Application of $\mathcal{E}_I$ -integral to Elasto-plastic Crack

Problems under Monotonic or Cyclic Loading*

(Physical Meanings of J (ΔJ) and Analysis of Crack Energy Density)

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In former papers, a concept of crack energy density $\mathcal{E}$ was proposed as the most fundamental parameter in fracture mechanics and it was shown that $\mathcal{E}$ can be evaluated by a path independent integral $\mathcal{E}_I$ without any restriction on constitutive law.

In this paper, $\mathcal{E}_I$ -integral is applied to elasto-plastic crack problems under monotonic or cyclic loading; and the availabilities of $\mathcal{E}_I$ -integral are demonstrated. First, it is shown that the physical meanings and applicable ranges of J and ΔJ can be clarified in the light of $\mathcal{E}_I$. Next, the method of the estimation of $\mathcal{E}$ based on $\mathcal{E}_I$ is established through the finite element analysis of elasto-plastic crack under monotonic loading. Moreover, an analysis under cyclic loading is carried out, and the basic characteristic of $\mathcal{E}$ and the relations between $\mathcal{E}$ and fatigue parameters ΔK and ΔJ are discussed.

Key Words: Fracture, Elasto-plastic crack, $\mathcal{E}_I$ -integral, J -integral, Cyclic loads, Crack energy density, Fatigue

1. Introduction

A concept of crack energy density has been advocated as the most principal parameter in fracture mechanics by one of the authors⁽¹⁾⁽²⁾, and it was shown that the crack energy density could be evaluated by a path independent integral $\mathcal{E}_I$ in the most generalized field considering inertia, body force and heat without any restriction on constitutive equation⁽³⁾. Crack energy density has always a clear physical meaning, therefore it has the possibility to become a fracture parameter describing the crack behaviors consistently over a wide range, and is expected to play an important role especially in elasto-plastic crack or creep crack in which the effects of the non-linearity of the relation between stress and strain can not be neglected.

This study purports to apply $\mathcal{E}_I$ -integral which estimates the crack energy density to elasto-plastic crack so as to clarify the physical meanings and the limits of J -integral and fatigue parameter ΔJ of which the physical meanings remain unexplained and newly position them as fracture mechanics parameters exactly, and to establish the way of estimating crack energy density by means of $\mathcal{E}_I$ -integral so as to testify the validity of crack energy density as an elasto-plastic crack parameter. First we consider the path independency and the physical meaning of J -

integral, clarify that J -integral has path independency and the physical meaning of crack energy density exactly and $J = \mathcal{E}_I$ holds when the constitutive law is given by deformation theory and that it does approximately and $J \cong \mathcal{E}_I$ holds when it is given by incremental theory provided by associated flow rule and unloading does not occur, and make clear the general relations between $\mathcal{E}_I$ -integral, J -integral and complementary J -integral (J_c). Next we discuss the amount of increase of crack energy density, $\Delta \mathcal{E}_{cycle}$, per unit loading cycle⁽⁴⁾ which is expected to play an important role in crack problems under cyclic loading and $\Delta \mathcal{E}$ which is newly defined in the light of the amount of change of strain and stress according to the change of loading, clarify the relation between $\Delta \mathcal{E}_{cycle}$, $\Delta \mathcal{E}$ and ΔJ which is a path integral derived from J -integral naturally, and show that ΔJ has an approximate path independency and gives $\Delta \mathcal{E}$ approximately when the path independency of J -integral holds approximately. Moreover, in order to verify the above theoretical expectations and demonstrate the validity of crack energy density by means of making its basic characteristic clear, we analyze $\mathcal{E}_I$ -integral and J -integral in elasto-plastic crack under monotonic loading by finite element method, consider fundamentally noteworthy points in estimation of crack energy density in reference to the estimation of conventional fracture mechanics parameters, and further analyze $\mathcal{E}_I$ -integral and ΔJ under cyclic loading to examine the relations between $\Delta \mathcal{E}_{cycle}$, $\Delta \mathcal{E}$ and conventional fatigue parameters.

2. Crack Energy Density $\mathcal{E}$ and $\mathcal{E}_I$ -integral

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Fig.1 Integration path around a notch

Consider a continuum in which a notch grows from the length a_0 at time 0 to the length a at time t (when it does not grow, $a=a_0$) as shown in Fig.1, and define Γ as an arbitrary path surrounding this notch and A as the area surrounded by Γ and the notch. $\mathcal{E}_J$ -integral is given⁽¹⁾ without any restriction on constitutive equation, in the static field where heat, inertia and body force are left out of consideration, by

$$\mathcal{E}_J = \int_{\Gamma} (W n_i - \int_0^t T_i du_{i,1}) ds - \int_{\Gamma_u - \Gamma_d + \Gamma_l + \Gamma_r} T_i du_{i,1} ds - \int_{A_0}^t \sigma_{ij,1} d\epsilon_{ij} dA \quad (1)$$

$$\mathcal{E}_J = \int_{\Gamma} (W n_i - T_i u_{i,1}) ds + \int_{A_0}^t (\epsilon_{ij,1} d\sigma_{ij} - \sigma_{ij,1} d\epsilon_{ij}) dA \quad (2)$$

where

$$dW = \sigma_{ij} d\epsilon_{ij}, \quad W = \int_0^t dW$$

Here, W , n_i , T_i , u_i , σ_{ij} and ϵ_{ij} are strain energy density, outward normal unit vector, traction force, displacement, stress and strain respectively. X_i is a fixed coordinate system and x_i is a moving coordinate system which always takes the origin at the center of the notch tip, and $(\cdot)_{,i} = \partial(\cdot)/\partial X_i$. Crack energy density $\mathcal{E}$ is given by $\mathcal{E}_J$ -integral as follows

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \int_{\Gamma_\rho} W dx_2 = \lim_{\rho \rightarrow 0} \mathcal{E}_J \quad (3)$$

and, especially when we choose Γ' as Γ in Fig.1 and define A' as the area surrounded by Γ' and the notch, it is given⁽¹⁾ by

$$\mathcal{E} = -\lim_{\rho \rightarrow 0} \left[\int_{\Gamma'} T_i u_{i,1} ds - \int_{A_0}^t (\epsilon_{ij,1} d\sigma_{ij} - \sigma_{ij,1} d\epsilon_{ij}) dA \right] = \lim_{\rho \rightarrow 0} \left(\tilde{\mathcal{E}}_J + \int_{A_0}^t (\epsilon_{ij,1} d\sigma_{ij} - \sigma_{ij,1} d\epsilon_{ij}) dA \right) \quad (4)$$

where

$$\tilde{\mathcal{E}}_J = \int_{A-A'} (\sigma_{ij} \epsilon_{ij,1}) dA - \int_{\Gamma'} T_i u_{i,1} ds$$

Crack energy density $\mathcal{E}$ is the strain energy per unit area absorbed at the point of crack tip in the continuum, that is, a quantity named, as it were, strain energy area density⁽¹⁾⁽²⁾. When we take it into consideration that the definition is given as an extremum value in case that the root radius ρ approaches 0, the estimation of $\mathcal{E}$ by applying $\mathcal{E}_J$ -integral for a notched specimen of which the root radius ρ is sufficiently small is considered realistic. We have two kinds of expressions of $\mathcal{E}_J$ -integral, that is, Eqs.(1) and (2), and it is convenient to employ the expression of Eq.(2), considering the correspondence with J -integral etc. But since the singularity of strain becomes larger than that of stress with the progress of plastic deformation in general though the singularities of these

two are equal as far as a linear elastic deformation arises, the evaluation of $\epsilon_{ij,1}$ is more difficult than that of $\sigma_{ij,1}$ in actual finite element analyses. Therefore, the evaluation of $\mathcal{E}_J$ by means of Eq.(1), which estimates line-integral $\int_{\Gamma_u - \Gamma_d + \Gamma_l + \Gamma_r} T_i du_{i,1} ds$ instead of area integral $\int_{A_0}^t \epsilon_{ij,1} d\sigma_{ij}$, is expected to give a more accurate value of $\mathcal{E}_J$ than by Eq.(2).

3. Reconsideration and Physical Positioning of J -integral from the Standpoint of $\mathcal{E}_J$ -integral

J -integral is originally introduced as a path independent integral under deformation theory, and it is clear that its physical meaning is nothing but crack energy density⁽¹⁾. Though, when it is applied under incremental theory, its physical meaning is not clear and even the path independency is not guaranteed, the evaluation of J -integral under incremental theory by means of finite element analysis is often performed, and the path independency of J -integral is judged to hold approximately when unloading does not occur⁽³⁾. In this chapter we consider the path independency and the physical meaning of J -integral under an arbitrary constitutive equation reconsidering J -integral from the standpoint of $\mathcal{E}_J$ -integral, and make clear the general relations between $\mathcal{E}_J$ -integral, J -integral and J_c -integral in order to consider the path independency.

3.1 On the path independency and physical meaning of J -integral

J -integral is defined as an integral independent of integration path under constitutive law is given by $\sigma_{ij} = (\partial W)/(\partial \epsilon_{ij})$, as follows⁽⁴⁾:

$$J = \int_{\Gamma} (W n_i - T_i u_{i,1}) ds \quad (5)$$

Though the path independency of the integral given by Eq.(5) is not guaranteed under arbitrary constitutive law, in this paper we define the quantity evaluated by Eq.(5) as J -integral for convenience, and then obtain the relation between $\mathcal{E}_J$ -integral and J -integral from Eq.(2) as shown in Eq.(6).

$$J = \int_{\Gamma} (W n_i - T_i u_{i,1}) ds = \mathcal{E}_J - \int_{A_0}^t (\epsilon_{ij,1} d\sigma_{ij} - \sigma_{ij,1} d\epsilon_{ij}) dA \quad (6)$$

Therefore, considering that $\mathcal{E}_J$ -integral is an integral independent of integration path under an arbitrary constitutive law, J -integral becomes a path independent integral and has the meaning of crack energy density in general when the second term of the right side of Eq.(6) becomes 0. Then we consider this second term. $\sigma_{ij,1}$ and $\epsilon_{ij,1}$ are X_1 -direction partial differentials when stress and strain are expressed as functions of position on condition that time is fixed, and $d\sigma_{ij}$ and $d\epsilon_{ij}$ are stress and strain increments at fixed position corresponding to time increment dt . Here, define C_{ijmn} as the coefficient matching $d\sigma_{ij}$ with $d\epsilon_{ij}$ as follows;

$$d\sigma_{ij} = C_{ijmn} d\epsilon_{mn} \quad (7)$$

Then the term in parentheses of the right side of Eq.(6) is expressed as

$$d\sigma_{ij}\varepsilon_{ij,1} - \sigma_{ij,1}d\varepsilon_{ij}$$

$$= C_{ijmn}d\varepsilon_{mn}\varepsilon_{ij,1} - \sigma_{ij,1}d\varepsilon_{ij} = C_{ijmn}d\varepsilon_{ij}\varepsilon_{mn,1} - \sigma_{ij,1}d\varepsilon_{ij} \quad (8)$$

If constitutive law $\sigma_{ij} = (\partial W / \partial \varepsilon_{ij})$ holds, the following relation as

$$C_{ijmn} = C_{mni j} \quad (9)$$

holds, and if constitutive law is given by incremental theory provided by associated flow rule where plastic potential is the same as yielding condition, Eq.(9) holds because the symmetry of the matrix connecting stress increment and strain increment is kept⁽⁷⁾. Then Eq.(8) can be expressed as

$$C_{ijmn}d\varepsilon_{ij}\varepsilon_{mn,1} - \sigma_{ij,1}d\varepsilon_{ij} = (C_{ijmn}\varepsilon_{mn,1} - \sigma_{ij,1})d\varepsilon_{ij} \quad (10)$$

and so we consider the meaning of terms in parentheses in Eq.(10). $\sigma_{ij,1}$ has the meaning of the quantity obtained by subtracting the stress at point M from the stress at point N when we choose an arbitrary point M inside integral path and a point N apart from M in x_1 -direction by unit length. On the other hand, $\varepsilon_{mn,1}$ has the meaning of the quantity obtained by subtracting the strain at point M from the strain at point N, consequently $C_{ijmn}\varepsilon_{mn,1}$ has the meaning of the quantity obtained by subtracting the stress after the change from the stress before the change at point M when the stress at point M changes, according to Eq.(7), until the state of strain at point M becomes equal to that at point N (we use the same C_{ijmn} given by Eq.(7) also in case of $\varepsilon_{mn,1} < 0$). Here, notice that the state of strain after the change at point M has turned out the same as the state of strain at point N.

Therefore, first, Eq.(8) becomes 0 when there is no influence of strain history on stress, in other words, the relationship between stress and strain is definite, namely, in a linear elastic region or when $\sigma_{ij} = (\partial W) / (\partial \varepsilon_{ij})$ holds, because $\sigma_{ij,1}$ agrees with $C_{ijmn}\varepsilon_{mn,1}$. This is naturally derived from the fact that J -integral is the quantity evaluating ε directly in this case⁽¹⁾. Next, Eq.(8) becomes approximately 0 even in the yielded region when constitutive law is given by incremental theory where Eq.(10) holds and unloading does not occur, because the difference between $C_{ijmn}\varepsilon_{mn,1}$ and $\sigma_{ij,1}$ caused by the difference of each strain history can be considered to be small enough numerically though it can not be testified in general to be a negligibly small quantity mathematically. On the other hand, even if Eq.(9) holds, the situation is different from the above mentioned when unloading occurs. Namely, when ε_{mn}^e and ε_{mn}^p are elastic and plastic strain components respectively ($\varepsilon_{mn} = \varepsilon_{mn}^e + \varepsilon_{mn}^p$) and we consider the existence of W^e ($\sigma_{ij} = (\partial W^e) / (\partial \varepsilon_{ij}^e)$), Eq.(7) becomes $d\sigma_{ij} = C_{ijmn}^e d\varepsilon_{mn}^e$, and Eq.(8) becomes as follows.

$$d\sigma_{ij}\varepsilon_{ij,1} - \sigma_{ij,1}d\varepsilon_{ij}$$

$$= C_{ijmn}^e d\varepsilon_{mn}^e \varepsilon_{ij,1} - \sigma_{ij,1}d\varepsilon_{ij}^e$$

$$= \{ (C_{ijmn}^e \varepsilon_{mn,1}^e - \sigma_{ij,1}) + C_{ijmn}^e \varepsilon_{mn,1}^p \} d\varepsilon_{ij}^e \quad (11)$$

Here $C_{ijmn}^e \varepsilon_{mn,1}^e - \sigma_{ij,1} = 0$ is true, so Eq.(8) does not become 0 because $C_{ijmn}^e \varepsilon_{mn,1}^p d\varepsilon_{ij}^e$ remains in Eq.(10) or (11).

Setting the results so far obtained in order, path independency of J -integral is exactly guaranteed on account of $J = \mathcal{E}_j$

when $\sigma_{ij} = (\partial W) / (\partial \varepsilon_{ij})$ holds, and an approximate path independency exists and what is estimated by J -integral is approximately crack energy density on account of $J \cong \mathcal{E}_j$ when Eq.(9) holds as in usual incremental theory and unloading does not occur. And path independency of J -integral does not exist because of $J \neq \mathcal{E}_j$ when unloading takes place. Besides when Eq.(7) is replaced by $d\varepsilon_{ij} = D_{ijmn}d\sigma_{mn}$, $(D_{ijmn}\sigma_{mn,1} - \varepsilon_{mn,1})d\sigma_{ij}$ can be obtained corresponding to Eq.(10), and in this case similar logic is applied to produce the same conclusion. In either case, it should be noted that only Eq.(9) is the condition to get Eq.(10) from Eq.(8) and no special assumptions are necessary.

Moreover we consider the case that the notch extends smoothly. The quantities related in this case can be expressed as functions consisting of x_1 , x_2 and a , and when we pay attention to a material point, $d/(da) = \partial / (\partial a) - \partial / (\partial x_1)$ holds except the point $(X_1, X_2) = (0, \pm \rho)^{(3)}$. Therefore Eq.(8) becomes as follows,

$$d\sigma_{ij}\varepsilon_{ij,1} - \sigma_{ij,1}d\varepsilon_{ij}$$

$$= \left\{ \left(\frac{\partial \sigma_{ij}}{\partial a} - \sigma_{ij,1} \right) \varepsilon_{ij,1} - \sigma_{ij,1} \left(\frac{\partial \varepsilon_{ij}}{\partial a} - \varepsilon_{ij,1} \right) \right\} da$$

$$= \left(\frac{\partial \sigma_{ij}}{\partial \sigma} \varepsilon_{ij,1} - \sigma_{ij,1} \frac{\partial \varepsilon_{ij}}{\partial a} \right) da \quad (12)$$

Therefore, when a crack of infinite length extends steadily in an infinite plate [$\partial / (\partial a) = 0$], $J = \mathcal{E}_j$ holds because Eq.(12) becomes 0 and path independency exists. Though a yielded region accompanied by unloading exists in this case, path independency is guaranteed because $\varepsilon_{mn,1}^p$ in Eq.(11) becomes 0 in steady crack extension.

3.2 The general relations between $\mathcal{E}_j$ -integral, J -integral, and complementary J -integral (J_c) and path independency of J -integral

When we consider a three dimensional continuum and an arbitrary closed curved surface S in it, the path independent integral based on the conservation law of energy and used to obtain $\mathcal{E}_j$ -integral is given⁽³⁾ by

$$w_k = \int_S \{ W n_k - \int_0^t (T_i u_i)_{,k} \} dS = 0 \quad (13)$$

A path independent integral w_c complementary to the above w_k can be newly introduced similarly as follows.

$$w_{c,k} = \int_S \{ W_c n_k - \int_0^t (u_i dT_i)_{,k} \} dS = 0 \quad (14)$$

where

$$dW_c = \varepsilon_{ij} d\sigma_{ij}, \quad W_c = \int_0^t dW_c$$

Therefore a path independent integral exclusive of time integral

$$\int_S \{ (W + W_c) n_k - (T_i u_i)_{,k} \} dS = 0 \quad (15)$$

can be obtained by means of adding Eq.(14) to Eq.(13). This integral coincides with what is called I_k -integral by Carlsson⁽⁸⁾. Then when we apply Eq.(15) to a two dimensional field and take it into consideration that traction force is zero on crack surface and that $\mathcal{E}_j = \int_{r_0} W dx_2$ is true, the following relation as

$$J = \mathcal{E}_j - \phi_c, \quad \phi_c = \int_{r_0+r+u-r_0} (W c n_1 - u_i T_{i,1}) ds \quad (16)$$

is obtained. We notice that the sum of J -integral and J_c -integral to the closed path including the notch becomes $-\mathcal{E}_j$ -inte-

gral. It goes without saying that the right side of Eq.(16) can be obtained directly from the right side of Eq.(6) by means of Gauß's theorem. As it is clear from Eq.(15) that the path independency of J -integral and that of J_c -integral are in a mutually necessary and sufficient relation, Eq.(16) suggests that it is right to judge the path independency of J -integral based on the comparison with $\mathcal{E}_j$ -integral such that it is path-independent when $J = \mathcal{E}_j$, it is approximately path-independent when $J \approx \mathcal{E}_j$, and it is not path-independent when $J \neq \mathcal{E}_j$. Besides, the path independent integral shown in Eq.(15) can be also introduced from the following Eqs.(17) and (18) which are obtained based on the principle of virtual work and that of complementary virtual work respectively, taking $(W + W_c)_k = \sigma_{ij,k} \epsilon_{ij} + \sigma_{ij} \epsilon_{ij,k}$ into account.

$$\int_V \sigma_{ij} \epsilon_{ij,k} dV - \int_S T_i u_{i,k} dS = 0 \quad (17)$$

$$\int_V \epsilon_{ij} \sigma_{ij,k} dV - \int_S u_i T_{i,k} dS = 0 \quad (18)$$

Here it should be noted that Eq.(15) is a path independent integral without any restriction on constitutive equation and loading history.

4. Reconsideration and Physical Positioning of ΔJ from the Standpoint of $\mathcal{E}_j$ -integral

Authors have made it clear that the amount of increase of crack energy density, $\Delta \mathcal{E}_{cycle}$, per unit cycle plays an important role in crack problems under cyclic loading through the consideration and positioning of Dugdale model under cyclic loading from a viewpoint of crack energy density⁽⁴⁾. In this chapter $\Delta \mathcal{E}_j$ -integral derived from $\mathcal{E}_j$ -integral is newly defined by means of contemplating the change of stress and strain from a standard state. In the light of $\Delta \mathcal{E}_j$ -integral, ΔJ derived from J -integral naturally in the same way is reconsidered, the relation between these parameters is made clear, and the roles and positionings of these parameters are examined in relation to $\Delta \mathcal{E}_{cycle}$.

4.1 $\Delta \mathcal{E}_j$ -integral and ΔJ -integral

When a state A changes to a state B after an arbitrary change in the field considered in chapter 2 (a stationary case of length a is considered), $\Delta \mathcal{E}_j^A$ is defined, if superscript attached indicates the state, as follows

$$\Delta \mathcal{E}_j^A = \int_r \left\{ \int_{i^A}^{i^B} (\sigma_{ij} - \sigma_{ij}^A) d\epsilon_{ij} - \int_{i^A}^{i^B} (T_i - T_i^A) du_{i,1} \right\} ds - \int_A \int_{i^A}^{i^B} (\sigma_{ij,1} - \sigma_{ij,1}^A) d\epsilon_{ij} dA \quad (19)$$

$$= \int_r \left\{ \int_{i^A}^{i^B} (\sigma_{ij} - \sigma_{ij}^A) d\epsilon_{ij} - \Delta T_i \Delta u_{i,1} \right\} ds - \int_A \left[\int_{i^A}^{i^B} (\sigma_{ij,1} - \sigma_{ij,1}^A) d\epsilon_{ij} - (\epsilon_{ij,1} - \epsilon_{ij,1}^A) d\sigma_{ij} \right] dA \quad (20)$$

where

$$\Delta T_i = T_i^B - T_i^A, \quad \Delta u_{i,1} = u_{i,1}^B - u_{i,1}^A$$

Then $\Delta \mathcal{E}_j^A$ has the meaning of $\mathcal{E}_j^A$ on the assumption that the values of stress and strain in the state A are regarded as those at origin, that is, strain energy density W^A in the state A is set at zero (therefore $\mathcal{E}_j^A = 0$), and so it has the characteristic of a path independent inte-

gral under an arbitrary constitutive law as a matter of course. On the other hand, if we consider the quantity given by the following relation,

$$\Delta J^{BA} = \int_r \left\{ \int_{i^A}^{i^B} (\sigma_{ij} - \sigma_{ij}^A) d\epsilon_{ij} - \Delta T_i \Delta u_{i,1} \right\} ds \quad (21)$$

this ΔJ^{BA} becomes an integral independent of integration path in the case that the constitutive equation follows deformation theory⁽⁹⁾. Further the general relation between $\Delta \mathcal{E}_j^A$ and ΔJ^{BA} similar to Eq.(6)

$$\Delta \mathcal{E}_j^A = \Delta J^{BA}$$

$$- \int_A \left[\int_{i^A}^{i^B} (\sigma_{ij,1} - \sigma_{ij,1}^A) d\epsilon_{ij} - (\epsilon_{ij,1} - \epsilon_{ij,1}^A) d\sigma_{ij} \right] dA \quad (22)$$

holds, therefore, considering the same way as in section 3.1, $\Delta \mathcal{E}_j^A = \Delta J^{BA}$ holds exactly on condition that the constitutive equation follows deformation theory (in this case, only monotonic loading process is considered), and the path independency of ΔJ^{BA} holds approximately because of $\Delta \mathcal{E}_j^A \approx \Delta J^{BA}$ when the change of external loading from the state A to the state B is monotonic (loading process or unloading process) on condition that the constitutive equation follows incremental theory provided by associated flow rule in which Eq.(9) holds. In these cases, ΔJ^{BA} comes to have the physical meaning of the changes of crack energy density $\Delta \mathcal{E}^{BA}$ corresponding to $\Delta \mathcal{E}_j^A$.

4.2 $\Delta \mathcal{E}_{cycle}$, $\Delta \mathcal{E}_j$ and ΔJ

Now consider a cycle in which load increases monotonically from a state A to a state B and then decreases to a state C monotonically. $\Delta \mathcal{E}_{cycle}$ is the quantity defined as $\mathcal{E}_j^B - \mathcal{E}_j^A$, and it can be known that $\Delta \mathcal{E}_j^A$ and $\Delta \mathcal{E}_j^B$ are positioned as the parameters which are given by multiplying $\Delta \mathcal{E}_{cycle}$ by constants (dependent on material constants in constitutive equation), considering the corresponding areas on the hysteresis loop in stress-strain curve. Meanwhile ΔJ^{BA} and ΔJ^{CB} come to be positioned as the quantities estimating approximately $\Delta \mathcal{E}_j^A$ and $\Delta \mathcal{E}_j^B$ respectively on condition that the constitutive equation follows incremental theory provided by associated flow rule. Especially, considering a small scale yielding, as $\mathcal{E} \approx \{(1-\nu^2)/E\} K^2$ (K : stress intensity factor, ν : Poisson's ratio (plane strain) or 0 (plane stress), E : Young's modulus) is true⁽¹⁾⁽²⁾ when monotonic loading is given from initial state, $\Delta \mathcal{E}^{BA} = \Delta \mathcal{E}_j^A (\approx \Delta J^{BA}) \approx \{(1-\nu^2)/E\} \Delta K^2 (\Delta K = K^B - K^A)$ comes true. Besides considering the Dugdale model under constant cyclic loading, $\Delta \mathcal{E}^{BA} (= \Delta J^{BA}) = \Delta \mathcal{E}^{CB} (= \Delta J^{CB}) = \Delta \mathcal{E}_{cycle}$ is true, and especially in a small scale yielding $\Delta \mathcal{E}_{cycle} \approx \Delta K^2/E$ holds, as shown before⁽⁴⁾. By the way, we have defined ΔJ as what is estimated by Eq.(22) so far. On the other hand, the quantity estimated from load-displacement curve from the state A to the state B is also called $\Delta J^{(10)}$ and this ΔJ becomes equivalent to $\Delta \mathcal{E}^{BA}$ exactly. This is testified in the same way as the quantity, called J , estimated from load-displacement curve in case that monotonic loading is given from initial state is shown to be nothing but $\mathcal{E}^{(2)}$.

5. Finite Element Analyses of Crack Energy Density under Monotonic Loading Based on $\mathcal{E}_J$ -integral

In this chapter, finite element analyses of elasto-plastic crack under monotonic loading are carried out to demonstrate the estimation of crack energy density by $\mathcal{E}_J$ -integral to be actually valid as well as to discuss fundamentally noteworthy points in case of the analyses.

5.1 Method and model for analyses

Finite element analyses of $\mathcal{E}_J$ -integral and J -integral performed on condition that the constitutive equation follows incremental theory. The estimation of $\mathcal{E}_J$ -integral is carried out by means of Eq.(1) for the reason mentioned in chapter 2. As for the element, a triangle constant strain element is employed. With regard to $\sigma_{ij,i}$ in the second term of Eq.(1), first we determine stress at each node as a weighted average concerning area of each element which surrounds the node, next we get stress distribution in each element by adopting a linear interpolation function to stress at each node determined above, and obtain their values. Time-integral terms are obtained as the sum of approximate values in each increment of load by means of a trapezoid formula. Here we analyze the case that uniform tension stress σ_0 is applied to the top and bottom surfaces of a center-cracked specimen on the assumption that crack is stationary and that plane strain state is realized. The model for analyses is shown in Fig.2(a) and integral paths along which path integral is evaluated are shown in Fig.2(b). Analyses are performed for a notched specimen with $\rho/W=0.0125$ for the reason mentioned in chapter 2, and additional analyses of a cracked specimen with $\rho=0$ are performed to compare the results. In case of the notched specimen 520 elements and 293 nodes are used and in case of the cracked specimen 327 elements and 188 nodes. As test material A533B steel is supposed, and Young's modulus E is 205.8GPa(21000 kgf/mm²), Poisson's ratio ν is 0.3, and yielding stress σ_Y is 480.2MPa(49kgf/mm²). As for the relation between equivalent stress and equivalent strain, approximation by two straight lines is made, and strain hardening rate $H'=d\bar{\sigma}/d\bar{\epsilon}_p$ is 2.058 GPa. Besides we referred to the literature (11) to make computer programming for analyses.

5.2 Result of analyses and its consideration

Values of $\mathcal{E}_J$ -integral for the notched specimen with $\rho/W=0.0125$ to each path are shown in Fig.3, and the variations of the values of both $\mathcal{E}_J$ -integral and J -integral obtained as average values of values for six paths in both notch and crack cases are shown against displacement obtained as an average value of loading-direction displacement at loading point in

Fig.2 Shape of specimens and integration paths for analyses

Fig.3 Path independency of $\mathcal{E}_J$ -integral under monotonic loading

Fig.4 Variation of $\mathcal{E}_J$ -integral and J -integral with displacement at loading point

Fig.4. It is observed that the path independency holds good to each loading step in Fig.3 and that the values of $\mathcal{E}_J$ -integral and J -integral for the notched specimen almost coincide with each other. So it is certified numerically that $J \cong \mathcal{E}_J$ holds in incremental theory provided by associated flow rule under monotonic loading as mentioned in section 3.1. Further, the value of J -integral for the notched specimen and that for the cracked specimen are

shown to agree with each other at least beyond perfect yielding of a section containing a crack ($\epsilon_f = 13.7 \text{ kJ/m}^2$), and consequently this shows that crack energy density to crack ($\rho \rightarrow 0$) can be estimated by means of analyses for notch with $\rho/W = 0.0125$ as performed in these analyses with practically sufficient accuracy.

By the way, the value of ϵ_f , when stress and strain for the cracked specimen are applied to Eq.(1) perfunctorily, is rather small as shown in Fig.4. This is considered to result from the fact that the contribution of area A' to $\int_{A'} \sigma_{ij} d\epsilon_{ij} dA$ in Eq.(1), which becomes finite value in the case of $\rho \rightarrow 0$ as is easily known by considering the linear elastic case, can be obtained by means of analyses for a notched specimen with sufficiently small ρ but can not be evaluated by means of usual analyses for a cracked specimen. Meanwhile in relation to the above fact, although $\bar{J}$ -integral includes an area integral similar to ϵ_f -integral⁽¹²⁾, its value for a cracked specimen is shown to become almost equal to $\bar{J}$ -integral⁽¹³⁾. This is considered to result from the fact that, as is known clearly from Eq.(4) taking that $\bar{J}$ -integral positioned as $\bar{J} = \lim_{\rho \rightarrow 0} \bar{\epsilon}_f$ into account⁽³⁾, $\bar{J}$ -integral should be originally estimated by $\int_{A-A'} \sigma_{ij} d\epsilon_{ij} dA$ when $\rho \rightarrow 0$, that is, the quantity in correspondence with area A' is excluded. Namely, it is considered that the quantity in correspondence with area A' including the singular point (crack tip) when $\rho \rightarrow 0$ can not be estimated for a usual cracked specimen, but actually the quantity in correspondence with area $A-A'$ excluding the singular point when $\rho \rightarrow 0$ can be estimated perfunctorily for area A of a cracked specimen ($\bar{J}$ is estimated by $\bar{\epsilon}_f$ in Eq.(4) in which $A-A'$ is replaced by A as $\rho \rightarrow 0$) as well as for area $A-A'$ of a notched one.

Consequently it is certified that crack energy density can be evaluated by finite element analyses of ϵ_f -integral for a notched specimen with $\rho/W = 0.0125$ with practically sufficient accuracy if the constitutive equation is given by incremental theory provided by associated flow rule and load increases monotonically. Moreover, even if unloading occurs where $\bar{J}$ -integral loses its meaning or the constitutive equation is given by incremental theory provided by non-associated flow rule, crack energy density is thought to be evaluated in the same way. Besides, the method of above mentioned analyses can be applied to the evaluation of crack energy density in creep crack⁽¹⁴⁾.

6. Finite Element Analyses of Crack Energy Density under Cyclic Loading Based on ϵ_f -integral

In this chapter, the evaluation of crack energy density in elasto-plastic fatigue crack is made by the method of analyses established in the former chapter to clarify the fundamental characteristic of crack energy density as well as to examine the relations between fatigue parameters

Fig.5 Cyclic loading for analyses

Fig.6 Path independency of ϵ_f -integral (0~196 MPa, plane stress, ①)

Fig.7 Path independency of ϵ_f -integral (0~196 MPa, plane stress, ①)

ΔK , ΔJ , $\Delta \epsilon$ and $\Delta \epsilon_{\text{cycle}}$.

6.1 Method and model for analyses

Crack energy density under cyclic loading is analyzed by means of ϵ_f in Eq. (1) and ΔJ in Eq.(21) using the same model, notch ($\rho/W = 0.0125$) and method as used in analyses under monotonic loading on the assumption that crack is stationary. First we assume plane stress state and analyze a small scale yielding case that uniform tension stress σ_0 is loaded cyclically between 0 and 196 MPa (0~20 kgf/mm²) in order to ①~⑦ as shown in Fig.5. We think of two test materials, that is, case ①: A533B steel which is the same as employed in analyses under monotonic loading and case ②: ideal elasto-plastic steel in which material constants are the same as A533B steel except strain hardening rate $H' = 1/100 \times 2.058 \text{ GPa}$. Further we analyze a large scale yielding case that σ_0 is loaded

cyclically between 0 and 323.4MPa (0~33 kgf/mm²) in order ①~④ to the above mentioned case ①. Next we assume plane strain state and analyze a small scale yielding case that σ_0 is loaded between 0 and 196MPa in order ①~⑦ as shown in Fig.5 to two test materials, that is, ① and ②, and a large scale yielding case that σ_0 is loaded between 0 and 323.4MPa in order ①~⑦ to test material ①.

6.2 Result of analyses and its consideration

In the beginning we show the result of analyses in the case of plane stress and discuss it. The values of $\mathcal{E}_j$ -integral to each path in the case that σ_0 is varied between 0 and 196MPa against case ① and case ② are shown in Fig.6 and Fig.7 respectively. The amount of increase of crack energy density, $\Delta\mathcal{E}_{cycle}$ per unit cycle obtained from the values of $\mathcal{E}_j$ -integral as an average value of values for six paths is shown at each step ①~⑦ and the ratios of $\Delta\mathcal{E}_{cycle}$ to $\Delta K^2/E = 2.853 \text{ kJ/m}^2$ where ΔK is the variation of stress intensity factor K obtained analytically and to ΔJ as an average value of values for six paths (ΔJ corresponding to ①~③ is determined as an average value of ΔJ corresponding to ①~② and ②~③) are shown in Table 1. According to Fig.6 and Fig.7, the path independency is maintained satisfactorily even when unloading occurs. It is observed in Table 1 that crack energy density does not increase so much after the state ③ in case ① and, on the other hand, that $\Delta\mathcal{E}_{cycle}$ has a steady value after the state ② in case ②. This is considered to result from the fact that $\Delta\mathcal{E}_{cycle} \cong 0$ because a shakedown state of notch tip elements in which crack energy density is charged is realized in case ① as isotropic hardening rule is assumed in these analyses and, on the other hand, that $\Delta\mathcal{E}_{cycle} \cong \text{const.}$ is realized in case ② as the test material behaves similarly to an ideal elasto-plastic one. The relation between ΔJ and $\Delta K^2/E$ is almost equal to what is expected from theory and, especially in case ② similar to Dugdale model, the relation between $\Delta\mathcal{E}_{cycle}$, ΔJ and $\Delta K^2/E$ becomes what is expected from theory. The path independency of $\mathcal{E}_j$ is also maintained satisfactorily in case that σ_0 is varied between 0 and 323.4MPa, and the approximate path independency of ΔJ holds numerically. The values of $\mathcal{E}_j$ and ΔJ as an average value of values for each path are shown for each in Table 2 (for example, ΔJ at ② means ΔJ corresponding to ①~②) and the ratio of $\Delta\mathcal{E}_{cycle}$ to ΔJ (an average value of ΔJ corresponding to loading process and unloading process) is shown in Table 3.

Next we show the result of analyses in case of plane strain state and discuss it. The path independency of $\mathcal{E}_j$ -integral and the approximate path independency of ΔJ hold satisfactorily in either case though we omit the relevant data. In case that σ_0 is varied between 0 and 196MPa, the value of $\mathcal{E}_j$ as an average value of values for each path is shown in Table 4, and the relation between $\Delta\mathcal{E}_{cycle}$, ΔJ and

Table 1 $\Delta\mathcal{E}_{cycle}$, ΔJ and $\Delta K^2/E$ (0~196 MPa, plane strain) (kJ/m²)

		①~③	②~④	③~⑤	④~⑥	⑤~⑦
I	$\Delta\mathcal{E}_{cycle}$	4.580	1.500	0.241	0.171	0.158
	$\Delta\mathcal{E}_{cycle}/\Delta\mathcal{E}_e$	1.605	0.526	0.084	0.060	0.055
	$\Delta J/\Delta\mathcal{E}_e$	1.147	1.087	1.075		
II	$\Delta\mathcal{E}_{cycle}$	4.476	2.460	2.410	2.556	2.370
	$\Delta\mathcal{E}_{cycle}/\Delta\mathcal{E}_e$	1.569	0.862	0.845	0.896	0.831
	$\Delta J/\Delta\mathcal{E}_e$	1.200	1.182	1.152	1.114	1.104

Table 2 $\mathcal{E}_j$ and ΔJ under cyclic loading (0~323.4 MPa, plane stress, ①)

	①	②	③	④
$\mathcal{E}_j$	0	53.60	56.35	59.93
ΔJ	0	52.02	8.972	8.974

Table 3 $\Delta\mathcal{E}_{cycle}$ and ΔJ (0~323.4 MPa, plane stress, ①)

		①~③	②~④
$\mathcal{E}_j$	$\Delta\mathcal{E}_{cycle}$	56.35	6.323
	$\Delta\mathcal{E}_{cycle}/\Delta J$	1.848	0.705

Table 4 $\mathcal{E}_j$ under cyclic loading (0~196 MPa, plane strain)

	①	②	③	④	⑤	⑥	⑦
I $\mathcal{E}_j$	0	2.768	3.167	4.257	4.547	5.500	5.756
II $\mathcal{E}_j$	0	2.769	3.183	4.276	4.651	5.691	6.052

Table 5 $\Delta\mathcal{E}_{cycle}$, ΔJ and $\{(1-\nu^2)/E\}\Delta K^2$ (0~196 MPa, plane strain)

		①~③	②~④	③~⑤	④~⑥	⑤~⑦
I	$\Delta\mathcal{E}_{cycle}$	3.167	1.489	1.380	1.243	1.209
	$\Delta\mathcal{E}_{cycle}/\Delta\mathcal{E}_e$	1.220	0.574	0.532	0.479	0.466
	$\Delta J/\Delta\mathcal{E}_e$	1.119	1.144	1.112	1.080	1.079
II	$\Delta\mathcal{E}_{cycle}$	3.183	1.507	1.468	1.415	1.371
	$\Delta\mathcal{E}_{cycle}/\Delta\mathcal{E}_e$	1.226	0.581	0.566	0.545	0.528
	$\Delta J/\Delta\mathcal{E}_e$	1.091	1.088	1.087	1.087	1.085

$\{(1-\nu^2)/E\}\Delta K^2$ is shown in Table 5. The relation is almost what is expected from theory. In case that σ_0 is varied between 0 and 323.4MPa, the values of $\mathcal{E}_j$ and ΔJ as an average value of values for each path are shown in Table 6, and the ratio of $\Delta\mathcal{E}_{cycle}$ to ΔJ is shown in Table 7.

Thus crack energy density is elasto-plastic crack under cyclic loading is evaluated by means of $\mathcal{E}_j$ -integral, moreover the relation between $\Delta\mathcal{E}_{cycle}$, ΔJ and ΔK in a small scale yielding and the relation between $\Delta\mathcal{E}_{cycle}$ and ΔJ in a large scale yielding have been numerically ascertained. The value of $\Delta\mathcal{E}_j(\Delta\mathcal{E})$ is not shown, because the relation between $\Delta\mathcal{E}$ and ΔJ under monotonic loading process is

Table 6 ϵ_f and ΔJ under cyclic loading
(0~323.4 MPa, plane strain, ①)

	①	②	③	④	⑤	⑥	⑦
ϵ_f	0	9.782	11.836	15.567	17.210	20.519	22.106
ΔJ	0	9.849	8.146	8.082	7.532	7.531	7.630

Table 7 $\Delta \epsilon_{cycle}$ and ΔJ
(0~323.4 MPa, plane strain, ①)

		①~③	②~④	③~⑤	④~⑥	⑤~⑦
ϵ_f	$\Delta \epsilon_{cycle}$	11.836	5.785	5.374	4.952	4.896
	$\Delta \epsilon_{cycle} / \Delta J$	1.315	0.713	0.688	0.658	0.646

similar to that between ϵ_f and J shown in Fig.4. Here we should like to emphasize again that the physical meaning of what is evaluated by ΔJ is nothing but $\Delta \epsilon$.

7. Conclusions

J -integral and fatigue parameter ΔJ have been newly positioned by applying ϵ_f -integral to elasto-plastic crack under monotonic and cyclic loading. Also a method for the estimation of crack energy density by ϵ_f -integral has been established and the validity of ϵ as an elasto-plastic crack parameter has been demonstrated through finite element analyses of elasto-plastic crack.

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