Tension of a Circular Solid Cylinder with Two Semicircular Grooves*  
(Interference Effects of Stress Concentration due to Two Notches)

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This paper is concerned with the stress concentration problem of an elastic solid cylinder with two semicircular grooves under tension. The principal results are as follows: Interference effects of stress concentrations due to two notches (a phenomenon leading to $a_2/a_1 < 1.0$) appear in the condition $b/C < 0.5$ and are larger for smaller values of $b/C$, where $C$ and $b$ are radii of the cylinder and the semicircular grooves and $a_1$ and $a_2$ are stress concentration factors for one and two notches, respectively. In the condition of $b/C > 0.5$, there are cases of $a_2/a_1 > 1.0$, as was suggested by Nisitani et al. for an infinite number of periodic notches. The interference effects of two notches are smaller than those of periodic infinite notches and larger in tension than in torsion.

The method of solution used here is an application of Green's functions for axisymmetric body force problems of a solid cylinder.

Key Words: Elasticity, Stress Concentration, Form Factor, Notch, Solid Cylinder, Tension, Semicircular Groove, Interference Effect of Notches

1. Introduction

This paper deals with the stress concentration problems of an elastic solid cylinder with two semicircular grooves under tension (compression). In previous papers (1), (2), Green's functions for axisymmetric body force problems of an elastic solid cylinder were shown and applied to the stress concentration problems of a solid cylinder with an annular groove of a semicircular (1) or a circular-arc (2) form under tension. In this paper, the method of solution proposed in the previous papers for one groove is applied to the problem of two grooves mentioned above. The method of solution used here is to distribute body forces so as to satisfy boundary conditions of the surfaces of two grooves. This may be the so-called body force method (3). For this purpose, we use an analytical solution (Green's functions mentioned above) which satisfies completely the boundary conditions of the surface of a circular solid cylinder with no notch.

It is well known (4), (5) that form factors (stress concentration factors) are relaxed by multiple notches by interference effects of stress concentrations. Recently, Nisitani and et al. (6) have investigated on the tenacity problem of a solid cylinder with periodic infinite grooves and suggested that there is a case where form factors for periodic infinite grooves are larger than that for a groove.

In a previous paper (7), we have investigated on the torsion problem of a solid cylinder with two grooves and pointed out that the relation $a_1 > a_2$ always holds, where $a_2$ represents the form factor for 1 groove. In this paper, it is shown that there is a case of $a_1 < a_2$ in tension of a solid cylinder with two grooves, similarly to periodic infinite grooves mentioned above. This paper also discusses on the effects of the distance between two grooves and the radius of semicircular notches on the form factors and shows that the interference effects of stress concentrations for two notches are less than those for periodic infinite notches.

2. Boundary conditions

Consider a circular solid cylinder of a radius $C$ with two semicircular grooves of a radius $b$ as shown in Fig. 1 expressed in cylindrical coordinates ($r$, $\theta$, $z$), where $2s$ represents the distance between the two grooves and $\phi$ is an angle between a normal (radius $b$) at a point ($r_1$, $z_1$) on the surface of a semicircular notch and the boundary $r = C$ of the cylinder. It is assumed that the cylinder is subjected to a uniform axial stress $\sigma_0$ at a sufficiently large distance from the grooves and all the surfaces of the cylinder with two grooves are free from applied surface forces. Hence, boundary conditions of the problem are expressed as follows:

(i) $|r-c| \leq s-b, s+bE<\infty: \sigma_0 = r_n = 0$

(ii) on the surface of a semicircular notch ($r_1, z_1$): $\sigma_0 = r_n = 0$

(iii) $0 \leq b \leq c, b \leq |z|: a_1 = b, a_1 = a_2 = r_n = r_\phi = 0$

............... (1)

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Fig. 1 A solid cylinder with two grooves under tension

where $\sigma_1$, $\tau_{1\phi}$ and $\tau_{1r}$ are normal, tangential in the $\phi$-direction and torsional stresses acting on the notched surfaces $(r_1, z_1)$ of a radius $b$, and are expressed as follows:

$$
\sigma_1 = r_1 \cos \phi + r_1 \sin \phi - 2r_1 \sin 2\phi \cos \phi
$$

$$
\tau_{1\phi} = (r_1 - r_1 \sin \phi \cos \phi + r_1 \cos \phi - r_1 \cos \phi) \cos \phi
$$

$$
\tau_{1r} = r_1 \sin \phi + r_1 \cos \phi \sin \phi
$$

Note that the stress components $\tau_{1\phi}$, $\tau_{1r}$ and $\tau_{1\phi}$ are always equal to zero because the present problem is an axisymmetric tension problem.

Fig. 3 An axial force acting on a circle of a radius $a$

and are expressed as follows:

$$
\sigma_k = \sigma_k \text{ of Eq.}(18) + \sigma_k \text{ of Eq.}(20) \text{ with } b \text{ of Eq.}(21)
$$

$$
\sigma_k = \sigma_k \text{ of Eq.}(19) + \sigma_k \text{ of Eq.}(20) \text{ with } b \text{ of Eq.}(22)
$$

where $\sigma_k$ represent stress components ($\sigma_r$, $\sigma_\phi$, $\sigma_\theta$, $\tau_{\phi r}$, $\delta()$) is a Dirac delta function and Eqs. (18) to (22) are shown in Appendix of the paper. The stresses $\sigma_k$ with a superscript $m$ or $3$ refer to the stresses at a point $(r, z)$ produced by the body force $F_1$ or $F_2$ acting on two circles $(r=a, z=\pm h)$ of Eq. (3) or (4), respectively.

From the symmetry of the problem with respect to $z=0$, we may consider only the domain $z \geq 0$ of the cylinder. Figure 4 shows a part of a notch of the cylinder with two grooves. The dotted line of a radius $a$ is the plane $(a_1, h_1)$ of distribution of the body forces and the one-dot chain line of a radius $b$ is the plane $(r_1, z_1)$ which becomes the surface of a semicircular notch. If we distribute the body forces of Eqs. (3) and (4) on the plane $(a_1, h_1)$ of a radius $a$, the stresses $\sigma_k$ at a point $(r, z)$ in the cylinder are expressed using $\sigma_k^m$ ($m=1, 3$) of Eqs. (6) and (7) as follows.

$$
\sigma_k = \int_0^\pi [\xi(\delta) \sigma_k + \xi(\delta) \sigma_k] d\alpha = a_1, n=1, h_1 d\phi + \sigma_k
$$

\begin{align*}
\sigma_k &= \frac{1}{2\pi} \int_0^\pi \left[ \delta(z-h) + \delta(z+h) \right] d\phi \quad (4) \\
\sigma_k &= \frac{1}{2\pi} \int_0^\pi \left[ \delta(z-h) - \delta(z+h) \right] d\phi \quad (5)
\end{align*}
where $\sigma_{k}$ is a uniform tension (stress)

$$\sigma_{r} = \rho, \sigma_{\theta} = \sigma_{z} = \tau_{r\theta} = 0 \hspace{1cm} (9)$$

and $\beta$ is an angle between a normal (radius $e$) at a point $(a = a_{1}, h = h_{1})$ on the plane of a radius $e$ and the boundary $r = C$ of the cylinder as shown in Fig. 4. In Eq. (9), $\zeta(\beta)$ and $\xi(\beta)$ are arbitrary functions corresponding to intensities of the body forces distributed. From the definition of Green’s functions used here, it is sufficient to consider only the boundary conditions (ii) in Eq. (1).

Application of the boundary conditions (ii) of Eq. (1) to Eq. (8) yields

$$\begin{align*}
-\rho \sin^{2} \varphi + \rho \sin \varphi \cos \varphi &= \int_{0}^{2\pi} \left[ \xi(\beta) \left( \frac{\partial}{\partial r_{\theta}} \right)_{r_{\theta}=r_{C}} + \zeta(\beta) \left( \frac{\partial}{\partial r_{z}} \right)_{r_{z}=r_{C}} \right]_{(r=r_{C}, \varphi = \varphi_{C})} d\beta \\
&\hspace{1cm} \text{(10)}
\end{align*}$$

where $\sigma_{r} = \sigma_{\theta} = \sigma_{z} = \tau_{r\varphi} = \tau_{\theta z} = \tau_{r\theta} = 0$, $m = 1, 3$ are expressed by

$$\begin{align*}
\sigma_{r} &= \sigma_{\theta} = \sigma_{z} = \tau_{r\varphi} = \tau_{\theta z} = \tau_{r\theta} = 0 \\
\tau_{r\varphi} &= (\sigma_{r} - \sigma_{\theta}) \sin \varphi \cos \varphi + r_{\varphi} (\cos^{2} \varphi - \sin^{2} \varphi) \\
&\hspace{1cm} \text{(11)}
\end{align*}$$

The expression (10) are dual integral equations with unknown functions $\xi(\beta)$ and $\xi(\beta)$, and must be realized at the points $r = r_{C}, z = z_{C}$ of the notched surface of a groove. To solve the dual integral equations by reducing them to a set of linear algebraic equations, we perform an appropriate discretization of the body forces distributed. That is, we shall assume the values $\xi(\beta)$ and $\xi(\beta)$ of the body forces as

$$\begin{align*}
\left\{ \xi(\beta) \right\} &= \sum_{j=1}^{N} \xi_{j} \delta(\beta - \beta_{j}), (0 < \beta_{j} < \pi : j = 1 \sim N) \\
\left\{ \zeta(\beta) \right\} &= \sum_{j=1}^{N} \zeta_{j} \delta(\beta - \beta_{j}) \hspace{1cm} \text{(12)}
\end{align*}$$

This means that the body forces are distributed on $N$ points (circles)

$$a_{j} = c - e \cos \beta_{j}, h_{j} = s - e \sin \beta_{j} \hspace{1cm} \text{(13)}$$

where $\xi_{j}$ and $\zeta_{j}$ are arbitrary constants corresponding to intensities of the body forces distributed. Substitution of Eq. (12) into Eq. (10) yields

$$\begin{align*}
-\rho \sin^{2} \varphi + \rho \sin \varphi \cos \varphi &= \sum_{j=1}^{N} \left[ \xi_{j} \left( \left( \frac{\partial}{\partial r_{\theta}} \right)_{r_{\theta}=r_{C}} \right)_{r=r_{C}, z=z_{j}} + \zeta_{j} \left( \left( \frac{\partial}{\partial r_{z}} \right)_{r_{z}=r_{C}} \right)_{r=r_{C}, \varphi=\varphi_{C}} \right] d\beta \\
&\hspace{1cm} \text{(14)}
\end{align*}$$

Since there are 2N arbitrary constants $\xi_{j}$ and $\zeta_{j}$ in Eq. (14), we shall consider $N$ points $r_{j} = c - b \cos \varphi_{j}, z_{j} = s - b \sin \varphi_{j}$

$$\begin{align*}
(0 < \varphi_{j} < \pi : j = 1 \sim N) \hspace{1cm} \text{(15)}
\end{align*}$$

on the plane shown by the two-dot chain line which will become the notched surface of a radius $b$, and choose the values $\xi_{j}$ and $\zeta_{j}$ such as to satisfy Eq. (14) at these selected points (circles). It follows from the above that we can construct a set of linear algebraic equations with 2N unknown constants $\xi_{j}$ and $\zeta_{j}$. Solving the linear equations and substituting the solutions obtained into the expressions

$$\sigma_{ki} = \left( \xi_{k} \sigma_{k} + \zeta_{k} \sigma_{k} \right)_{r=r_{C}, z=z_{C}} \hspace{1cm} \text{(16)}$$

then we can find the stresses of any point $(r, z)$ of the cylinder with two circumferential notches.

4. Numerical results

Based on the method of solution described in the previous chapter, numerical calculations were performed under the assumption of Poisson’s ratio $\nu = 0.3$ of the cylinder. As the N points $(a = a_{j}, h = h_{j})$ are $j = 1 \sim N$, the body forces are distributed and as the N points $(r_{j} = r_{C}, z_{j} = z_{C})$ are $j = 1 \sim N$, the boundary conditions (ii) of Eq. (1) are to be satisfied, we took 70 points corresponding to the angles $\varphi_{j}$ and $\beta_{j}$ determined by the deviation of $\pi$ into $(N + 1)$ equal angles, where $\pi$ is the central angle of a semicircular notch.

Figure 5 shows the position (angles $\varphi$) where the maximum stress $(\sigma_{k})_{\text{max}}$ of $\sigma_{k}$ occurs. The stress $(\sigma_{k})_{\text{max}}$ for the case of two notches occurs not at the bottom $(\varphi = 90^\circ)$ of a notch but at a point of $\varphi \geq 90^\circ$ on the notched surface, though the stress $(\sigma_{k})_{\text{max}}$ for cases of one notch and periodic infinite notches occurs at the bottom $(\varphi = 90^\circ)$. It is seen from Fig. 5 that the angle $\varphi$, where $(\sigma_{k})_{\text{max}}$ is found, depends on the radius $b$ and the distance between two notch points (circle). The largest value of the angle $\varphi$ occurs when the notch sizes are about $b/C = 0.2$.

Table 1 shows the form factors $a_{k} = (\sigma_{k})_{\text{max}}/\sigma_{\text{nom}}$ obtained in the paper, where $\sigma_{\text{nom}}$ is the mean value of the axial stress.
$\sigma_s$ at the cross section where $(\sigma_s)_{\text{max}}$ is found. In Table 1, the numbers (1), (2) and (3) represent the values of $\alpha_s$ calculated by (1); $(\sigma_s)_{\text{max}}$ and $\sigma_{\text{nom}}$ at the values of the minimum cross section ($\varphi = 90^\circ$), (2); real $(\sigma_s)_{\text{max}}$ and $\sigma_{\text{nom}}$ at the cross section ($\varphi > 90^\circ$) of real $(\sigma_s)_{\text{max}}$ and (3); real $(\sigma_s)_{\text{max}}$ and $\sigma_{\text{nom}}$ at the minimum cross section ($\varphi = 90^\circ$), respectively.

Fig. 6 Interference effects of stress concentrations

![Graph](image)

Fig. 7 Interference effects of stress concentrations

![Graph](image)

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Table 1 Form factors $\alpha_s$ for two notches
If we compare as a reference $a_2$ in the case of $b/C = s/C = 0.05$ with $a_1$ of a notch, it is seen that $a_2$ is smaller about 14.1% than $a_1$. This means that the interference effects (a phenomenon leading to $a_2/a_1 < 1.0$) of stress concentration for the case of two notches under tension are larger than those (8.51%) under tension shown in a previous paper.\(^{(7)}\) In Table 1, $a_1$ for the case $(s/C = \infty)$ of one notch\(^{(1)}\) under tension is also shown and the values in parentheses are results shown by Nisitani \& al.\(^{(6)}\)

Figure 6 shows the relation between $a_2/a_1$ and $b/C$. The dotted lines (black circles) in Fig.6 represent the results for periodic notches.\(^{(1)}\) We can see from Fig.6 that (i) the interference effects of two notches are less than those of periodic notches, and (ii) the interference effects are larger for smaller values of $b/C$ and occur in the condition of $b/C < 0.5$: in the condition of $b/C \geq 0.5$, there are cases of $a_2/a_1 > 1.0$ rather than the interference effects. This fact mentioned above (ii) is also observed in Fig. 7 which shows the relation between $a_2/a_1$ and $s/C$ as a parameter of $b/C$. It is seen from Fig. 7 that the relation of $a_2/a_1$ at $s/C = 2.0$ is realized in the condition of $b/C \geq 0.5$. The interference effects ($a_2/a_1 < 1.0$) which occur in the condition of $b/C < 0.5$ are larger for smaller values of the distance $s$ between two notches and $a_2/a_1 > 1.0$ with an increase of $s$.

Figures 8 and 9 show the distributions of the stresses $(a_i/\sigma_{nom})_{max}$ in the case of $s/b = 1.0$ (Fig.8) and $s/b = 2.0$ (Fig.9) for several values of $b/C$, where $\sigma_{nom}$ is a mean stress of $\sigma_i$ at the smallest cross section $(x = a)$ and is expressed by

$$\sigma_{nom} = p(1 - b/c)^2$$

There is little difference between these distributions in Figs.8 and 9, but it seems that the stresses for $s/b = 1.0$ are more likely to make slightly uniform distributions than those for $s/b = 2.0$.

By the way, it has been said\(^{(9)}\) about stress concentrations due to shallow notches ($b/C \approx 1.0$) that there are strong resemblances in the stress distributions at the vicinity of the bottom of a notch. This fact is easily confirmed in Figs. 10 and 11. That is, these figures show the distributions of the stresses $(a_i/\sigma_{max})_{max}$ at the vicinity of the bottom of a notch in the cases of $s/b = 1.0$ (Fig.10) and $s/b = 2.0$ (Fig.11). The notation $x$ in the abscissa $x/b$ of these figures represents a distance from the bottom of a notch to a point on the smallest cross section $(z = a)$.
From these figures we see that the distributions of \(a_i(\alpha_{\text{max}})\) have a great resemblance to one another under the condition of \(b/C < 0.3\) or \(0.4\), regardless of the values of \(x/b\) and \(b/C\).

Discussions on the convergence of solution have been made in the previous papers. (1) (2) (7)

5. Concluding remarks

The principal results of this paper are summarized as follows:

1) The interference effects (a phenomenon leading to \(\alpha_2/\alpha_1 < 1.0\)) of stress concentrations for the case of two notches are larger for smaller values of \(b/C\) and they occur in the condition of \(b/C < 0.5\).

Appendix

Expressions of Eqs. (18) \& (22) mentioned in Eqs. (6) and (8) are as follows:

\[ a_i^+ = \frac{1}{Br} z_i^+ \left( (3-2\nu)Q_{ia} - \frac{1}{x_i-1} \left( x - \frac{r}{a} \right) Q \right) - 2(3-4\nu)Q_{ia} + \frac{2x_i^2}{2ar(x_i^2 - 1)} A_i \]

\[ a_i^+ = \frac{1}{Br} z_i^+ \left[ 2(3-4\nu)Q_{ia} + 2\nu \left( \frac{1}{x_i-1} \left( x - \frac{r}{a} \right) Q + \frac{2x_i^2}{2ar(x_i^2 - 1)} Q \right) \right] \]

\[ a_i^+ = \frac{1}{Bar} \frac{1}{x_i-1} \left( A_i + 2(1-\nu)Q \right) \]

\[ a_i^+ = \frac{1}{Bar} \frac{1}{x_i-1} \left( -A_i + (x_i^2 - 1)Q_{ia} + 2\nu Q' - \left( x - \frac{r}{a} \right) Q \right) \]

\[ a_i^+ = \frac{1}{Bar} \frac{1}{x_i-1} \left( (x_i^2 - 1)Q_{ia} - 2\nu Q' - \left( x - \frac{r}{a} \right) Q \right) \]

\[ a_i^+ = \frac{1}{Bar} \frac{1}{x_i-1} \left( A_i - 2(1-\nu)Q' \right) \]

\[ a_i^+ = \frac{1}{Ba} \frac{1}{x_i-1} \left[ \left( (1-2\nu)Q_{ia} + \frac{1}{2ar(x_i^2 - 1)} \right) \left( x - \frac{r}{a} \right) Q + \frac{2x_i^2}{2ar} A_i \right] \]

\[ \sigma_r = -\frac{1}{2} \int_0^1 \left( [1 - (\nu I_0(\beta r)+ I_0(\beta r)) b_0 + (1-2\nu)I_0(\beta r)+\beta I_0(\beta r)] b_1 \right) \cos \beta z \, db \]

\[ \sigma_r = -\frac{1}{2} \int_0^1 \left( \frac{2}{\beta r} I_0(\beta r) b_0 + (1-2\nu)I_0(\beta r) b_1 \right) \cos \beta z \, db \]

\[ \tau_{rz} = \int_0^1 \left( I_0(\beta r) b_0 + \left( (1-2\nu)I_0(\beta r)+\beta I_0(\beta r) \right) b_1 \right) \cos \beta z \, db \]

where \( I_0(\beta r) \) is a modified Bessel function of the order \( m \) of the first kind and \( \nu \) is a Legendre function of the second kind of the order \( m \), and \( x \) represents

\[ x = \frac{r^2 + a^2 + z^2}{2ar} \]

\( z_1 = z - h, z_2 = z + h \ldots \ldots \ldots (23) \]
In these expressions, B, Q, Q', D, q and A are expressed as follows:

\[ B = 16\frac{r}{\mathcal{A}r} \]  
\[ Q = rQ_1(x) - Q_{-1}(x), \quad Q' = rQ_{-1}(x) - Q_{1}(x) \]  
\[ D = (J_0(\beta c) + iJ_1(\beta c))(2\left(1 - \nu\right)I_1(\beta c) + \beta cL_1(\beta c)) \]  
\[ -2I_1(\beta c)(2(1 - \nu)I_0(\beta c) + \beta cL_0(\beta c)) \]  
\[ q_1 = \left\{ (1 - 2\nu)I_1(\beta a) - \frac{\beta a}{2}I_0(\beta a) \right\} \{ K_0(\beta c) + K_1(\beta c) \} \]  
\[ + \left\{ \beta cK_1(\beta c) + K_0(\beta c) + 2
\nu K_0(\beta c) \right\} \]  
\[ q_1 = -2(1 - \nu)I_1(\beta a)K_1(\beta c) - \beta cL_1(\beta a)K_0(\beta c) + \beta aI_0(\beta a)K_1(\beta c) \]  
\[ q_2 = 2(1 - \nu)I_0(\beta a)K_0(\beta c) + \frac{2}{\beta c}I_1(\beta a)K_0(\beta c) + \frac{\beta a}{2}I_0(\beta a) \{ K_0(\beta c) + K_1(\beta c) \} \]  
\[ - \left\{ \beta cL_0(\beta a)K_0(\beta c) + K_1(\beta c) \right\} \]  
\[ q_3 = \beta cL_1(\beta a)K_0(\beta c) - \beta aI_1(\beta a)K_1(\beta c) - 2(1 - \nu)I_0(\beta a)K_1(\beta c) \]  
\[ A_1 = 2Q + A_3 \]  
\[ A_3 = \frac{4}{x^2 - 1}Q + \frac{\nu}{a}Q' \]  
\[ A_3 = \left(1 - \frac{2x}{x^2 - 1}\right)Q + \frac{x^2}{2ar}Q_{-1/2} \]  
where \( K_n(\beta c) \) is a modified Bessel function of the order \( n \) of the second kind.

References