Analysis of Unsteady State Problems in Nonlinear

Heat Conduction Materials*

(1st. Report: One-dimensional Exponential

Models of κ =constant)

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Development of the new engineering materials such as ceramics, amorphous semiconductors, synthetic polymers, extreme-low and high temperature resisting materials, etc. has made it more and more important to analyze the unsteady nonlinear heat conduction problems concerning these materials.

This paper deals with two cases of one-dimensional exponential models of κ =constant: (1) λ and $c\rho$ are functions of coordinate x (model A) and (2) λ and $c\rho$ are functions of temperature T (model B). The transient features of heat transfer for both models have been mathematically analyzed and numerically calculated according to exact solutions.

Key Words: Thermal Conduction, Thermophysical Properties, Industry

1. Introduction

The boundary value problems of nonlinear heat conduction are encountered in several new-electronic materials⁽¹⁾ such as ceramics, amorphous semiconductors, synthetic polymers, extreme-low and high temperature resisting materials, etc..

Some papers⁽²⁾⁻⁽¹⁰⁾ have been published concerning the boundary value problems of nonlinear heat conduction. These studies, however, did not deal with the structures of exact solutions of the nonlinear heat conduction.

The purpose of this paper is to present exact solutions of the following two boundary value problems considering κ =constant:

(1) Thermal conductivity λ and thermal capacity $c\rho$ are functions of $\exp(\alpha_1 x)$.

(2) Thermal conductivity λ and thermal capacity $c\rho$ are functions of $\exp(\alpha_2 T)$.

Nomenclature

c : specific heat
 l : length of heat conduction model
 m : geometrical symmetric order ($m=0$: slab, $m=1$: cylinder, $m=2$: sphere)
 q : heat flux
 T : temperature K
 t : time
 x : coordinate

α_i : exponential coefficient of nonlinear model ($i=1, 2$)

ξ : dimensionless coordinate in Eq.(25)

θ : dimensionless temperature in Eq.(25)

κ : thermal diffusivity $\equiv \lambda/c\rho$

λ : thermal conductivity

ρ : density of nonlinear heat conduction material

τ : dimensionless time in Eq.(25), Fourier number

2. Physical Models

The following two models have been used.

(1) Model A

$$\left. \begin{aligned} \lambda &= \lambda_0 e^{\alpha_1 x}, \quad c\rho = c_0 \rho_0 e^{\alpha_1 x} \\ \kappa &= \lambda_0 / c_0 \rho_0 \end{aligned} \right\} \dots\dots\dots (1)$$

Fig.1 Thermal diffusivity vs. temperature for heat conduction materials of κ =constant⁽¹¹⁾

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$$\begin{aligned}\lambda(\text{Ni}) &= 5.43 \exp(2.93 \times 10^{-4} T), \quad \lambda(\text{Ti}) = 1.55 \exp(2.96 \times 10^{-4} T) \\ \lambda(\text{Pt}) &= 6.52 \exp(1.99 \times 10^{-4} T), \quad \lambda(\text{V}) = 2.90 \exp(2.86 \times 10^{-4} T) \\ \lambda(\text{Nb}) &= 5.04 \exp(2.43 \times 10^{-4} T), \quad \lambda(\text{U}) = 2.26 \exp(6.62 \times 10^{-4} T)\end{aligned}$$

Fig.2 Temperature dependence on thermal conductivity of heat conduction materials⁽¹⁹⁾

(2) Model B

$$\left. \begin{aligned}\lambda &= \lambda_0 e^{\alpha_2 T}, \quad c\rho = c_0 \rho_0 e^{\alpha_2 T} \\ x &= \lambda_0 / c_0 \rho_0\end{aligned} \right\} \dots\dots\dots (2)$$

where α_i ($i=1,2$) is the exponential coefficient of the two models A and B.

Using α_1 and α_2 for both models A and B respectively, it is possible to compare the exact solutions of the two models A and B. Equations (1) and (2) show that λ and $c\rho$ are constant in the limit of $\alpha_i \rightarrow 0$.

3. Basic Equations

The basic equations of nonlinear heat conduction of geometrically symmetric bodies are given as⁽²⁰⁾

$$c\rho \frac{\partial T}{\partial t} = \frac{1}{x^m} \frac{\partial}{\partial x} \left[x^m \lambda(x, T) \frac{\partial T}{\partial x} \right] \dots\dots\dots (3)$$

In general, the thermal conductivity $\lambda(x, T)$ of Eq.(3) is specified for the following cases: (1) $\lambda = \lambda(x)$, (2) $\lambda = \lambda(T)$ and (3) $\lambda = \lambda(x, T)$ type.

Therefore, Eq.(3) is classified into the following three basic equations of heat conduction.

(1) Basic equation of nonlinear heat conduction of the first kind:

$$c\rho \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + m \frac{\lambda}{x} \frac{\partial T}{\partial x} + \frac{d\lambda}{dT} \frac{\partial T}{\partial x} \dots\dots\dots (4)$$

(2) Basic equation of nonlinear heat conduction of the second kind:

$$c\rho \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + m \frac{\lambda}{x} \frac{\partial T}{\partial x} + \frac{d\lambda}{dT} \left(\frac{\partial T}{\partial x} \right)^2 \dots\dots\dots (5)$$

(3) Basic equation of nonlinear heat conduction of the third kind:

$$\begin{aligned}c\rho \frac{\partial T}{\partial t} &= \lambda \frac{\partial^2 T}{\partial x^2} + m \frac{\lambda}{x} \frac{\partial T}{\partial x} + \frac{\partial \lambda}{\partial x} \frac{\partial T}{\partial x} \\ &+ \frac{\partial \lambda}{\partial T} \left(\frac{\partial T}{\partial x} \right)^2 \dots\dots\dots (6)\end{aligned}$$

4. Basic Solutions

4.1 Basic solution of model A

Under the assumption of the basic equation of nonlinear heat conduction of the first kind, the equation of model A is solved for certain boundary and initial condition.

Using the variable separation method, the basic solution can be obtained as follows. Putting $\tau = \lambda_0 t / c_0 \rho_0$, from Eq.(4), we have

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{m}{x} \frac{\partial T}{\partial x} + \alpha_1 \frac{\partial T}{\partial x} \dots\dots\dots (7)$$

Introducing the following new variable transform defined by

$$T(x, \tau) = x^{(1-m)/2} e^{-\alpha_1 x/2} e^{-\alpha_1^2 \tau/4} \Psi(x, \tau) \dots\dots\dots (8)$$

and substituting Eq.(8) into Eq.(7), we have the fundamental basic equation of nonlinear heat conduction for the model A as follows:

$$\frac{\partial \Psi}{\partial \tau} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{x} \frac{\partial \Psi}{\partial x} - \frac{m\alpha_1}{2x} \Psi - \frac{(1-m)^2}{4x^2} \Psi \dots\dots\dots (9)$$

Therefore, assuming that $\Psi(x, \tau)$ is $\Psi(x, \tau) = X(x)Z(\tau)$ and using the variable separation method, we get:

$$\begin{aligned}\frac{1}{Z(\tau)} \frac{dZ}{d\tau} &= \frac{1}{X(x)} \left[\frac{d^2 X}{dx^2} + \frac{1}{x} \frac{dX}{dx} \right. \\ &\left. - \frac{m\alpha_1}{2x} X - \frac{(1-m)^2}{4x^2} X \right] = -\mu^2 \dots\dots\dots (10)\end{aligned}$$

Equalizing the constant $-\mu^2$ in both sides of Eq.(10), we can obtain a general solution of $Z(\tau)$ as:

$$Z(\tau) = A_0 e^{-\mu^2 \tau} \dots\dots\dots (11)$$

where A_0 is an arbitrary constant.

Meanwhile, the differential equation concerning $X(x)$ of Eq.(10) is expressed by the wave equation in parabolic coordinate⁽²¹⁾

Now, $F(\alpha, \gamma; Z)$ is defined as a confluent hypergeometric function and the arbitrary constants are A and B . Then we can obtain the following solution:

$$\begin{aligned}X(x) &= A e^{-i\mu x} x^{(1-m)/2} \\ &\times F\left(1 - \frac{m}{2} - i \frac{m\alpha_1}{4\mu}, 2 - m; 2i\mu x\right) \\ &+ B e^{i\mu x} x^{-(1-m)/2} \\ &\times F\left(\frac{m}{2} + i \frac{m\alpha_1}{4\mu}, m; -2i\mu x\right) \dots\dots\dots (12)\end{aligned}$$

Also, we find a steady state solution⁽²²⁾ (S.S.S.) for $\partial T / \partial \tau = 0$ as:

$$T(x, \tau \rightarrow \infty) = \int \frac{C_1}{x^m \lambda(x)} dx + C_2 \dots\dots\dots (13)$$

where C_1, C_2 are integral constants. And then the solution of Eq.(7) is obtained as:

$$\begin{aligned}T(x, \tau) &= \sum_{\mu=1}^{\infty} x^{(1-m)/2} e^{-(\alpha_1/2)x} e^{-[\mu^2 + (\alpha_1^2/4)]\tau} \\ &\times \left[A_{\mu} e^{-i\mu x} x^{(1-m)/2} F\left(1 - \frac{m}{2} - i \frac{m\alpha_1}{4\mu}, 2 - m; 2i\mu x\right) \right. \\ &\left. + B_{\mu} e^{i\mu x} x^{-(1-m)/2} F\left(\frac{m}{2} + i \frac{m\alpha_1}{4\mu}, m; -2i\mu x\right) \right] \\ &+ \int \frac{C_1}{x^m \lambda(x)} dx + C_2 \dots\dots\dots (14)\end{aligned}$$

4.2 Basic solution of model B

It is established that Van Dusen's heat function for λ and $c\rho$ can play an important role in obtaining the solution of the nonlinear heat conduction problem of the second kind.

$$U = \int_0^T \lambda(T) dT = \int_0^T e^{a_1 T} dT \\ = \frac{1}{a_2} (e^{a_2 T} - 1) \quad (15)$$

Now substituting Eq.(15) into Eq.(5) and putting $\tau = \lambda_0 t / c_0 \rho_0$, we get

$$\frac{\partial U}{\partial \tau} = \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial U}{\partial x} \right) \\ = \frac{\partial^2 U}{\partial x^2} + \frac{m}{x} \frac{\partial U}{\partial x} \quad (16)$$

Using the variable separation type $U(x, \tau) = X(x)Z(\tau)$, Eq.(16) is given by

$$\frac{1}{Z(\tau)} \frac{dZ}{d\tau} = \frac{1}{X(x)} \left(\frac{d^2 X}{dx^2} + \frac{m}{x} \frac{dX}{dx} \right) \\ = -\mu^2 \quad (17)$$

Therefore, the result makes it possible to write $Z(\tau) = A_0 \exp(-\mu^2 \tau)$. On the other hand, $X(x)$ is given by the solution for the next differential Eq.(18).

$$\frac{d^2 X}{dx^2} + \frac{m}{x} \frac{dX}{dx} + \mu^2 X = 0 \quad (18)$$

Applying the transformation to $y = \mu x$ and $X = y^{(1-m)/2} Q(y)$, this equation can lead to the well-known Bessel differential equation. Therefore, a solution of Eq.(18) is obtained by using the Bessel function J and the Neumann function Y as:

$$U(x, \tau) = \sum_{m=1}^{\infty} e^{-\mu^2 \tau} \left\{ A_{\mu} \frac{J_{(m-1)/2}(\mu x)}{(\mu x)^{(m-1)/2}} \right. \\ \left. + B_{\mu} \frac{Y_{(m-1)/2}(\mu x)}{(\mu x)^{(m-1)/2}} \right\} + \text{S.s.s.}, (m \geq 1) \\ = \sum_{m=1}^{\infty} e^{-\mu^2 \tau} (A_{\mu} \sin \mu x + B_{\mu} \cos \mu x) \\ + \text{S.s.s.}, (m=0) \quad (19)$$

where A_{μ} and B_{μ} are arbitrary constants.

Now, solving Eq.(15) for T , we can obtain the following relation:

$$T(x, \tau) = \frac{1}{a_2} \ln(1 + a_2 U) \quad (20)$$

Also, using Eq.(16), a steady state solution can be found for the relation $\partial U / \partial \tau = 0$.

Assuming the integral constants as C_1, C_2 , the solution of Eq.(16) is:

$$U(x, \tau \rightarrow \infty) = \int \frac{C_1}{x^m} dx + C_2 \quad (21)$$

Accordingly substituting Eqs.(19), (21) into Eq.(20), we obtain the unsteady basic solution for the nonlinear heat conduction problem of the second kind:

$$T(x, \tau) = \frac{1}{a_2} \ln \left\{ 1 + a_2 \left[\sum_{m=1}^{\infty} e^{-\mu^2 \tau} \left(A_{\mu} \right. \right. \right. \\ \left. \left. \times \frac{J_{(m-1)/2}(\mu x)}{(\mu x)^{(m-1)/2}} + B_{\mu} \frac{Y_{(m-1)/2}(\mu x)}{(\mu x)^{(m-1)/2}} \right) \right] \right\}$$

$$\left. \begin{aligned} &+ \int \frac{C_1}{x^m} dx + C_2 \right\}, (m \geq 1) \\ &= \frac{1}{a_2} \ln \left\{ 1 + a_2 \left[\sum_{m=1}^{\infty} e^{-\mu^2 \tau} (A_{\mu} \sin \mu x \right. \right. \\ &\left. \left. + B_{\mu} \cos \mu x) + C_1 x + C_2 \right] \right\}, (m=0) \end{aligned} \right\} \quad (22)$$

5. Initial and Boundary Conditions

Since it is impossible to solve these problems with the general boundary condition of the third kind, we assume the following two boundary conditions for solving the nonlinear heat conduction problem.

(1) The condition of both sides having temperature fixed (Condition 1- boundary condition of the first kind):

$$T(x=0, \tau) = T_1, T(x=l, \tau) = T_2, \\ T(x, \tau=0) = T_2 \quad (23)$$

(2) The condition of one-side being heated and the other side adiabatic (Condition 2- boundary condition of the second kind):

$$T(x=0, \tau) = T_1, q(x=l, \tau) = -\lambda \\ \times (\partial T / \partial x) = 0, T(x, \tau=0) = T_2 \quad (24)$$

6. Exact Solutions of the Unsteady State

6.1 Exact solution of the condition 1 for the model A

Now, we intend to solve Eq.(14) in order to satisfy Eq.(23) for the model A of $x = \text{constant}$. Putting $x = \lambda_0 / c_0 \rho_0$, we introduce the following dimensionless numbers and substitute τ ,

$$\theta = \frac{T}{T_1}, \quad \zeta = \frac{x}{l}, \quad \tau = \frac{\kappa t}{l^2}, \\ \theta_1 = (\theta)_{\tau=0}, \quad \theta_2 = (\theta)_{\tau=\tau_1} \quad (25)$$

In this case the unsteady exact solution could be expressed by Eq.(14) and we can obtain the following solution for $m=0$:

$$\theta(\zeta, \tau) = \sum_{m=1}^{\infty} \zeta^{1/2} e^{-(a_1/2)\tau} e^{-[\mu^2 + (a_1/2)\tau]\tau} \\ \times [A_{\mu} e^{-i\mu\zeta} \zeta^{1/2} F(1, 2; 2i\mu\zeta) \\ + B_{\mu} e^{i\mu\zeta} \zeta^{-1/2} F(0, 0; -2i\mu\zeta)] \\ + \text{S.s.s.} \quad (26)$$

Equation (26) is related to the confluent hypergeometric function (Kummer's formula).

$$F(\alpha, \gamma; Z) = e^Z F(\gamma - \alpha, \gamma; -Z) \quad (27)$$

and the following relation between F function and Bessel function J_{ν} ,

$$F(\nu + \frac{1}{2}, 2\nu + 1; 2iZ) = J_{\nu}(Z) \frac{\Gamma(\nu + 1)}{(Z/2)^{\nu} e^{-iZ}} \quad (28)$$

Substituting Eqs.(27) and (28) into Eq.(26) and rearranging Eq.(26), we obtain

$$\theta(\zeta, \tau) = \sum_{\mu=1}^{\infty} e^{-(\alpha_1/2)\zeta} e^{-[\mu^2 + (\alpha_1^2/4)]\tau} \times (A_{\mu} \sin \mu\zeta + B_{\mu} \cos \mu\zeta) + \text{S.s.s.} \quad (29)$$

The two constants A_{μ} and B_{μ} of Eq.(29) and steady state solution are determined for the condition of Eq.(23).

From the relation $\sin \mu = 0$, we then find $\mu = n\pi$ and:

$$A_{\mu} = -\frac{2(\theta_1 - \theta_2)n\pi}{(n\pi)^2 + (\alpha_1^2/4)}, \quad B_{\mu} = 0 \quad (30)$$

$$\text{S.s.s.} = \int C_1 e^{-\alpha_1 \zeta} d\zeta + C_2 = (\theta_1 - \theta_2) e^{-(\alpha_1/2)\zeta} \frac{\sinh[\alpha_1(1-\zeta)/2]}{\sinh(\alpha_1/2)} + \theta_2 \quad (31)$$

So, substituting Eqs.(30) and (31) into Eq.(29), we can get the following unsteady state exact solution:

$$\theta(\zeta, \tau) = (\theta_1 - \theta_2) e^{-(\alpha_1/2)\zeta} \left\{ \frac{\sinh[\alpha_1(1-\zeta)/2]}{\sinh(\alpha_1/2)} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{n^2 \pi^2}{(n\pi)^2 + (\alpha_1^2/4)} \sin n\pi\zeta e^{-[(n\pi)^2 + (\alpha_1^2/4)]\tau} \right\} + \theta_2 \quad (32)$$

Also, to express the heat flux, by nondimensionalizing the steady heat flux $q_0(\alpha_1 \rightarrow 0) = \lambda_0$, we get:

$$\bar{q}(\zeta, \tau) = \frac{q}{q_0} = \frac{\alpha_1}{2} e^{(\alpha_1/2)\zeta} (\theta_1 - \theta_2) \left\{ \frac{\sinh[\alpha_1(1-\zeta)/2] + \cosh[\alpha_1(1-\zeta)/2]}{\sinh(\alpha_1/2)} - \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin n\pi\zeta}{n\pi} - \frac{2 \cos n\pi\zeta}{\alpha_1} \right) \frac{n^2 \pi^2}{(n\pi)^2 + (\alpha_1^2/4)} e^{-[(n\pi)^2 + (\alpha_1^2/4)]\tau} \right\} \quad (33)$$

From Eq.(33) at infinity ($\tau \rightarrow \infty$), we can obtain the steady state solution of heat flux as.

$$\bar{q}(\zeta, \tau \rightarrow \infty) = \frac{\alpha_1}{2} e^{(\alpha_1/2)\zeta} \frac{(\theta_1 - \theta_2)}{\sinh(\alpha_1/2)} = \text{constant} \quad (34)$$

In this case, the steady state heat flux of the model A is expressed by a constant value independently of the coordinate.

6.2 Exact solution of the condition 2 for model A

Similarly, using the same model A, an exact solution for the condition Eq.(24) will be found. When the general solution of this problem is given by Eq.(29), the steady state solution of the temperature is given by:

$$\text{S.s.s.} = \int C_1 e^{-\alpha_1 \zeta} d\zeta + C_2 = \theta_1 \quad (35)$$

Now, substituting Eq.(35) into Eq.(29), A_{μ} and B_{μ} for Eq.(24) of the condition 2 are determined as follows:

$$A_{\mu} = -2(\theta_1 - \theta_2) \left(\frac{1 - \cos \nu_{\mu}}{\nu_{\mu}} \right) / \left[\frac{2(1 - e^{-\alpha_1/2})}{\alpha_1} - \frac{1}{(2\nu_{\mu})^2 + (\alpha_1^2/4)} \right] \left[(2\nu_{\mu} \sin 2\nu_{\mu} - \frac{\alpha_1}{2} \cos 2\nu_{\mu}) e^{-\alpha_1/2} + \frac{\alpha_1}{2} \right], \quad B_{\mu} = 0 \quad (36)$$

where ν_{μ} is the root of $\tan \nu_{\mu} = \nu_{\mu} / (\alpha_1/2)$. Therefore, substituting Eq.(36) into Eq.(29) and rearranging this equation, we can obtain the next expression.

$$\theta(\zeta, \tau) = \theta_1 - 2(\theta_1 - \theta_2) \sum_{\mu=1}^{\infty} \frac{1 - \cos \nu_{\mu}}{\nu_{\mu}} \frac{e^{-\alpha_1 \zeta/2} \sin \nu_{\mu} \zeta e^{-[\nu_{\mu}^2 + (\alpha_1^2/4)]\tau}}{[2(1 - e^{-\alpha_1/2})/\alpha_1] - [1/((2\nu_{\mu})^2 + (\alpha_1^2/4))] \{ [2\nu_{\mu} \sin 2\nu_{\mu} - (\alpha_1/2) \cos 2\nu_{\mu}] e^{-\alpha_1/2} + (\alpha_1/2) \}} \quad (37)$$

Similarly, by differentiating Eq.(37) and nondimensionalizing, we can obtain the heat flux as follows:

$$\bar{q}(\zeta, \tau) = 2(\theta_1 - \theta_2) e^{(\alpha_1/2)\zeta} \sum_{\mu=1}^{\infty} \frac{1 - \cos \nu_{\mu}}{\nu_{\mu}} \frac{[\nu_{\mu} \cos \nu_{\mu} \zeta - (\alpha_1/2) \sin \nu_{\mu} \zeta] e^{-[\nu_{\mu}^2 + (\alpha_1^2/4)]\tau}}{\{ [2(1 - e^{-\alpha_1/2})/\alpha_1] - 1/[(2\nu_{\mu})^2 + (\alpha_1^2/4)] \} \times \{ [2\nu_{\mu} \sin 2\nu_{\mu} - (\alpha_1/2) \cos 2\nu_{\mu}] e^{-\alpha_1/2} + (\alpha_1/2) \}} \quad (38)$$

6.3 Exact solution of the condition 2 for model B

Using the general solution of the exponential model B for $\kappa = \text{constant}$ (given by Eq.(2)) given by Eq.(22) assuming that $m=0$, the steady state solution can be given as:

$$\text{S.s.s.} = \int C_1 d\zeta + C_2 = \frac{1}{\alpha_2} [e^{\alpha_2 \zeta} + (e^{\alpha_2 \theta_1} - e^{\alpha_2 \theta_2})(1 - \zeta) - 1] \quad (39)$$

Determination of A_{μ} and B_{μ} for Eq.(23) and $\mu = n\pi$ from the relation $\sin \mu = 0$ yields,

$$A_{\mu} = -\frac{2(e^{\alpha_2 \theta_1} - e^{\alpha_2 \theta_2})}{n\alpha_2 \pi}, \quad B_{\mu} = 0 \quad (40)$$

Substituting Eqs.(39) and (40) into Eq.(22) and rearranging, we get:

$$\theta(\zeta, \tau) = \frac{1}{a_2} \ln \left[e^{a_2 \theta_1} + (e^{a_2 \theta_1} - e^{a_2 \theta_2})(1 - \zeta) - \frac{2}{\pi} (e^{a_2 \theta_1} - e^{a_2 \theta_2}) \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi\zeta e^{-(n\pi)^2 \tau} \right] \quad (41)$$

On the other hand, using the heat function of Eq.(15), the heat flux q can be expressed as:

$$q(\zeta, \tau) = -\lambda \frac{\partial \theta}{\partial \zeta} = -\frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial \zeta} = -\frac{\partial U}{\partial \zeta} \quad (42)$$

Therefore, the nondimensional heat flux can be expressed as:

$$\bar{q}(\zeta, \tau) = \frac{e^{a_2 \theta_1} - e^{a_2 \theta_2}}{a_2} (1 + 2 \sum_{n=1}^{\infty} \cos n\pi\zeta e^{-(n\pi)^2 \tau}) \quad (43)$$

From Eq.(43) at infinity ($\tau \rightarrow \infty$), we can obtain the steady state solution of heat flux.

$$\bar{q}(\zeta, \tau \rightarrow \infty) = \frac{e^{a_2 \theta_1} - e^{a_2 \theta_2}}{a_2} = \text{constant} \quad (44)$$

Again the above value is constant independently of the coordinate.

6.4 Exact solution of the condition 2 for model B

Here the adiabatic condition of model B ($\kappa = \text{constant}$) is equal to the condition of $\partial U / \partial \zeta = 0$. Therefore, considering this condition, we obtain the unsteady state exact solution from Eq.(22). Similarly, by defining $m=0$, the steady state solution and the two arbitrary constants A_μ , B_μ are determined for Eq.(24).

(a)

(b)

Fig.3 Unsteady state temperature profile at condition 1

$$S.s.s. = \int C d\zeta + C_2 = \frac{e^{a_2 \theta_1} - 1}{a_2} \quad (45)$$

$$A_\mu = -\frac{4(e^{a_2 \theta_1} - e^{a_2 \theta_2})}{(2n-1)a_2\pi}, \quad B_\mu = 0 \quad (46)$$

Using the relation $\cos \mu = 0$ of Eq.(24), taking $\mu = (2n-1)\pi/2$ and substituting Eqs.(45), (46) into Eq.(22), we can obtain the following:

$$\theta(\zeta, \tau) = \frac{1}{a_2} \ln \left[e^{a_2 \theta_1} - \frac{4(e^{a_2 \theta_1} - e^{a_2 \theta_2})}{\pi} \times \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{2n-1}{2} \pi\zeta e^{-((2n-1)/2)^2 \pi^2 \tau} \right] \quad (47)$$

Now, in order to obtain the unsteady state heat flux in the material of the model B, using the differential gradient of the heat function with respect to ζ from Eq.(42) and the nondimensional heat flux, we can obtain:

$$\bar{q}(\zeta, \tau) = \frac{2}{a_2} (e^{a_2 \theta_1} - e^{a_2 \theta_2}) \sum_{n=1}^{\infty} \cos \left(\frac{2n-1}{2} \pi\zeta \right) \times e^{-((2n-1)/2)^2 \pi^2 \tau} \quad (48)$$

The above exact solutions mentioned in 6.1 through 6.4 are identical to those mentioned in reference (20) in the limit $a_{1,2} \rightarrow 0$. So the exact solutions for the two models A and B ($\kappa = \text{constant}$) can be considered general solutions for the constant thermophysical properties models.

7. Analytical Result of the Exact Solutions

In order to clarify the unsteady heat conduction feature, the above mentioned

(a)

(b)

Fig.4 Unsteady state temperature profile at condition 2

Fig.5 Steady state temperature profile at condition 1

exact solutions can be applied for the values $\theta_1=1$, $\theta_2=0$ and $0 \leq \zeta \leq 1$, $0 \leq \tau \leq 3$.

Figure 3 shows the analytical unsteady temperature distributions of the model A from Eqs. (32), (41) for the exact solutions of the condition 1 (Eq.(23)) and Fig.4 shows the analytical unsteady temperature distributions of the model B from Eqs. (37), (47) of the condition 2 (Eq.(24)).

The results show that for $\alpha_1 < 0$ of the model A the temperature is higher than that in the model of the constant thermo-physical properties according to each coordinate value and for $\alpha_1 > 0$ the temperature tendency is reversed.

Meanwhile, the model B shows a reverse feature of the unsteady temperature distributions compared with model A. In order to clarify the transient behavior of those models, Fig.5 illustrates the steady temperature distributions of models A and B. These results show that models A and B are in reverse tendency and they converge to the steady temperature distribution of the constant thermo-physical properties model ($\alpha_{1,2}=0$) varying with the exponential coefficients $\alpha_{1,2}$.

In addition, Fig.6 shows the analytical heat flux distributions of the model A from Eqs.(33), (43) of the condition 1 (Eq.(23)) and Fig.7 shows the analytical heat flux distributions of the model B from Eqs.(38), (48) of the condition 2 (Eq.(24)).

Fig.6 Unsteady state heat flux profile at condition 1

Fig.7 Unsteady state heat flux profile at condition 2

Table 1 The relative errors of the steady state temperature and heat flux for the models A and B

$\alpha_{1,2}$	ϵ_θ					ϵ_q
	0.1	0.3	0.5	0.7	0.9	$0 \leq \xi \leq 1$
0.5	$\times 10^3$ 1.36	$\times 10^3$ 2.34	$\times 10^3$ 5.64	$\times 10^3$ -4.61	$\times 10^3$ -1.40	$\times 10^{-2}$ -2.06
-0.5	$\times 10^3$ 2.01	$\times 10^3$ 2.55	$\times 10^4$ -7.23	$\times 10^3$ -6.98	$\times 10^{-2}$ -1.56	$\times 10^{-2}$ -2.06

Fig.8 Steady state heat flux at condition 1

These results also show that for $\alpha_{1,2} > 0$ of the models A and B the unsteady heat flux is higher than the constant model of the thermophysical properties varying with each coordinate value and for $\alpha_{1,2} < 0$ both models have lower heat flux than the constant model for thermophysical properties in these coordinate regions.

These results show the characteristic behavior of both models for $x = \text{constant}$.

It is of interest to investigate the correlations of models A and B for the region $|\alpha_{1,2}| \leq 0.5$.

From the analytical results of the previous condition 1, the calculations of the relative errors ($\epsilon_\theta, \epsilon_q$) for the steady temperature and the heat flux are given in Table 1.

Comparing these analytical results, the temperature and the heat flux of the models A, B approximately coincide in the range of $|\epsilon_{\theta \max}| = 1.56 \times 10^{-2}$ and $|\epsilon_{q \max}| = 2.06 \times 10^{-2}$.

8. Concluding Remarks

After an analysis of nonlinear heat conduction problems of the exponential models studied in this paper with constant thermal diffusivity, the following concluding remarks can be made.

(1) The unsteady state solutions for two conditions of both sides having constant temperature and of one side being heated, while the other side is adiabatic could be obtained through the mathematical analysis.

(2) The two relations obtained for the two models analyzed in this work have been found approximately equivalent and can be expressed in the following form:

$\theta(\alpha_1, \xi, \tau) \approx \theta(-\alpha_2, \xi, \tau)$ and $\bar{q}(\alpha_1, \xi, \tau) \approx \bar{q}(\alpha_2, \xi, \tau)$ for the region $|\alpha_{1,2}| \leq 0.5$.

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