A Numerical Method of Free Jet from a Cross-flow Turbine Nozzle

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The nozzle of a cross-flow turbine has to give a certain circumferential velocity and an optimum angle to the flow at the nozzle exit (runner inlet). Therefore, the nozzle shape has an important influence upon the turbine performance. Its shape is asymmetric and complicated, and the exit flow from it has free boundaries. The flow from such a nozzle has not been analyzed enough until now because of complexity of its flow. In this report, the flow from a nozzle with arbitrary asymmetric curved surfaces is calculated numerically by Schwarz-Christoffel method. In order to estimate the accuracy of this method, the calculation results of the flow from a cross-flow turbine nozzle are compared with experimental ones. Furthermore, the effects of nozzle shape on the exit flow are investigated.


1. Introduction

The nozzle of a cross-flow turbine has to give a certain circumferential velocity and an optimum angle to the flow at the nozzle exit (runner inlet). Therefore, the nozzle shape has an important influence upon the turbine performance. In the previous paper, we reported on the basis of comparison between with and without runner that the influence of runner on the flow condition (flow angle and pressure distributions) is relatively small at the nozzle exit. Therefore, the study of the flow condition at the nozzle inside and outside for a simple nozzle is important for the investigation of the turbine performance.

Generally, the nozzle exit flow has no solid boundary and the boundary condition of nozzle exit flow is that the velocity on a free streamline is constant and its pressure is equal to atmospheric pressure. The two-dimensional jet with free streamlines has been dealt with as a potential flow and calculated by a conformal mapping method. It is common knowledge that the theoretical values about the two-dimensional jet through a straight wall nozzle coincide with the experimental ones exactly.

For a curved wall jet, the exit flow has been obtained by a streamline curvature method for one solid wall and by a conformal mapping and a singular point method for a symmetric curved wall. But, in such a case of cross-flow turbine nozzle with asymmetric and complicated shape, a numerical method to calculate the free streamline shape from two curved side walls has not been shown until now.

In this report, the flow from a nozzle with arbitrary asymmetric curved surfaces is calculated numerically by Schwarz-Christoffel method. In order to estimate the accuracy of this method, the calculation results of the flow from a cross-flow turbine nozzle are compared with experimental ones. Furthermore, the effects of nozzle shape on the exit flow are investigated.

2. A Numerical Calculation of the Flow through Nozzle

Let \( W \) denote a complex velocity potential and consider \( \Omega \) plane defined with the following equation.

\[
Q = \log(-\frac{d\Omega}{dW})
\]

Using \( \theta \) for the angle between velocity \( q \) and \( z \) axis, we have

\[
Q = \log r^{-1} + i\theta
\]

For a flow with straight walls and free streamlines, the straight wall (\( \theta = \text{const.} \)) and the free streamline on the \( z \) plane correspond to the straight lines parallel to real axis and imaginary axis on the \( \Omega \) plane respectively.

Using a and b for the two bend points on the \( z \) plane, the function to transform this \( z \) plane to the upper plane of the \( t \) plane is given as the following equation by Schwarz-Christoffel method.

Fig. 1 z plane and t plane
\[ Q = 2B \log(\sqrt{-a} + \sqrt{-b}) + c \]  

Next a new \( \Omega \) plane can be made adding many \( \Omega \) planes together as shown in Fig. 2. 

\[ \Omega = \sum \Omega_\pi \]  

Then at each range of \( t \), 

- \( t = A_1 \) : plane wall
- \( A_1 < t < A_2 \) : curved wall
- \( A_2 < t < A_3 \) : free streamline
- \( A_3 < t < A_4 \) : curved wall
- \( A_4 < t < A_5 \) : plane wall

so that a curved wall can be obtained. Its new \( \Omega \) plane is written in the following expression.

\[ Q = \frac{2B}{\pi} \log(\sqrt{A_1} + \sqrt{A_2}) \]

\[ + 2 \sum_{n=1}^{\infty} B_n \log(\sqrt{A_{n+1}} + \sqrt{A_n}) \]

\[ + \log -ia \]  

Then the angle of inclination to a horizontal line at each point on the nozzle wall in Fig. 1 is obtained by the following equations.

At \( t = A_1 \)
\[ \theta_1 = -a \]

At \( t = A_2 \) (2p \( \pi \) n)
\[ \theta_2 = -a + \frac{2B_1}{\pi} \sqrt{A_2} \tan^{-1} \left( \sqrt{A_1 - A_2} / \sqrt{A_2 - A_{n+1}} \right) \]

\[ + 2 \sum_{n=1}^{\infty} B_n \tan^{-1} \left( \sqrt{A_1 - A_2} / \sqrt{A_{n+1} - A_n} \right) \]

At \( t = A_3 \) (n + 1) \( \pi \) \( \leq t \leq A_{n+1} \)
\[ \theta_3 = -a + \frac{2B_1}{\pi} \sqrt{A_3} \tan^{-1} \left( \sqrt{A_1 - A_3} / \sqrt{A_3 - A_{n+1}} \right) \]

\[ + 2 \sum_{n=1}^{\infty} B_n \tan^{-1} \left( \sqrt{A_1 - A_3} / \sqrt{A_{n+1} - A_n} \right) \]

At \( t = A_{n+1} \)
\[ \theta_n = -a + \frac{2B_1}{\pi} \sqrt{B_n} \]

Given the angles at each point as \( \theta_1, \theta_2, \ldots, \theta_\pi \) and assuming \( A_1, A_2, \ldots, A_{\pi} \), the simultaneous equations about \( B_1, B_2, \ldots, B_{\pi-1} \) are obtained and \( B_1, B_2, \ldots, B_{\pi-1} \) can be obtained by solving these equations.

Otherwise, \( w \) plane is also transformed by a Schwarz-Christoffel transformation to upper half plane of \( t \) plane by the following equation.

\[ W = (m/r) \log(t - \iota) \]  

where, \( \iota \) indicates a sink and \( m \) the last width of jet.

Connecting Eqs. (5) and (9) with \( \Omega \) plane.

- \( t = A_1 \) Curved wall
- \( t = A_2 \) Curved wall
- \( t = A_4 \) Curved wall
- \( t = A_5 \) Plane wall

Fig. 2 \( \Omega \) plane

Parameter \( \gamma \), the relation between \( W \) and \( Z \) can be obtained. Meanwhile, we have

\[ dW/dt = (m/r)(t - \iota) \]  

Hence the length \( L_{\pi} \) between two points \( A_{\pi-1} \) and \( A_{\pi} \) on the solid wall in Fig. 1 is obtained as follows:

\[ L_{\pi} = \int_{A_{\pi-1}}^{A_{\pi}} \frac{1}{\sqrt{1 - (\psi A_{\pi} - A_{\pi-1})}} \sqrt{(A_{\pi-1} - A_{\pi})^2} \]

\[ \times \int_{t_{A_{\pi-1}}}^{t_{A_{\pi}}} \frac{1}{\sqrt{1 - (t_{A_{\pi}} - t_{A_{\pi-1}})^2}} \sqrt{1 - (A_{\pi-1} - A_{\pi})^2} \frac{dt}{dt} \]

\[ + \sqrt{1 - (A_{\pi-1} - A_{\pi})^2} \]  

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\[ \times \int_{t_{A_{\pi-1}}}^{t_{A_{\pi}}} \frac{1}{\sqrt{1 - (t_{A_{\pi}} - t_{A_{\pi-1}})^2}} \sqrt{1 - (A_{\pi-1} - A_{\pi})^2} \frac{dt}{dt} \]

\[ + \sqrt{1 - (A_{\pi-1} - A_{\pi})^2} \]  

Then the position of \( A_{\pi+1} \), with the position of \( A_{\pi} \) as the origin, is

\[ \frac{dZ}{dt} = \frac{dW}{dt} \frac{d\psi}{dW} = -\frac{m}{r} \frac{d\psi}{dW} \]  

so that

\[ Z_{\pi+1} = Z_{\pi} + \frac{m}{r} \int_{t_{A_{\pi}}}^{t_{A_{\pi+1}}} \frac{1}{\sqrt{1 - (t_{A_{\pi+1}} - t_{A_{\pi}})^2}} dt \]

2.1 Numerical calculation procedure

(1) Assume \( \iota \) and \( m \)

As \( m \) is the last width of jet, it can be assumed appropriately by predicting the free streamline in Fig. 1.

(2) Calculate the distance between each two points from Eqs. (11), (12) using the assumed values \( A_1, A_2, \ldots, A_{\pi} \) and the calculated values \( B_1, B_2, \ldots, B_{\pi-1} \).

(3) Calculate \( A_1, A_2, \ldots, A_{\pi} \) from the following equation using the real distance \( S \) and the calculated one \( L \).

\[ A_{\pi} = G_{\pi} + \frac{1}{S/\pi}(G_{\pi} - G_{\pi-1}) \]  

where \( G \) is the value of \( A \) of the last iteration. And the values of \( A_1 \) and \( A_{\pi+1} \) are fixed at 1.0, -1.0 respectively.

(4) Repeat (2) ~ (3) till the values \( A_1, A_2, \ldots, A_{\pi} \) do not change.

(5) Calculate the position of \( A_{\pi+1}, \) as the position of \( A_\pi \) is the origin, from Eq. (14) using the assumed values \( \iota \) and \( m \) and the calculated values \( A_1, A_2, \ldots, A_{\pi} \) and \( B_1, B_2, \ldots, B_{\pi-1} \). Assume again new values of \( \iota \) and \( m \) using the real position and the calculated one. Increase \( m \) for an enlargement of the exit width of nozzle and bring \( \iota \) close to 1.0 for an extension of the asymmetry of nozzle.

(6) Repeat the calculations of (1) ~ (5) till the values of \( \iota \) and \( m \) do not change.

2.2 Estimation of the velocity and pressure

As \( Q \) has been calculated as Eq. (2), \( q \) and \( \phi \) can be obtained from its real part and imaginary part respectively.

Putting the velocity on a free
streamline \( \alpha = 1.0 \), as \( 2g/H \alpha = 1.0 \), the pressure \( P \) of an arbitrary point is obtained from the following equation.

\[
P(\rho g/\alpha H) = 1.0 - \alpha^2/(2gH) = 1.0 - \alpha^2 \quad \cdots \cdots (16)
\]

3. Jets from Slit

At first, simple jets from two-dimensional slit are calculated. The theoretical configuration of free streamlines of jet is given by the solid line in Fig. 3. The circular marks are the results obtained by this numerical calculation method in case of \( n = 5 \). The calculated and theoretical values about the contraction coefficient \( C_c \) (last jet width slit width) to each half angle \( \theta \) (in Fig. 3) are presented in Table 1. Both numerical results agree with the theoretical values exactly.

4. Calculation results and Comparison with Experimental values

The detail of experimental apparatus is the same as shown in the previous paper. The nozzle shape is shown in Fig. 4. The nozzle exit radius \( R_e = 124 \text{mm} \) and the nozzle span \( b = 100 \text{mm} \). The parameter which prescribe the nozzle shape are given in the following items.

1. Nozzle entry arc \( (\delta) \)
2. Nozzle throat width \( (S_t) \)
3. Nozzle upper wall shape \( (S/S_t) \) (in Fig. 5)
   - A-type: Circular arc shape with nozzle tip angle \( \alpha = 15^\circ \)
   - B-type: The nozzle width \( (S) \) which decreases at constant rate from the nozzle throat to the nozzle tip.
   - C-type: Log spiral shape
   The isobar distributions inside of the nozzle and the free streamline configurations obtained from the wall pressure distributions are presented in Fig. 6 for \( \delta = 90^\circ \). \( S_t/R_e \theta = 0.28 \). A-type nozzle. At the inside of nozzle, the fluid flows turning along the upper wall and therefore the isobar shows a concentric shape with its center near the center of a circle indicating the upper wall arc and the static pressure increases with an increasing radius. The calculated results coincide with the experimental ones. The calculation were conducted with \( n = 10 \).

The static pressure and flow angle \( \sigma \) (the angle between a peripheral velocity component \( V \) and a radial velocity component \( V_r \), \( \sigma = \tan^{-1}(V/V_r) \)) distributions along the nozzle exit are shown in Fig. 7 and Fig. 8 respectively. The numerical results agree well with the experimental ones in both cases. In Fig. 6, the pressure along the nozzle exit arc increases.

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**Table 1** Contraction coefficient for various nozzle angles \( \theta \) (in Fig. 3)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>22.5°</th>
<th>45°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory (^1)</td>
<td>0.855</td>
<td>0.745</td>
<td>0.611</td>
</tr>
<tr>
<td>Calc.</td>
<td>0.853</td>
<td>0.746</td>
<td>0.610</td>
</tr>
</tbody>
</table>

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Fig. 3 Jet from slit

Fig. 4 Schematic diagram of the test nozzle

Fig. 5 Changes of \( S/S_t \) along peripheral position \( \theta/\delta \)

Fig. 6 Isobar distributions and free streamline configurations
with peripheral position because the radius from the center of upper wall increases with peripheral position from throat to tip. But the pressure has to decrease to atmospheric pressure at nozzle tip and therefore it shows an asymmetric distribution with a peak on the way as shown in Fig. 7. Thus, the flow at the nozzle exit of a cross-flow turbine does not recover to atmospheric pressure immediately but has a considerable value of pressure. The static pressure and flow angle a distributions along the nozzle exit for various nozzle entry arcs are presented in Fig. 7 and Fig. 8 respectively. The exit pressure has a larger value with an increase of the nozzle entry arc. Therefore the nonuniformity of pressure distribution along the peripheral position at the nozzle exit increases with an increase of δ. Their pressure distributions are the same regardless of the size of nozzle entry arc and the pressure peaks are situated near \( δ/δ = 0.25 \) in all cases. For the flow angle \( α \), the uniformity is good for small nozzle entry arc.

Fig. 9 shows the influence of nozzle throat width on the pressure and flow angle. In the case of large \( S_{n}/R_{o} \), the flow angle \( α \) is large, especially at the first half of nozzle exit. The nozzle throat width \( S_{n}/R_{o} \) is a significant parameter because an increase of flow angle results in a shock loss at the blade inlet. The exit pressure also increases with an increase of \( S_{n}/R_{o} \).

The influence of upper wall shape on the pressure and flow angle is shown in Fig. 10. In the case of C-type the flow angle \( α \) at the first half of nozzle exit arc is large because in that region the decrease rate of \( S \) is large as shown in Fig. 5. In contrast at the nozzle exit \( α \) is small and the uniformity along peripheral position is not good. A-type, which has a circular arc shape, has the best uniformity along the peripheral position. There is hardly any difference between these three types about the exit pressure, that is, the exit velocity.

5. Conclusions

(1) A flow through an asymmetric nozzle with an arbitrary curved wall is calculated numerically and it is shown that calculated results about the nozzle of a cross-flow turbine agree well with the experimental ones.

(2) At the nozzle exit of a cross-flow turbine, a flow does not drop to an atmospheric pressure immediately even if it is a potential one and it has a considerable value of pressure owing to the turning along the nozzle upper wall.

(3) The influence of nozzle shape (nozzle entry arc, nozzle throat width and nozzle wall shape) on the flow at nozzle exit can be investigated using this calculation method.
FACOM M-360 computer was used for above numerical calculation and the execution time was about one hundred seconds.

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References
(2) Kobayashi, R., ZAMM, 42-7/8 (1962), Seite 281-293.