Finite Bending of Curved Beam with Thin Wall Cross Section

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It is well known that a curved beam with a thin wall subjected to bending moment, has its bending stiffness reduced due to a flattening of its cross section, until local buckling occurs at last. Restricted cases, for example, circular tubes had been treated by the analytical method. Present paper gives a convenient representation for the stiffness matrix of finite element in order to analyze the finite bending of a thin walled curved beam with arbitrary cross section. Here the buckling of a circular tube is examined and compared with the proposed simplified solutions, and several representative cases are computed.

Key Words: Structural Analysis, Strength of Materials, Thin Walled Curved Beam, Finite Bending, Finite Element Method, Numerical Analysis

1. Introduction

Curved beams and tubes are widely employed as the elements composing the general structures and the pipe lines and it is important to know about their behaviors for the designing structures. It has been well known that when a curved beam is subject to bending, the shape of its cross section is flattened and a collapsing occurs at smaller bending moment than in case of a straight pipe or beam that has the same shape of the cross section because the rigidity is reduced and a higher stress generates. With regard to this problem, since Brazier first undertook its theoretical analysis, modifications have been made toward the solutions of closer approximation. However, most of these analyses were made in dealing with a circular tube and among the results of these analytical methods, some differences have been observed. As a matter of fact, there are difficulties in the method introducing the deformation of finite displacement in a strict sense, and development of a practical method that could be applied to the cross section of any shape is desirable. Also in the estimation of stiffness of the curved beam, the effect of the secondary stresses should be taken into consideration, but very little has been known about the reduction of stiffness in its finite bending.

In recent years, through the introduction of the method of repeated increment by the finite element method, the finite deformation has come to be dealt with without any difficulties. In these circumstances, the authors have tried to

clarify the method of analyzing the finite bending of the curved beam. By employing this proposed method, it is, needless to say, possible to analyze the collapsing phenomena of a straight beam with any shape of the cross section and besides, for a curved beam, not only the analysis of the finite bending within its center plane but also that of the off-center-plane bending can be easily made. In case of the symmetrical cross section, such problems as the finite bending around the axis oblique to the principal axes may be claimed to have for the first time been made clear in this study.

2. Theory

2.1 Fundamental equation based upon the principle of virtual work

Under the action of external force R , internal force S and stress σ produced are in equilibrium, the condition of equilibrium is represented by the principle of virtual work with finite degrees of freedom approximately by

$$\bar{r}^T R = \sum_i \bar{s}^T S = \sum_i \int \bar{\epsilon}^T \sigma dV \quad \dots\dots\dots (1)$$

where, r is the nodal displacement and the infinitesimal virtual displacement is written as $\bar{r}$. R is the force corresponding to $\bar{r}$ and $\bar{s}$ is the virtual deformation that occurs corresponding to $\bar{r}$. Then, if an increment of the force is given as in $R' = R + \bar{R}$, a displacement $\bar{r}$ generates to make a new condition of equilibrium as represented by the internal force $S' = S + \bar{S}$ and the stress $\sigma' = \sigma + \bar{\sigma}$. The principle of virtual work at this condition becomes

$$\begin{aligned} \bar{r}^T R' &= \sum_i \int \bar{\epsilon}^T \sigma' dV \\ &= \sum_i \int (\bar{\epsilon} + \bar{\bar{\epsilon}})^T (\sigma + \bar{\sigma}) dV \\ &\approx \sum_i \int (\bar{\epsilon}^T \sigma + \bar{\epsilon}^T \bar{\sigma} + \bar{\bar{\epsilon}}^T \sigma) dV \quad \dots\dots\dots (2) \end{aligned}$$

where, picking up $\epsilon_k = \epsilon_k(r_i)$ which is one of the elements of the strain ϵ and applying Taylor's expansion to it, the following equations are obtain ;

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$$\begin{aligned}\bar{\epsilon}_k &= (\partial \epsilon_k / \partial r_i)_{r=0} \bar{r}_i, \quad \bar{\epsilon} = C \bar{r} \\ \bar{\epsilon}'_k &= \bar{\epsilon}_k + \dot{\epsilon}_k = (\partial \epsilon_k / \partial r_i)_{r=0} \bar{r}_i + \dot{\epsilon}_i \\ &= \left\{ \left(\frac{\partial \epsilon_k}{\partial r_i} \right)_{r=0} + \left(\frac{\partial^2 \epsilon_k}{\partial r_i \partial r_j} \right)_{r=0} \bar{r}_j \right\} \bar{r}_i \quad \dots\dots\dots (3)\end{aligned}$$

Now the element $\partial^2 \epsilon_k / \partial r_i \partial r_j$ is written as the i -th row and j -th column of B_k .

$$\dot{\epsilon}_k = \left(\frac{\partial^2 \epsilon_k}{\partial r_i \partial r_j} \right)_{r=0} \bar{r}_j \bar{r}_i = \bar{r}^T B_k \bar{r} \quad \dots\dots\dots (4)$$

Also, the relation between stress and strain is represented by

$$\sigma = H \epsilon \quad \dots\dots\dots (5)$$

where, H is the matrix of the elastic coefficient.

Substituting the equations (3), (4) and (5) into the equation (2), the following rearrangement can be made:

$$\bar{r}^T R' = \bar{r}^T \sum_i \int C^T \sigma dV + \bar{r}^T (K_E + K_G) \bar{r} \quad \dots\dots\dots (6)$$

where

$$K_E = \sum_i \int C^T H C dV, \quad K_G = \sum_i \int \sigma_k B_k dV \quad \dots\dots\dots (7)$$

Since $\bar{r}$ is arbitrarily chosen in the above equation, we have

$$R' - \sum_i \int C^T \sigma dV = K \bar{r}, \quad K = K_E + K_G \quad \dots\dots\dots (8)$$

$\bar{r}$ is gained by solving the equation (8). Then, σ and $\dot{S}$ are obtained in the previous equations and the position of equilibrium $r + \bar{r}$, the internal force $S + \dot{S}$ and the stress $\sigma + \dot{\sigma}$ at the next stage are determined. By repeating the above procedures, the finite deformation is gained.

2.2 Strain-displacement relation in a member of thin walled cross section

It will be assumed that the locus of the fiducial point properly chosen in a thin walled member with a constant cross section is a circular arc with the radius R_0 . Then, as shown in Fig.1, taking the origin at the fiducial point, the z -axis is chosen in the direction of the transverse axis of the plane vertical to the pipe axis and the plane of the circular arc. The y -axis is chosen normal to the z -axis within the cross section.

Two cross sections that include an

Fig.1 Coordinates and displacement

infinitesimal angle of aperture $d\phi_0$ may be supposed each to keep a plane after the increment of deformation is added. After the deformation the angle changes to $d\phi_1$, as shown in Fig.1. The displacement (v', w') of the point (y, z) is represented as the sum of the displacement as rigid body (v_0, w_0, α) of the cross section and the displacement (v, w) of each point due to the strain of the cross section in the following

$$\begin{pmatrix} v' \\ w' \\ \theta' \end{pmatrix} = \begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \\ \alpha \end{pmatrix} + \begin{pmatrix} v \\ w \\ \theta \end{pmatrix} \quad \dots\dots\dots (9)$$

Also, the length at the point (y, z) along the x -axis ds_0 becomes ds_1 after the increment is added. Namely,

$$\begin{aligned}ds_0 &= (R_0 + z) d\phi_0 \\ ds_1 &= (R_0 + w_0 + \alpha y + z + w) d\phi_1 \end{aligned} \quad \dots\dots\dots (10)$$

As the axial strain, $\epsilon = ds_1/ds_0 - 1$ is rearranged into the following:

$$\epsilon = \left(\epsilon_0 + \frac{w}{R_0} + z\psi + \frac{\alpha y}{R_0} + w\psi + y\alpha\psi \right) / \left(1 + \frac{z}{R_0} \right) \quad \dots\dots\dots (11)$$

where $\epsilon_0 = (R_0 + w_0) d\phi_1 / R_0 d\phi_0 - 1$, $d\phi_1/d\phi_0 = 1 + \phi_0$. The strain in the direction of wall thickness of the element cross section is represented by

$$\epsilon = \epsilon_t + \zeta \kappa_t$$

where ϵ_t is the strain at the middle plane of wall thickness and ζ is the coordinate taken in the normal direction to the middle plane.

Fig.2 Local coordinates

We find the following expressions for the strain components:

$$\begin{aligned}\epsilon_t &= \left(\epsilon_0 + \frac{w}{R_0} + z\psi + \frac{y_1 \alpha}{R_0} \right) / \left(1 + \frac{z_1}{R_0} \right) \\ \epsilon_r &= \left((w + y_1 \alpha) \bar{\psi} + (w + y_1 \alpha) \psi \right) / \left(1 + \frac{z_1}{R_0} \right) \\ \kappa_t &= \left(\psi \cos \phi - \frac{\alpha \sin \phi}{R_0} \right) / \left(1 + \frac{z_1}{R_0} \right) \\ \kappa_r &= - \left(\alpha \bar{\psi} + \bar{\alpha} \psi \right) \sin \phi / \left(1 + \frac{z_1}{R_0} \right) \end{aligned} \quad \dots\dots\dots (12)$$

where, ϕ is the angle which the center line of the thickness makes with the y -axis. The coordinates may be written as $y = y_1 - \zeta \sin \phi$, $z = z_1 + \zeta \cos \phi$.

Next, the transverse strain due to the displacement within the cross section is represented. The cross section of thin wall is properly divided into the elements and the nodal points binding them are defined. For the purpose of simplification, the element is considered to be a plate. By

denoting the displacement of the nodes at both ends of the element with $\{v_i, w_i, \theta_i\}_{i=1,2}$, the displacement at the middle plane of the element are represented

$$\begin{pmatrix} V_i \\ W_i \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

as follows:

$$\left. \begin{aligned} V(\xi) &= V_1(1-\xi) + V_2\xi \\ W(\xi) &= W_1(1-3\xi^2+2\xi^3) + W_2(3\xi^2-2\xi^3) \\ &\quad + \theta_1 l \xi(1-\xi)^2 + \theta_2 l(\xi^3-\xi^2) \end{aligned} \right\} \dots\dots\dots (13)$$

where, l is the nodal distance $\xi = s/l$, $\sin \phi = (z_2 - z_1)/l$, $\cos \phi = (y_2 - y_1)/l$. If the normal strain is represented by $\epsilon_{ss} = \epsilon_s + \xi \kappa_s$ as the sum of that at middle plane and the linear variation in the direction of thickness, ϵ_s and κ_s are written as

$$\epsilon_s = \frac{\partial V}{\partial s} + \frac{1}{2} \left(\frac{\partial W}{\partial s} \right)^2, \quad \kappa_s = -\frac{\partial^2 W}{\partial s^2} + \frac{d}{ds} \left(\frac{\partial V}{\partial s} \frac{\partial W}{\partial s} \right)$$

Therefore,

$$\left. \begin{aligned} \epsilon_s &= \frac{\partial V}{\partial s} = \frac{V_2 - V_1}{l}, \quad \kappa_s = -\frac{\partial^2 W}{\partial s^2} = -\frac{1}{l} \left\{ \theta_2 - \theta_1 - 3(1-2\xi)(\theta_2 + \theta_1 - 2\frac{W_2 - W_1}{l}) \right\} \\ \dot{\epsilon}_s &= \frac{\partial \dot{W}}{\partial s} \frac{\partial W}{\partial s} = \left\{ \theta_1(1-\xi) + \theta_2\xi - 3\xi(1-\xi)(\theta_2 + \theta_1 - 2\frac{W_2 - W_1}{l}) \right\} \\ &\quad \times \left\{ \theta_1(1-\xi) + \theta_2\xi - 3\xi(1-\xi)(\theta_2 + \theta_1 - 2\frac{W_2 - W_1}{l}) \right\} \\ \dot{\kappa}_s &= \frac{\partial \dot{V}}{\partial s} \frac{\partial^2 W}{\partial s^2} + \frac{\partial V}{\partial s} \frac{\partial^2 \dot{W}}{\partial s^2} \\ &= \frac{1}{l^2} \left[(V_2 - V_1) \left\{ \theta_2 - \theta_1 - 3(1-2\xi)(\theta_2 + \theta_1 - 2\frac{W_2 - W_1}{l}) \right\} \right. \\ &\quad \left. + (\dot{V}_2 - \dot{V}_1) \left\{ \theta_2 - \theta_1 - 3(1-2\xi)(\theta_2 + \theta_1 - 2\frac{W_2 - W_1}{l}) \right\} \right] \end{aligned} \right\} \dots\dots\dots (14)$$

2.3 Stiffness matrix of member with thin walled cross section

The stiffness matrix is obtained from the equation (7), but in its calculation, each element of the stiffness matrix includes an integration with respect to $1+z_1/R_0$. In the case of this study, we obtained the stiffness matrix in a convenient form for practical use by making approximate integration.

Because of the thin plate element, the second term of the right side of the equation(2) become

$$\int \bar{\epsilon}^T \sigma dV = \int \left\{ \begin{pmatrix} \bar{\epsilon}_s \\ \bar{\kappa}_s \end{pmatrix}^T \begin{pmatrix} T_s \\ M_s \end{pmatrix} \right\} \times (1+z_1/R_0) R_0 l d\xi d\varphi_0 \dots\dots\dots (15)$$

Furthermore, using the mean value $1+z_m/R_0$ which means disregarding the change of $1+z_1/R_0$ within the element, the virtual deformations of the member and internal forces corresponding to it are expressed as

$$\left. \begin{aligned} \bar{s}_1 &= \bar{\epsilon}_s \left(1 + \frac{z_m}{R_0} \right), & \dot{S}_1 &= \dot{T}_s l \\ \bar{s}_2 &= \bar{\epsilon}_s l, & \dot{S}_2 &= \dot{T}_s \left(1 + \frac{z_m}{R_0} \right) \\ \bar{s}_3 &= \bar{\kappa}_s \left(1 + \frac{z_m}{R_0} \right), & \dot{S}_3 &= \dot{M}_s l \\ \bar{s}_4 &+ 3(1-2\xi)\bar{s}_3 = \bar{\kappa}_s l \\ & & \dot{S}_4 &+ (1-2\xi)\dot{S}_3 = \dot{M}_s \left(1 + \frac{z_m}{R_0} \right) \end{aligned} \right\} \dots\dots\dots (16)$$

Then, the equation(15) becomes

$$\int \bar{\epsilon}^T \sigma dV \approx \sum_i \bar{s}_i \dot{S}_i R_0 d\varphi_0 \dots\dots\dots (17)$$

Applying Hooke's law, we find

$$\begin{pmatrix} \dot{T}_s \\ T_s \end{pmatrix} = C \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \begin{pmatrix} \dot{\epsilon}_s \\ \epsilon_s \end{pmatrix}, \quad \begin{pmatrix} \dot{M}_s \\ M_s \end{pmatrix} = D \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \begin{pmatrix} \dot{\kappa}_s \\ \kappa_s \end{pmatrix} \dots\dots\dots (18)$$

where $C \equiv Et/(1-\nu^2)$, $D \equiv Et^3/12(1-\nu^2)$ and E is Young's modulus, ν is Poisson's ratio and t is the thickness of plate.

From equation (16) and (18) we obtain

$$\begin{pmatrix} \dot{S}_1 \\ S_2 \end{pmatrix} = C \begin{pmatrix} l' & \nu \\ \nu & 1/l' \end{pmatrix} \begin{pmatrix} \dot{s}_1 \\ s_2 \end{pmatrix} = k_c \begin{pmatrix} \dot{s}_1 \\ s_2 \end{pmatrix} \dots\dots\dots (19)$$

$$\begin{pmatrix} \dot{S}_3 \\ S_4 \end{pmatrix} = D \begin{pmatrix} l' & \nu & 0 \\ \nu & 1/l' & 0 \\ 0 & 0 & 3/l' \end{pmatrix} \begin{pmatrix} \dot{s}_3 \\ s_4 \\ s_5 \end{pmatrix} = k_D \begin{pmatrix} \dot{s}_3 \\ s_4 \\ s_5 \end{pmatrix} \dots\dots\dots (20)$$

where $l' = l/(1+z_m/R_0)$

The deformation of members s_i is represented from the definitions as follows;

$$\left. \begin{aligned} s_1 &= \epsilon_0 + \dot{w}_m/R_0 + z_m \dot{\phi} + y_m \dot{\alpha}/R_0 \\ s_2 &= \cos \phi (\dot{v}_2 - \dot{v}_1) + \sin \phi (\dot{w}_2 - \dot{w}_1) \\ s_3 &= \cos \phi \dot{\phi} - \sin \phi \dot{\alpha}/R_0 \\ s_4 &= \dot{\theta}_1 - \dot{\theta}_2 \\ s_5 &= \dot{\theta}_1 + \dot{\theta}_2 + \frac{2 \sin \phi}{l} (\dot{v}_2 - \dot{v}_1) \\ &\quad - \frac{2 \cos \phi}{l} (\dot{w}_2 - \dot{w}_1) \end{aligned} \right\} \dots\dots\dots (21)$$

The nodal displacement of element is expressed as follow:

$$r_e = \{v_1, w_1, \theta_1, v_2, w_2, \theta_2, \epsilon_0, \phi, \alpha\}$$

Then the deformation of member-nodal displacement relations can be written

$$\dot{s} = a \dot{r}_e = \begin{pmatrix} a_c \\ a_o \end{pmatrix} \dot{r}_e \dots\dots\dots (22)$$

$$\begin{pmatrix} a_c \\ \vdots \\ a_D \end{pmatrix} = \begin{pmatrix} 0 & 1/2R_0 & 0 & 0 & 1/2R_0 & 0 & 1 & z_m & y_m/R_0 \\ -\cos \phi & -\sin \phi & 0 & \cos \phi & \sin \phi & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos \phi & -\sin \phi/R_0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 \sin \phi/l & 2 \cos \phi/l & 1 & 2 \sin \phi/l & -2 \cos \phi/l & 1 & 0 & 0 & 0 \end{pmatrix} \quad (23)$$

Therefore, the elastic stiffness matrix is gained as follows:

$$\bar{s}^T \dot{S} = \begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \\ \bar{s}_4 \\ \bar{s}_5 \end{pmatrix}^T \begin{pmatrix} \dot{S}_1 \\ \dot{S}_2 \\ \dot{S}_3 \\ \dot{S}_4 \\ \dot{S}_5 \end{pmatrix} + \begin{pmatrix} \bar{s}_3 \\ \bar{s}_4 \\ \bar{s}_5 \end{pmatrix}^T \begin{pmatrix} \dot{S}_3 \\ \dot{S}_4 \\ \dot{S}_5 \end{pmatrix} = \bar{r}^T (a_c^T k_c a_c + a_D^T k_D a_D) \dot{r} \quad (24)$$

$$k_e = a_c^T k_c a_c + a_D^T k_D a_D \quad (25)$$

The geometrical stiffness matrix can be arranged in the same way. Namely, considering

$$\int \bar{\epsilon}^T \sigma dV = \int (\bar{\epsilon}_t T_t + \bar{\epsilon}_s T_s + \bar{\epsilon}_i M_i + \bar{\epsilon}_s M_s) (1 + z_i/R_0) R_0 l d\xi d\varphi_0 \quad (26)$$

Then we have

$$\int \bar{\epsilon} \sigma dV = (S_1 \bar{s}_1 + S_2 \bar{s}_2 + S_3 \bar{s}_3 + S_4 \bar{s}_4 + S_5 \bar{s}_5) R_0 d\varphi_0 = \sum S_i \bar{s}_i R_0 d\varphi_0 \quad (27)$$

where $\bar{s}_i$ is shown as follows:

$$\left. \begin{aligned} \bar{s}_1 &= \bar{\epsilon}_t (1 + z_m/R_0) = (\dot{w}_m + y_m \dot{\alpha}) \bar{\psi} + (\dot{w}_m + y_m \bar{\alpha}) \bar{\psi} \\ \bar{s}_2 &= \int \frac{\partial \bar{W}}{\partial \xi} \frac{\partial \bar{W}}{\partial \xi} \frac{d\xi}{l} \approx \frac{1}{l} (\bar{W}_2 - \bar{W}_1) (\bar{W}_2 - \bar{W}_1) \\ &= \frac{1}{l} \{-\sin \phi (\bar{v}_2 - \bar{v}_1) + \cos \phi (\bar{w}_2 - \bar{w}_1)\} \{-\sin \phi (\bar{v}_2 - \bar{v}_1) + \cos \phi (\bar{w}_2 - \bar{w}_1)\} \\ \bar{s}_3 &= \bar{\epsilon}_i (1 + z_m/R_0) = -\sin \phi (\bar{\alpha} \bar{\psi} + \bar{\alpha} \bar{\psi}) \\ \bar{s}_4 &= -\frac{1}{l} \{(\bar{V}_2 - \bar{V}_1)(\bar{\theta}_1 - \bar{\theta}_2) + (\bar{V}_2 - \bar{V}_1)(\bar{\theta}_1 - \bar{\theta}_2)\} = -\frac{1}{l} (\bar{s}_2 \bar{s}_4 + \bar{s}_2 \bar{s}_4) \\ \bar{s}_5 &= -\frac{1}{l} \{(\bar{V}_2 - \bar{V}_1) \left[\bar{\theta}_1 + \bar{\theta}_2 - \frac{2}{l} (\bar{W}_2 - \bar{W}_1) \right] + (\bar{V}_2 - \bar{V}_1) \left[\bar{\theta}_1 + \bar{\theta}_2 - \frac{2}{l} (\bar{W}_2 - \bar{W}_1) \right]\} \\ &= -\frac{1}{l} (\bar{s}_2 \bar{s}_5 + \bar{s}_2 \bar{s}_5) \end{aligned} \right\} \quad (28)$$

From these, $\sum S_i \bar{s}_i$ can be put together with $\bar{r}^T \sum S_i B_i \dot{r}$ and the geometrical stiffness matrix of the element is obtained.

2.4 Method of calculation

By employing the elastic stiffness matrix and the geometrical stiffness matrix of the element which are obtained in the foregoing paragraph and summing them up for the whole structure, the global stiffness is gained as below

$$R' - \sum a^T S = K \dot{r} \quad (29)$$

where, $\dot{r}$ consists of the nodal displacements $(\dot{v}_i, \dot{w}_i, \dot{\theta}_i)$ and $(\dot{\epsilon}_0, \dot{\psi}, \dot{\alpha})$. Three out of them that are supposed to be appropriate may be determined arbitrarily in order to make the rigid body displacements $(\dot{v}_0, \dot{w}_0, \dot{\alpha})$ independent. (Of course, in some cases other conditions of constraint such as symmetrical deformation etc. might be added.) The effect of the way of choice of these three is usually not large, but strictly speaking, it is desirable to take a more rational method. Suppose the solution is gained by dropping three degrees of freedom appropriately chosen. Then if a certain rigid body displacement $(\dot{v}_0', \dot{w}_0', \dot{\alpha}')$ is to be deducted from the displacement $\dot{v}, \dot{w}$ obtained so that the equation (30) is made minimum,

$$I = \int (\dot{v} - \dot{v}_0' + z \dot{\alpha}')^2 ds + \int (\dot{w} - \dot{w}_0' - y \dot{\alpha}')^2 ds \quad (30)$$

$(\dot{v}_0', \dot{w}_0', \dot{\alpha}')$ can be readily calculated from the equation below,

$$\frac{\partial I}{\partial \dot{v}_0'} = \int \dot{v} ds - \dot{v}_0' \int ds + \alpha' \int z ds = 0$$

$$\frac{\partial I}{\partial \dot{w}_0'} = \int \dot{w} ds - \dot{w}_0' \int ds - \alpha' \int y ds = 0$$

$$\frac{\partial I}{\partial \dot{\alpha}'} = \int (\dot{v} z - \dot{w} y) ds - \dot{v}_0' \int z ds$$

$$+ \dot{w}_0' \int y ds + \alpha' \int (y^2 + z^2) ds = 0$$

.....(31)

and the rigid body displacement $(\dot{v}_0 + \dot{v}_0', \dot{w}_0 + \dot{w}_0', \dot{\alpha} + \dot{\alpha}')$ is determined. The curvature becomes then

$$\frac{1}{R_1} = \frac{1}{R_0 + w_0} \approx \frac{1}{R_0} (1 - \epsilon_0) + \psi \quad (32)$$

and it must be noted that we should proceed to the next step with the nodal coordinates of

$$y' = y + \dot{v} - z \dot{\alpha} \quad z' = z + \dot{w} + y \dot{\alpha} \quad (33)$$

As for the components of the load vector R' , only the terms corresponding to $\dot{\psi}, \dot{\alpha}$, are not zero. When equilibrium holds under the external force as illustrated in Fig.3, considering the virtual work by external forces, we have

$$(P' \dot{\epsilon}_0 + M' \dot{\psi} + N' \dot{\alpha}/R_0) R_0 d\varphi_0 \quad (34)$$

and R' is formed by using the axial force

$P'=0$, the bending moment $M' = M + \dot{M}$ around the y-axis and $N' = N + \dot{N}$ z-axis. In solving the stiffness equation two out of ϕ , $\dot{\alpha}$, $\dot{M}$ and $\dot{N}$ are given and the rest are to be calculated. 1) Finding M' and N' for given ϕ and $\dot{\alpha}$ and 2) finding M' and $\dot{\alpha}$ for given ϕ and $\dot{N}=0$ would be the most practical problems.

Fig.3 System of external forces

3. Examples of Numerical Calculation

At first, analysis with regard to the straight beam consisting of a circular tube is performed for the purpose of making comparison with the results by other methods of analysis, and then the cases of curved beams with various forms of cross section are analyzed. The numerical values used in the calculation are $Et/(1-\nu^2)=10^3$ and $Et^3/12(1-\nu^2)=10^2$.

Fig.4 Relationship between bending moment and curvature of straight beam with circular cross section

Table 1. Curvature and bending moment in straight beam with circular cross section

	$(a^2/tR)_{max}$	$(M/E\alpha t^2)_{max}$
Chwalla	0.806	0.378
Brazier	0.494	0.329
Reissner	0.435	0.303
present method	0.504	0.320

3.1 Circular tube

It is assumed that the initial shape is a straight beam by putting $1/R_0=0$. The number of elements taken is 36. In Fig.4 the relation between the bending moment M and the curvature $1/R$ is indicated in terms of $E\alpha t^2$ and a^2/t respectively with the quantities made dimensionless. Besides, the results of analyses by Chwalla, Brazier and Reissner are included in the Figure. In these figures, the analyses by Brazier, Reissner and the present method have almost the same trend, while that of Chwalla shows a quite different trend. In Table 1, the representative values by each method are compared. Figure 5 shows the relation between the bending moment and the curvature when a curved beam with the initial curvature of $1/R_0=0.01$ is subjected to the bending within the plane of curvature. Figures 6 and 7 present the deformation of the cross sections and the stress distributions in the longitudinal and circumferential directions at the points 1, 2 and 3 as shown in Fig.5. The conditions of the cross sections being flattened and the behavior of the changing sign stress as deformation takes place are illustrated. Figures 8-10 show the results of analysis for a curved beam with the initial curvature $1/R_0=-0.01$. What is interesting with them is that in the process of deformation of the cross section shown in Fig.9, flattening grows within the plane in the beginning and the

Fig.5 Relationship between bending moment and curvature of circular tube with initial curvature $1/R_0=0.01$

Fig.6 Deformation of the cross section of circular tube with initial curvature $1/R_0=0.01$

trend of flattening changes toward outside of the plane in the vicinity of the curvature $1/R=0$. From the stress distribution in Fig.10, it is known that the circumferential stress is dominated by that due to bending

Fig.7 Variation of stress in longitudinal and circumferential directions of circular tube with initial curvature $1/R_0=0.01$

Fig.8 Relationship between bending moment and curvature of circular tube with initial curvature $1/R_0=-0.01$

Fig.9 Deformation of the cross section of circular tube with initial curvature $1/R_0=-0.01$

of the wall, while the membrane stress is dominant in the longitudinal stress which is very close to the stress distributions in the straight beam. Figure 11 shows the relation between the ultimate bending moment and the initial curvature. A curved beam having a positive initial curvature collapses at smaller bending moment than that of a straight beam, while a curved beam with a negative initial curvature collapses at larger bending moment than that of a straight beam.

Fig.10 Variation of stress in longitudinal and circumferential directions of circular tube with initial curvature $1/R_0=-0.01$

Fig.11 Relationship between ultimate bending moment and initial curvature

3.2 Square tube

A numerical calculation is performed with respect to the cross sectional shape as shown in Fig.12. The number of elements is 40. In Fig.12, the relation between the bending moment and the curvature $1/R$ for

Fig.12 Relationship between bending moment and curvature of square tube with initial curvature $1/R_c = 0.01$

Fig.13 Deformation of the cross section of square tube with initial curvature $1/R_c = 0.01$

Fig.14 Variation of stress in longitudinal and circumferential directions of square tube with initial curvature $1/R_c = 0.01$

a square tube with the initial curvature 0.01 is presented in terms of the quantities EIt/a^3 and a^2/t that are made dimensionless. Figures 13-14 show the conditions of deformation of the cross section and the stress distribution. The longitudinal stress is dominated by the membrane stress and involves very little effect from bending in the beginning. The effect of bending gradually increases as deformation grows. The maximum stress generates at the corners. Then, the numerical calculation for a curved beam with the cross section shown in Fig.15 is performed with the initial curvature is 0.01. Figure 15 shows

Fig.15 Asymmetrical bending of square tube with initial curvature $1/R_c = 0.01$

Fig.16 Deformation of the cross section of asymmetrical bending of square tube with initial curvature $1/R_c = 0.01$

Fig.17 Relationship between bending moment and curvature of channel section with initial curvature $1/R_c = 0.01$

the result of calculation for the moments M and N around the y -axis and the z -axis when an increment of curvature within the z -axis plane is given. Figure 16 presents the condition of deformation of the cross section.

3.3 Asymmetrical cross section

Taking a curved channel section and a curved z section shown in Figs.17 and 20, a numerical calculation is performed. The initial curvatures $1/R_0 = 0.01$ and $R_0/a = 10$ are taken respectively and the number of elements is 20. Figures 17-19 show the results of calculation for the channel section. From the stress distribution diagram, it is known that the membrane stress is dominant in the longitudinal stress and the influence of bending is very small, while the circumferential stress is dominant due to bending. The result for the z section is shown in Figs.20 and 21.

4. Conclusions

1) For the purpose of performing the analysis of the finite bending applied to the curved beam with an arbitrarily cross section, the authors derived an incremental stiffness matrix in a convenient form for

Fig.18 Deformation of the cross section of channel section with initial curvature $1/R_0 = 0.01$

Fig.19 Variation of stress in longitudinal and circumferential directions of channel section with initial curvature $1/R_0 = 0.01$

Fig.20 Relationship between bending moment and curvature of z -section with initial curvature $1/R_0 = 0.01$

Fig.21 Deformation of the cross section of z -section with initial curvature $1/R_0 = 0.01$

application and made it possible to analyze the problems with less labor for calculation and with sufficient accuracy for practical use.

2) The finite bending of a beam with an asymmetrical cross section and the finite bending of a symmetrical beam subjected to the bending moment around an axis oblique to the principal axes were made clear in this study for the first time.

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