An Analysis of Two-dimensional Elastic-plastic
Stress Problem Using the Quadratic Programming*
(The case of Adopting Prandtl-Reuss Equation
and von Mises Yield Function)

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G.Maier has investigated problems in which normality conditions with regard to plastic strain increments pertaining to the yield surface in stress space are not satisfied and work-softening effects are considered by the aid of quadratic programming concepts.

Authors applied this method for solving the elastic-plastic plane stress problem as a quadratic optimization in terms of displacement rates and plastic multiplier rates. The matrices are formulated in explicit forms for the problem of a perforated strip under tension.

A comparison is made between the results obtained by the present method and by the ordinary matrix method. It is shown that the elastic-plastic problem is fairly tractable to the present method.

Key Words: Structural Analysis, FEM, Elastic-plastic Stress Problem, Plane Stress, Quadratic Programming

1. Introduction

The advent of the finite element method made it easy to solve the elastic-plastic problem which had been generally difficult to deal with. Regarding the finite element method, Yamada⁽¹⁾ introduced a constitutive plastic stress and strain matrix method of the incremental strain theory for the materially nonlinear problem, and Marcal and King⁽²⁾ have solved the elastic-plastic stress problem by a variable stiffness approach. These methods have been applied to various actual engineering problems.

Maier⁽³⁾⁽⁴⁾ has investigated a general elastic-plastic stress problem by using the mathematical programming theory for the case, in which the normality conditions with regard to plastic strain increment pertaining to the yield surface are not satisfied and the work-softening effects are considered. In one of the applications of this method, Kaneko⁽⁵⁾ has studied a reinforced concrete frame problem using a new numerical algorithm of linear complementarity method.

In the present paper, the method of elastic-plastic stress analysis by means of quadratic programming formulation based on Maier's method is discussed and applied to a numerical example. The explicit forms of the elastic-plastic constitutive laws are treated for the

purpose of applying them to various materials which exhibit complicated yielding behaviors.

The elastic-plastic stress problem can be reduced to an optimization problem and be solved by using the quadratic programming algorithm. The unknown parameters in the problem are the nodal displacement velocities and plastic multiplier rates. The matrix equations derived here as a first step are restricted to those for the case where only the plane stress state is dealt with and the Prandtl-Reuss equations and the von Mises yield function are adopted.

As a typical numerical example, an elastic-plastic stress problem of a perforated strip under tension is solved, and the results obtained are compared with the results by Yamada's matrix method (this method will be called the ordinary method in the following), attention being focused especially on the stress distributions in the minimum cross section. Also, the progressive forms of the boundary of plastic regions are compared with those by Marcal and King.

2. Theory

2.1 Elastic-plastic constitutive laws

Let us consider a finite element under the condition of constant strain and stress field.

The strain velocity $\dot{\epsilon}$ is assumed to be the sum of the elastic strain velocity $\dot{\epsilon}^e$ and the plastic strain velocity $\dot{\epsilon}^p$

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p \quad \dots\dots\dots (1)$$

where $\dot{\epsilon}$ is written as

$$\dot{\epsilon}^T = [\dot{\epsilon}_x \dot{\epsilon}_y \dots \dot{\gamma}_{xy} \dot{\gamma}_{yz}] \quad \dots\dots\dots (2)$$

* Received 31st May, 1984

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in which the superscript T denotes the transposition of matrix.

ϵ^e is expressed, using the elastic stress strain matrix S and the stress velocity $\dot{\sigma}$, as

$$\epsilon^e = S^{-1} \dot{\sigma} \quad (3)$$

where the superscript -1 means the inverse of matrix, and the stress velocity $\dot{\sigma}$ is given by

$$\dot{\sigma}^T = [\dot{\sigma}_x \dot{\sigma}_y \dots \dot{\tau}_{xx} \dot{\tau}_{xy}] \quad (4)$$

Plastic strain velocity ϵ^p is assumed to be proportional to V , the gradient of the plastic potential ϕ on the yield surface, such that

$$\epsilon^p = V \dot{\lambda} \quad (5)$$

where

$$V^T = \left\{ \frac{\partial \phi}{\partial \sigma} \right\}^T = \left[\frac{\partial \phi}{\partial \sigma_x} \frac{\partial \phi}{\partial \sigma_y} \dots \frac{\partial \phi}{\partial \tau_{xx}} \frac{\partial \phi}{\partial \tau_{xy}} \right] \quad (6)$$

The coefficient $\dot{\lambda}$ is called the plastic multiplier rates.

The strain velocity $\dot{\epsilon}$ is given by the nodal displacement velocity $\dot{u}$ as

$$\dot{\epsilon} = B \dot{u} \quad (7)$$

where B is a matrix relating the strains to the nodal displacements.

Substituting Eqs.(3), (5) and (7) into Eq.(1), and solving for $\dot{\sigma}$, we obtain

$$\dot{\sigma} = SB \dot{u} - SV \dot{\lambda} \quad (8)$$

The yield function is assumed to be a function of stresses and plastic strains as $\phi(\sigma, \epsilon^p)$, and its velocity $\dot{\phi}$ to be given by

$$\dot{\phi} = N^T \dot{\sigma} - H \dot{\lambda} \quad (9.a)$$

and

$$\dot{\lambda} \geq 0, \quad \dot{\phi} \leq 0, \quad \dot{\phi} \dot{\lambda} = 0 \quad (9.b)$$

where N is the gradient of the yield function and H is called the interaction work-hardening matrix, and N and H are given as

$$N^T = \left\{ \frac{\partial \phi}{\partial \sigma} \right\}^T = \left[\frac{\partial \phi}{\partial \sigma_x} \frac{\partial \phi}{\partial \sigma_y} \dots \frac{\partial \phi}{\partial \tau_{xx}} \frac{\partial \phi}{\partial \tau_{xy}} \right] \quad (10)$$

$$H = - \left\{ \frac{\partial \phi}{\partial \epsilon^p} \right\}^T \left\{ \frac{\partial \phi}{\partial \sigma} \right\} \quad (11)$$

The equilibrium equation which was presented by Maier is written as

$$B^T \dot{\sigma} f = \dot{F} \quad (12)$$

where f denotes the volume of an element and the equivalent nodal force velocity is written as $\dot{F}$.

From Eqs.(8), (9.a), (9.b) and (12), the following equation and inequalities are obtained

$$\left. \begin{aligned} B^T S B \dot{u} - B^T S V \dot{\lambda} &= \dot{F} \\ N^T S B \dot{u} - (N^T S V + H) \dot{\lambda} &\leq 0 \end{aligned} \right\} \quad (13.a)$$

and

$$\dot{\lambda} \geq 0 \quad (13.b)$$

Unknown parameters in these equations are $\dot{u}$ and $\dot{\lambda}$. The solution of Eqs.(13.a) and (13.b) may be obtained by solving the following minimization problem.⁽¹³⁾⁽⁴⁾

Minimize the objective function,

$$\left. \begin{aligned} Q(\dot{u}, \dot{\lambda}) &= \frac{1}{2} \dot{\lambda} H \dot{\lambda} f + \\ &\frac{1}{2} (\dot{u}^T B^T - \dot{\lambda} N^T) S (B \dot{u} - V \dot{\lambda}) f - \dot{F}^T \dot{u} \end{aligned} \right\} \quad (14.a)$$

under the subsidiary condition

$$\dot{\lambda} \geq 0 \quad (14.b)$$

2.2 Equations for the plane stress state

The explicit forms of each matrix are presented for a triangular element under the condition of the plane stress state. Displacement components are assumed to be a linear function of x and y as follows,

$$\left. \begin{aligned} u &= a_1 + a_2 x + a_3 y \\ v &= a_4 + a_5 x + a_6 y \end{aligned} \right\} \quad (15)$$

It is assumed that the material obeys the flow rule of the Prandtl-Reuss equations, and von Mises yield function and the isotropic work-hardening are adopted. Then the yield function ϕ becomes identical to the plastic potential ϕ , and ϕ is written as

$$\phi^2 = (\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3 \tau_{xy}^2) = \bar{\sigma}^2 \quad (16)$$

where $\bar{\sigma}$ denotes the equivalent stress.

V and N in Eqs.(6) and (10) are obtained as

$$V = N = \frac{3}{2 \bar{\sigma}} \begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ 2 \tau_{xy} \end{Bmatrix} \quad (17)$$

where the stress components with primes indicate deviatoric stresses. The matrix B of Eq.(7) relating the strains to the nodal displacements is given by

$$B = \frac{1}{2 \Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \quad (18)$$

where

$$\left. \begin{aligned} b_i &= y_j - y_k, \quad b_j = y_k - y_i \\ b_k &= y_i - y_j, \quad c_i = x_k - x_j \\ c_j &= x_i - x_k, \quad c_k = x_j - x_i \\ 2 \Delta &= c_k b_j - c_j b_k, \quad f = t \Delta \end{aligned} \right\} \quad (19)$$

The subscripts i , j and k indicate vertices of the triangular finite element, and x_i and y_i denote Cartesian coordinates of the node i . The thickness and the area of the element are denoted by t and Δ respectively.

The elastic stress strain matrix S is given by using Young's modulus E and Poisson's ratio ν as follows

$$S = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \quad (20)$$

The interaction work-hardening matrix H is written using the work-hardening coefficient H' and the stress components as

$$H = -\frac{H'}{\sigma} \sigma^T N \quad (21)$$

where

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (22)$$

3. Numerical Method and an Example

As already stated in the previous sections, the elasto-plastic stress problem is reduced to an optimization problem based on Eqs. (14.a) and (14.b). In order to solve this problem, a new objective function may be formulated with the Lagrange multiplier, and the optimization problem is now to find out the saddle point of the new objective function. For the numerical calculation, the subroutine of the quadratic programming by Wolfe with upper-bounded variables is used.⁽⁷⁾ The unknown nodal displacements are unbounded variables ($-\infty < u < \infty$), whereas the plastic multiplier rates are assumed to be nonnegative variables ($0 \leq \lambda < \infty$).

Calculating the above mentioned problem which is concerned with Eq. (14.a) using Eqs. (17) to (22), table of matrices $B^T S B f$, $B^T S V f$, $N^T S B f$ and $(N^T S N + H) f$ are prepared, and the optimal solutions are searched by using a linear programming algorithm. Hence the peculiarity of this method lies in its direct minimization of the objective function, while the solution of the simultaneous equation is used for the ordinary method.

A step-by-step loading procedure is adopted and the incremental load in each step is calculated by the so-called $\gamma_{\min}$ method to find the multiplier ratio for the assumed load velocity of the element in which the equivalent stress level is nearest to the yielding surface.

As a numerical example, a perforated tensile strip with a central hole is analyzed by the present method. Stress distributions are compared with those obtained by the ordinary method. Forms of

Fig.1 Finite element model of perforated strip under tension

plastic enclaves are compared with those obtained by Marcal and King.

Taking account of the symmetrical condition, only the first quadrant of the whole region is considered, of which a finite element model is shown in Fig.1. The dimensions, symbols and element numbers will be referred to in the following section. The problem involves 149 finite elements and 94 nodal points. The material properties used here are as follows: Young's modulus $E=68.65$ GPa, Poisson's ratio $\nu=0.2$, the uni-axial yield stress $\sigma_r=238$ MPa, and the work-hardening coefficient $H'=2.197$ GPa.

4. Results and Discussions

The numerical calculation of the perforated tension strip problem is carried out up to loading stage 51. The seven stages are selected in order to compare the results with those by Marcal and King. The results are exhibited by using the nondimensional parameter $\sigma_{\text{mean}}/\sigma_r$, where σ_{mean} stands for the mean stress in the loading cross section. The distributions of the dimensionless

Fig.2 Stress distributions of σ/σ_r in the minimum cross section by the present method, $\sigma_m = \sigma_{\text{mean}}$

Fig.3 Stress distributions of σ/σ_r in the minimum cross section by the ordinary method, $\sigma_m = \sigma_{\text{mean}}$

Table 1 Plastic multiplier rates λ by two methods at the first three stages

STAGE NUMBER	$\sigma_{\text{mean}}/\sigma_y$	PLASTIC ELEMENT NUMBER	PRESENT METHOD λ (Pa) ¹	ORDINARY FEM METHOD λ (Pa) ¹
2	0.55	1	1.210×10^4	1.262×10^4
3	0.666	1	0.901	0.944
		15	0.850	0.891
4	0.67	1	1.184	1.228
		15	0.914	0.936
		2	0.911	0.936

stresses σ_1/σ_y obtained by the present method for 7 cases are shown in Fig.2, where σ_1 is the principal stress in the y direction. And further, the stress distributions of σ_1/σ_y are given in Fig.3 calculated by the ordinary method. It may be observed that the stress distributions obtained by the two methods are almost similar, especially in the part where σ_1/σ_y is greater than unity. In other words, the work-hardening effects are evaluated almost equally by both methods.

The values of the plastic multiplier rates at the first three stages obtained by the two methods are compared in Table 1. The hypothetical incremental load is assumed to be 0.98 N in both the methods. The stage numbers at three stages after the initial yielding, $\sigma_{\text{mean}}/\sigma_y$, the element numbers of the plastic range, and the values of λ by the two methods are given in this table. The values of λ by the present method are about 3 to 5 percent lower than those by the ordinary one.

The plastic zones at several loading levels are illustrated in Fig.4. In the first quadrant (A), the plastic enclaves calculated by the present method are shown at 7 loading stages. In the second quadrant (B), the results calculated by Marcal and King are shown at 6 stages. The results by the present method show that the initial yielding occurs at the bottom of a circular hole in the minimum cross section at the level of $\sigma_{\text{mean}}/\sigma_y = 0.5$, and that the plastic enclaves expand outwards near the minimum cross section. The plastic core grows in the element 116 on the outer straight boundary of case 5. The nose of the plastic zone reaches the outer boundary of case 7. Finally, it is evident that the expansion of the plastic zones due to the progressive loading by the present method is almost similar to that obtained by Marcal and King.

5. Conclusions

The new approach to the elastic-plastic structural problem proposed by Maier was investigated in the present paper. In this method the elastic-plastic stress problem may be transformed into an optimization problem and solved by the quadratic programming technique with bounded variables.

The equations in this method are concretely formulated in explicit forms in the case of plane stress state by using the incremental strain theory.

Fig.4 Progressive plastic zones in several loading cases of $\sigma_{\text{mean}}/\sigma_y$
(A) Present method (B) Marcal and King

The numerical example of a perforated tension strip was examined by comparing the results obtained by the present method and the ordinary one. The stress distributions of σ_1/σ_y in the minimum cross section calculated by the two methods satisfactorily agreed with each other. The plastic multiplier rates near the first three stages obtained by the present method were slightly lower than those by the ordinary method.

This numerical example requires a slightly longer computing time when this method is applied. The reason may be that the sparseness of matrices was not taken into account in the quadratic programming and the iteration process was needed in the quadratic programming algorithm.

It might be emphasized that the present method is applicable to the elastic-plastic stress problems for various materials which exhibit complicated yielding behaviors, because it is easy to handle different types of yielding functions and plastic potentials by the present method.

The authors would like to acknowledge kind guidance and advice of Prof. K. Hata of the Hokkaido Institute of Technology.

The numerical calculations were executed with FACOM M-140 at the Computer Center of the Hokkaido Institute of Technology and with HITAC M-200 H at Hokkaido University Computing Center.

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