Vibrational Characteristics of Friction Between Gear Teeth*

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A lot of investigations have been made on the friction between gear teeth, but most of them discuss its coefficient and few do its vibrational characteristics. As an element in a vibrational system, the friction has two aspects, that is exciting source and damping. In this paper, its vibrational characteristics are discussed.

The results obtained in this study are : (1) The peak value of vibrational amplitude response curve caused by the friction, where main component of the vibration is synchronized with meshing frequency, is almost independent of lubricating oil viscosity and transmitting power. (2) Its effect as damping element to forced vibration depends on the kind of exciting source.

Key Words: Gear, Vibration, Vibration of Rotating Body, Friction, Lubrication

1. Introduction

In a power transmitting system which contains gears, a friction force acts between gear teeth. This force is usually regarded as loss of power. But when the force increases, it may cause damage to teeth and even excite a vibration. From a view point of vibrational analysis, this force has two aspects, that is exciting force and damping effect. As gears are used under high load, it is necessary to make clear the dynamic characteristics of the friction force.

Usually gear teeth contact each other through a thin oil film, and the lubricating condition changes with speed, load, and viscosity of the oil. Sasaki⁽¹⁾ examined experimentally the friction force between two cylinders, and expressed the friction coefficient as a function of viscosity, rolling speed, and load. Terauchi⁽²⁾ measured the friction force between gear teeth using strain gauges, and introduced an experimental formula for calculating the friction force. But these studies focused on how to get an exact friction coefficient and therefore vibrational characteristics were not discussed. In this paper, it is discussed how the friction force between gear teeth under mixed and boundary lubrication influences the vibrational system.

2. Numerical Analysis

2.1 Analytical model and its coordinate system

* Received 2nd October, 1983

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In this paper, only the vibration in tooth sliding direction will be discussed because the friction force is mostly likely to influence this direction. And to simplify the analytical model, vibration in other directions will be ignored. Figure 1 shows the analytical model and the coordinate system, where r_i indicates the base circle radius of gear i ($i=1, 2$), L and M denote starting and end points on the contact line respectively. C_p indicates the pitch point. Here only shaft I is assumed to be flexural and the friction effect on torsional vibration is ignored as it seems to be small. If the tooth surface has an ideal involute curve, the tooth speed of each gear at contact point is expressed as follows.

$$\left. \begin{aligned} U_1 &= -(r_1 \tan \alpha + e)\omega_1 + \dot{x} \\ U_2 &= -(r_2 \tan \alpha - e)\omega_2 \end{aligned} \right\} \dots \dots \dots (1)$$

where e is the distance between the contact point and the pitch point (it is taken as positive when the contact point is on the end side of contact line).

Furthermore, a symbol S is introduced to express the ratio of sliding speed $U_s = U_2 - U_1$ to rolling speed $U_R = |U_1 + U_2|/2$.

$$S = U_s/U_R \dots \dots \dots (2)$$

Fig. 1 Analytical model

2.2 Friction coefficient between gear teeth

It is not easy to analyze precisely the frictional mechanism under mixed lubrication as there are a lot of factors involved. As the purpose of this paper is not to analyze the friction mechanism but to grasp the vibrational characteristics of the friction, the relationship between sliding ratio and the friction force will be assumed to be able to be expressed in a simple formula referring to previous studies.

The relationship between the friction coefficient and the sliding speed has been obtained experimentally. It can be shown by a curved full line in Fig. 2. The broken lines show the approximate relation.

The approximate lines are expressed as follows.

$$\begin{aligned}\mu &= a_1 U_s \quad (0 \leq U_s \leq U_{s0}) \\ \mu &= \mu_0 + a_2 (U_s - U_{s0}) \quad (U_s > U_{s0})\end{aligned} \quad (3)$$

where, U_{s0} and μ_0 are the sliding speed and the friction coefficient at the turning point, respectively. a_1 and a_2 denote gradients of the two lines and they are functions of load, viscosity and roughness of the tooth surface.

There are a lot of experimental studies on the mean friction coefficient between gear teeth under mixed lubrication. According to them, the coefficient is expressed as follows.

$$\mu_m = c_0 (\eta_s U_R / Q)^\beta \quad (4)$$

where η_s is the viscosity of lubricating oil at atmospheric pressure, Q is the transmitting force per unit face-width, and β , c_0 are constant coefficients.

Maximum frictional coefficient μ_0 in Fig. 2 can be expressed using a similar equation.

$$\mu_0 = c_1 (\eta_s U_R / Q)^\beta \quad (5)$$

Using the previously defined symbol S , Eq. (5) can be rewritten as follows.

$$\begin{aligned}\mu &= a_1 U_R S \quad (|S| \leq U_{s0} / U_R) \\ \mu &= \mu' \text{SIGN}(S) + a_2 U_R S \quad (|S| > U_{s0} / U_R)\end{aligned}$$

where $\mu' = \mu_0 - a_2 U_{s0}$ (6)

2.3 Changes in contact force between teeth

In an involute spur gear system, the

Fig. 2 Relationship between sliding speed and frictional coefficient

number of meshing tooth pairs changes from 1 to 2 and 2 to 1 with the rotation of gears. And depending on the number, the contact force changes even when the total transmitting force is the same. For example, in the two-pair-contact region, half of the transmitting force acts on the tooth surface while in the one-pair-contact region the total force acts on the surface. But actually, this contact force does not change suddenly but continuously. So to express this continuous change, a force moving coefficient ξ is introduced. Figure 3 shows the change in the contact force on a tooth. From this figure, it is understood that the instantaneous contact force per unit face-width $P(t)$ can be expressed as a function of time. Using this force, the friction coefficient and the friction force can be obtained.

2.4 Equations of motion

From Eqs. (1)-(6), the friction force under mixed lubrication can be calculated. And putting $U_{s0} = 0$, $a_2 = 0$ in Eq. (5), the friction force under boundary lubrication can be obtained.

Using these friction forces, equations of motion are expressed as follows.

$$\begin{aligned}m_1 \ddot{x} + kx &= \mu P(t)b \quad (\text{one-pair-contact region}) \\ m_1 \ddot{x} + kx &= (\mu_1 P_1(t) + \mu_2 P_2(t))b \quad (\text{two-pair-contact region})\end{aligned} \quad (7)$$

where μ_1 and μ_2 are friction coefficients, and $P_1(t)$ and $P_2(t)$ are the contact forces at each contact point in the two-pair-contact regions, respectively. b is the face width of the gear.

Numerical result is obtained by integrating Eq. (7) using Runge Kutta method until steady state response appears.

2.5 Numerical results

Details of the numerical model are shown in Table 1. The force moving coefficient ξ and the coefficients c_1 , β and a_2 for calculating the frictional coefficient are decided using the results shown in the references⁽²⁾ assuming that the accuracy of the gears is in JIS 0 class. Similarly μ_0 is decided from the references⁽¹⁾ as $\mu = \mu_0$ when $s = 0.06$.

Fig. 3 Active force on a tooth

2.5.1 Frictional response

Figure 4 shows the relationship between the frictional force and the shaft deflection when the non-dimensional speed f_m/f_n (meshing frequency/natural frequency) is 0.543. The bottom line indicates the number of tooth pairs in contact. Depending on the change of this number, the friction force changes. The change is periodic, but it is not sinusoidal. Therefore an excited deflection of the shaft is not sinusoidal, containing higher harmonic components.

2.5.2 Steady state amplitude response and its vibrational component

Figure 5 shows a steady state amplitude response of the shaft caused by the friction force. The magnitude of harmonic components are also shown in the figure. As mentioned before, the response contains higher harmonic components, and from the figure it is understood that the

Table 1 Details of the numerical model

ITEMS	SYMBOL	VALUE
Polar moment of inertia of Motor	J_1	$7.29 \times 10^{-2} \text{ kgm}^2$ (7.43 kgfms ²)
Polar moment of inertia of load	J_2	$9.63 \times 10^{-1} \text{ kgm}^2$ (98.2 kgfms ²)
Polar moment of inertia of Gear I	I_1	$3.49 \times 10^{-4} \text{ kgm}^2$ (3.56 $\times 10^{-2}$ kgfms ²)
Polar moment of inertia of Gear II	I_2	$3.70 \times 10^{-2} \text{ kgm}^2$ (3.77 kgfms ²)
Module	m	3
Number of teeth (Gear I)	z_1	23
Number of teeth (Gear II)	z_2	45
Base circle radius of Gear I	r_1	$3.24 \times 10^{-2} \text{ m}$
Base circle radius of Gear II	r_2	$1.26 \times 10^{-1} \text{ m}$
Mass of Gear I	m_1	1.64 kg
Stiffness of Shaft I	k	$7.28 \times 10^5 \text{ N/m}$ (74.2 kgf/mm)

Fig.4 Frictional force and deflection of shaft

amplitude of each component changes with the meshing frequency. Although the time response, such as shown in Fig.4, becomes almost sinusoidal at $f_m/f_n=1$, the amplitude response has a peculiar peak as shown in the figure. The reason is not clearly known, but it seems that when the amplitude exceeds a certain value, the friction force acts as a damping factor rather than an exciting one.

2.5.3 Relationship between $\eta_s U_R/Q$ and steady state response

It is clear from Eq.(5) that the value of friction coefficient μ is a function of non-dimensional parameter $\eta_s U_R/Q$. In Fig. 6, the relationship between this parameter and the amplitude response is shown. So far, this parameter is used as an index of lubricating condition, but depending on the roughness of the tooth surface, the lubricating condition might differ even where the value of parameter is the same. So in this calculation, the lubricating condition is assumed to be always a mixed one independently of the value.

Depending on the change in $\eta_s U_R/Q$, the friction coefficient μ changes, and as a result the amplitude response changes. Values of $\eta_s U_R/Q$ themselves are a function of rotating speed even if the

Fig.5 Magnitude of vibrational component

Fig.6 Relationship between $\eta_s U_R/Q$ and the response

load and viscosity are constant. In the calculation, $\eta_s U_R/Q$ was assumed to be constant. From some points of view the results are examined as follows.

At first, a case where η is constant and f_m/f_n is nearly equal to unity is discussed. In this case, $\eta_s U_R$ can be regarded almost constant, and the change in $\eta_s U_R/Q$ is regarded as a change of Q . So it is understood that the peak value of the response is nearly independent of the transmitting load Q . But the shape of the peak changes with Q . And excepting $f_m/f_n=1$ region, the amplitude increases with the load.

Next, a case where the transmitting force and the rotational speed are constant is discussed. Examining Fig. 6 from this point of view, it is clear that the amplitude decreases with an increase in the viscosity coefficient η except in the region where f_m/f_n is nearly equal to unity. This is explained that metallic contact region decreases with an increase in the oil film thickness due to an increase in viscosity. This explanation seems to be better than the contention that the damping effect increases with the oil viscosity.

2.5.4 Relationship between the shaft stiffness and the amplitude response

Figure 7 shows the relationship between the shaft stiffness and the amplitude response. Paying attention to $f_m/f_n=1$ region, the maximum amplitude does not change with the shaft stiffness. But the width of the peak becomes narrow when the shaft stiffness increases.

2.5.5 Relationship between contact ratio and amplitude response

Figure 8 shows the relationship between contact ratio and amplitude response. Numerically this result was obtained by changing the center distance of meshing gears as contact ratio is a function of the distance. As it is mentioned in the examination of Fig. 4, a change in the number of contact pairs affects greatly the time response. But on

Fig.7 Relationship between shaft stiffness and the response

the amplitude response, its effect seems to be small except in $f_m/f_n=1$ region as shown in Fig.8. In this region the maximum amplitude decreases with an increase in contact ratio.

3. Experiment

3.1 Experimental equipment

Figure 9 is a schematic drawing of the experimental equipment, and the detail is shown in Table 2. This equipment consists of two shafts with one pair of spur gears. To investigate the flexural deflection easily, the driven shaft is set rigid enough compared with the driving shaft so that only the deflection of the driving shaft appears.

Power is supplied from a motor to shaft I, and transmitted to shaft II through a pair of meshing gears. Then it is absorbed by an electric generator which is connected with shaft II. Maximum transmitting force per unit face-width of the gear is 2.4×10^4 N/m. Details of the test gears are as follows: module 3, pressure angle 20° , face width 10mm. Number of the teeth is 23 for gear I and 45 for gear II. These gears have an

Fig.8 Relationship between contact ratio and the response

Fig.9 Experimental equipment

accuracy of JIS 0 class.

As lubricating condition influences greatly the vibration, a sufficient amount of oil is supplied to the meshing region through a nozzle.

In the experiment, the deflection of the driving shaft is measured near gear I in the direction of sliding between teeth by a non-contact type pick up.

3.2 Experimental results

3.2.1 Vibrational component of tooth sliding direction

Figure 10 shows the spectra of response in a three dimensional space. Amplitude, meshing frequency and frequency component are taken on axes Z, Y, X respectively. It is understood from this figure that the response consists of three main components, namely, the meshing frequency, its harmonics and the component synchronized with the rotation.

Fig.11 shows the time response of the shaft. It is clear that the response is not sinusoidal when the shaft is excited by the friction force between meshing teeth.

3.2.2 Relationship between amplitude response and transmitting force

Table 2 Details of the experimental equipment

ITEMS	SYMBOL	VALUE
Module	m	3
Number of teeth (Gear I)	z_1	23
Number of teeth (Gear II)	z_2	45
Base circle radius of Gear I	r_1	$3.24 \times 10^{-2} \text{ m}$
Base circle radius of Gear II	r_2	$1.26 \times 10^{-2} \text{ m}$
Mass of Gear I	m_1	1.11 Kg
Stiffness of Shaft I	k	$3.58 \times 10^5 \text{ N/m}$ (36.5 kgf/mm)
Increment of distance between the gear shafts	Δa	$5.0 \times 10^{-4} \text{ m}$
Face width	b	$1.0 \times 10^{-2} \text{ m}$
Coefficient	c_1	0.0063
Coefficient	β	-0.18
Coefficient	a_2	-0.5

Fig.10 Amplitude spectrum of shaft vibration
(Teeth sliding direction)

Fig.12 shows the amplitude response curves where the transmitting force is a parameter. Only a synchronized component of the vibration with meshing frequency is extracted as amplitude in this figure, because it is dominant. Since an electric generator is used as a device for loading, the transmitting force changes with the rotation. To express the magnitude of load, the exciting voltage of the generator is shown in the figure. The exact values of load for voltages 48, 24 and 15V are 9.8×10^3 , 6.8×10^3 and $2.1 \times 10^3 \text{ N/m}$ respectively when the meshing frequency is equal to a resonance frequency of 103Hz. Comparing the 48V curve with the 15V one, it is understood that there is a great difference in the amplitude ratio between at resonance speed and at the others. For example, the amplitude ratio is about 4:1 at $f_m=50\text{Hz}$, while it is 1.2:1 at the resonance ($f_m=103\text{Hz}$). This means that the peak value is not influenced greatly by the transmitting force, and this tendency coincides qualitatively with numerical results shown in Fig.6.

3.2.3 Relationship between amplitude response and oil viscosity

Figure 13 shows the relationship between the amplitude response and the oil viscosity. Maximum value of the amplitude is not influenced greatly by oil viscosity similarly to the relationship between the response and the transmitting force shown in Fig.12. This tendency coincides qualitatively with numerical result shown

Fig.11 Deflection of shaft in teeth sliding direction ($f_m=49.2 \text{ Hz}$)

Fig.12 Relationship between transmitting force and the response

in Fig.6. In the experiment, a sufficient amount of backlash was given between the gear teeth, and therefore the characteristic peak shape of the response curve is not a result of the limitation of the deflection.

4. Influence of the Friction between Gear Teeth on Vibrational Characteristics

The friction response has been discussed numerically and experimentally. But from a view point of vibrational characteristics, it is necessary to examine the damping effect as well as the exciting effect. Analytical model shown in Fig.1 seems to be a good model as numerical results show good agreement with experimental ones. But a weak point of it is that to get numerical result it requires long computing time, and so it is difficult to grasp general vibrational characteristics using this model. Hence a new simple model which expresses fundamental characteristics of the friction will be suggested next, and using it how the friction force influences the system will be discussed.

4.1 Simplified analytical model

Under mixed lubrication and boundary lubrication, dry frictional characteristics seem to be dominant. But even in a dry frictional system, still the friction coefficient is a function of roughness of the tooth surface, sliding speed, rolling speed, and viscosity. So to discuss fundamental characteristics of the system, here we put the following three assumptions.

- (1) Friction coefficient is constant under constant rotating speed.
- (2) Contact ratio is unity, and the speed at contact point changes sinusoidally.
- (3) Only the driving shaft is flexural and the vibration in the direction of sliding of teeth is considered.

Using these assumptions, the friction model between gear teeth can be a simple model as shown in Fig.14. It consists of a spring (shaft stiffness) and a mass (mass of the gear), where the frictional element moves sinusoidally under dry contact

Fig.13 Relationship between oil viscosity and the response

condition.

4.1.1 Equation of motion and the solution

Equation of motion of this model is derived as follows.

$$m_1 \ddot{x} + kx - f_0 \text{SIGN}(A\omega \cos \omega t - \dot{x}) \dots (9)$$

The solution to Eq.(9) is Eq.(10).

$$x = B \sin(\omega_n t + \phi) + x_s \dots (10)$$

In Eq.(10), the amplitude B and the phase ϕ change discontinuously at $t=t_{01}$ when the speed $\dot{x}$ decreases to zero. The time t_{01} is expressed as follows.

$$t_{01} = \cos^{-1}(\dot{x}/A\omega)/\omega \dots (11)$$

Using suffix 1 when $t < t_{01}$ and 2 when $t > t_{01}$, next equations are derived in the case of $B_1 > f_0/k$ from the condition of continuation of speed and displacement.

$$\begin{aligned} x_1(t=t_{01}) &= x_2(t=t_{01}) \\ x_1 &= B_1 \sin(\omega_n t + \phi_1) + x_s \\ x_2 &= B_2 \sin(\omega_n t + \phi_2) - x_s \\ B_1 \cos(\omega_n t_{01} + \phi_1) &= B_2 \cos(\omega_n t_{01} + \phi_2) \dots (12) \end{aligned}$$

where

$$\begin{aligned} x_s &= f_0/k \text{SIGN}(A\omega \cos \omega t - \dot{x}_1) \\ &= -f_0/k \text{SIGN}(A\omega \cos \omega t - \dot{x}_2) \end{aligned}$$

From Eq.(12), B_2 and ϕ_2 are expressed with B_1 , ϕ_1 , x_{10} and $\dot{x}_{10}$ as follows.

$$\left. \begin{aligned} B_2 &= \sqrt{(x_{10} + x_s)^2 + (\dot{x}_{10}/\omega_n)^2} \\ \phi_2 &= \tan^{-1} \frac{\omega_n(x_{10} + x_s)}{\dot{x}_{10}} \end{aligned} \right\} \dots (13)$$

where $x_{10} = x_1(t=t_{01})$

When $B_1 < f_0/k$, the frictional speed at t_2 is 0 and x_2 is expressed as follows.

$$x_2 = A \sin \omega t + (x_{10} - A \sin \omega t_{01}) \dots (14)$$

This condition holds until t_{02} when the absolute sum of forces produced by the spring and the inertia becomes equal to f_0 , and then sliding begins to occur again. Putting a suffix of 3 after $t=t_{02}$, next equations are derived from the condition of continuation.

$$\begin{aligned} x_2(t=t_{02}) &= x_3(t=t_{02}) \\ &= A(\sin \omega t_{02} - \sin \omega t_{01}) + x_{10} \\ &= B_2 \sin(\omega_n t_{02} + \phi_2) \\ &\quad + f_0/k \text{SIGN}(\cos \omega t_{02}) \\ \dot{x}_2(t=t_{02}) &= \dot{x}_3(t=t_{02}) \\ &= A\omega \cos \omega t_{02} \\ &= B_3 \omega_n \cos(\omega_n t_{02} + \phi_3) \end{aligned} \dots (15)$$

Fig.14 Simplified analytical model

From Eq.(15), B_3 and ϕ_3 are determined. So this system can be solved as a partial linear system.

4.1.2 Numerical result

It is understood from Eq.(10) and (11) that the steady state response of this system is determined by the natural frequency ω_n , the moving frequency of friction element ω , its amplitude A , and the ratio of friction force to the stiffness of spring f_0/k . And it does not depend on the initial values of B and ϕ . Therefore figures which show relationship between amplitude of the response and ω/ω_n with some parameters are discussed. Half of peak to peak value of periodic cycle of the response is defined as an amplitude.

Figure 15 shows the amplitude response where the ratio f_0/k is taken as a parameter. Amplitude of friction element A is assumed constant. This corresponds to the relationship between the response and the oil viscosity or the response and the transmitting force in an actual system. Numerical quantities in Table 1 are used in the calculation. It is understood from this figure that maximum amplitude of the response is almost the same as A , and the shape of peak looks like a high land with an increase in f_0/k . Excepting this resonance area, the amplitude increases with f_0/k . This result has qualitative agreement with the one shown before. This means that this simple model is a sufficiently suitable one to analyze the basic characteristics of the system.

4.2 Damping effect of the friction force

So far, a vibration excited by the friction force with meshing frequency of gear teeth has been discussed. And a simple model was proposed to express the basic characteristics of the system sufficiently. But generally, spectra of the vibration in gear system contain not only meshing frequency (including its harmonics) but also other components (mainly shaft rotational frequency).

Fig.15 Relationship between f_0/k and the response

Hence, dynamic characteristics of the system excited by other frequencies than meshing frequency is discussed next, and the effect of friction force is examined using this simple model.

4.2.1 System with the exciting force acting directly on the mass

Figure 16(a) shows a model where an exciting force acts directly on the mass. This corresponds to an actual gear system where gears or rotors on the shaft have an unbalance masses. In such a system, only the vibration around $\omega_1 = \omega_n$ seems to be serious, so the response around this frequency is discussed.

In this case, vibration amplitude $|x_0|$ can be assumed sufficiently larger than A_0 , and next equation is introduced.

$$m_1 \ddot{x} + kx + f_0 \text{SIGN}(\dot{x}) = A_1 \sin \omega_1 t \quad \dots (16)$$

A solution to the equation can be obtained by making the absorbing energy of friction force equal to that of an equivalent damping. That is, introducing an equivalent damping coefficient $c_e = 4f_0/\pi\omega x_0$, the solution is expressed as follows.

$$x_0 = \frac{A_1}{k} \frac{\sqrt{1 - (4f_0/\pi A_1)^2}}{1 - (\omega_1/\omega_n)^2} \quad \dots (17)$$

This shows that the friction force has little damping effect around the resonance speed when the system is excited by unbalance mass.

4.2.2 System with exciting force acting through frictional element

Figure 16(b) shows a model where an exciting force acts through a friction element. This corresponds to an actual gear system where gears have a geometrical eccentricity. The equation of motion of the system is expressed as follows.

$$m_1 \ddot{x} + kx = f_0 \text{SIGN}(A_0 \omega_0 \cos \omega_0 t + A_1 \omega_1 \cos \omega_1 t - \dot{x}) \quad \dots (18)$$

The amplitude response obtained from the solution to Eq.(18) can be shown as a function of A_0 , ω_0 , A_1 and ω_1 . Figure 17 shows one of the numerical examples. Although a peak exists at the point where

Fig.16 Simplified analytical model considering the exciting force

the exciting frequency ω_1 is equal to the natural frequency ω_n , the amplitude is relatively small compared with the exciting amplitude A_1 .

As understood from the above explanation, what kind of the damping effect the friction force has depends on the kind of exciting force in the gear system.

5. Conclusion

Under mixed and boundary lubricating conditions, vibrational characteristics of the frictional force between gear teeth have been discussed numerically and experimentally. The results are summarized as follows.

(1) Maximum amplitude of the response excited by the friction force between gear teeth under mixed and boundary lubricating condition is influenced neither by the transmitting force nor by the oil viscosity. But with an increase in the transmitting force and also with a decrease of the oil viscosity, the shape of peak of the response curves comes to look like a high land.

(2) Under constant rotational speed, the friction force can be expressed with a simple model where the frictional element moves sinusoidally without lubricant. And the damping effect of the friction force on the forced vibration depends on the kind of the exciting force.

Fig.17 Response to exciting force which acts through frictional element

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