both the stable domain and the amplitude and phase characteristics, it may be concluded that these two results agree quite well with each other. Fig. 5(a) and Fig. 5(b) correspond to the system with $h=0.03$ and $h=0.0$ in Fig.3, respectively. In Fig. 5(a) the stable region by the present theory is in $\nu \gtrsim 1.60$, whereas that by the numerical integration is in $\nu \gtrsim 1.52$. In Fig.5(b) where h is equal to zero, according to the discrimination by the numerical integration, the region $\nu \gtrsim 1.56$ is neutral (i.e., infinitesimally stable) and the region $\nu \lesssim 1.56$ is unstable.

Consequently, although there is a small difference between the results of this theory and those of the numerical integration with regard to the limits of the stable region itself, it is understood that there is good agreement between both results with respect to the amplitude and phase characteristics.

4. Conclusions

In succession to the preceding paper⁽¹⁾ the subharmonic oscillations of order $1/2$ of a system with a Duffing soft type restoring characteristic are studied by making use of a harmonic balance method under the assumption that both the periodic exciting force and the constant force are working. Furthermore, the existence and validity of the present theory have been confirmed by means of numerical integration. The results are summarized as follows:

- (1) As regards the subharmonic vibrations of order $1/2$ in such an oscillating system considered here, the characteristics of a hard spring system considerably differ from those of a soft one.
- (2) When the nonlinearity q of the system decreases, the response curves similarly are depressed and they shift in the direction of low frequency.
- (3) The critical points between stable states and unstable ones do not necessarily correspond to the vertical tangents of the curves.
- (4) Even though the exciting force does not include a constant component, the subharmonic vibrations of order $1/2$ are likely to occur, but they may take place in quite a limited frequency region.
- (5) In the systems with $G \neq 0$, the amplitudes of C of the $1/2$ -th subharmonic component may rise from zero regardless of the value of the damping h .
- (6) In the systems with $G=0$, the amplitudes of C of the $1/2$ -th subharmonic component rise from zero only in the case of $h=0$. Then, if h increases, the figures of their response curves change into a kind of peninsula and their stable regions become narrower, simultaneously shifting in the direction of low frequency.
- (7) In the systems without the G component, a threshold of the damping h exists against an occurrence of the subharmonic oscillations of order $1/2$.
- (8) Comparing the results by the present theory with those by the numerical integrations, some differences may be recognized between the two so far as the limits of the stable region are concerned. It may be, however, concluded that the two results quantitatively coincide quite well with respect to the amplitude and phase characteristics.

Finally, it should be noted that for numerical computations in this paper, the authors were allowed to utilize the data processing centers of Ohta University and also of Kyushu University, and they wish to thank the members of these centers.

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