Numerical Analysis of Low-Reynolds Number Flow around a Sphere*

By Teruo KUMAGAI** and Akihide KOBAYASHI***

Numerical solutions to the Navier-Stokes equation are obtained by using a successive-over-relaxation method for a steady incompressible flow past a sphere at the Reynolds numbers below 1.0, where the Reynolds number is related to the sphere diameter. The numerically-obtained streamlines and the numerically-determined sphere drag agree well with the recent experimental streamlines around a sphere and the recent measurements of the sphere drag respectively at the Reynolds numbers below 0.4. The analytical solutions, the present numerical solutions and the recent experimental results lead to the conclusion that the first-order solution to Oseen's equation represents the most simple and suitable flow-model among the analytical solutions for the steady incompressible flow past a sphere at the Reynolds numbers below 0.4.

Key words: Fluid Mechanics, Sphere, Low Reynolds Numbers, Numerical Analysis, Stokes Solution, Oseen Solution, Chester-Breath Solution

1. Introduction

To study the behavior of a suspension or a motion of small particles in multiphase flow is a classical subject in an excellent textbook by Happel & Brenner(1983) (1), and in a good review by Batchelor(1976) (2). This subject is also closely related to a significant problem in the recent flow visualization applying a tracer method to turbulent flows, that is, how accurately a tracer particle follows the actual turbulent flows. In the low-Reynolds number hydrodynamics to consider these problems without considering the effect of Brownian motion and the electric or magnetic effects, the Stokes drag of a small particle and the Stokes approximation have been usually adopted for simplicity(1). However the recent measurements of a sphere drag, first carried out by Maxworthy(1965)(4), make us feel that there is left room to doubt the validity of the Stokes approximation even at very low Reynolds numbers.

The steady incompressible flow past a sphere at the Reynolds numbers below 1.0 is a classical subject in an extensive literature. A large number of reviews have been written on this subject. Therefore here are only summarized the significant developments in Table 1 in the form of a chronological table of authors. The previous investigations may be classified into the following three categories; that is, analytical, numerical and experimental treatments. The analytical works seem to predominate. Stokes', Oseen's and Chester-Breath(1969)(5)'s solutions may be considered the representative solutions. Stokes' and Oseen's solutions are well-known and have been described in detail by Lamb(1932)(6), Froudman & Pearson(1957)(7), Imai(1973)(8), and so on. The solutions obtained by using the procedure of matched-asymptotic-expansions were established by Chester & Breath through the preceding work of Froudman & Pearson or of Van Dyke(1964) (9). In the experimental treatments, Perry(1950)(10) summarized the classical measurements of the sphere drag, first carried out by Allen(1900)(11) and succeeding by many authors before 1950. Perry showed the validity of the Stokes drag at low Reynolds numbers. After the work of Perry the measurements by Maxworthy, Pruppacher & Steinberger(1968)(12) and Kumagai & Fujiwara(1983)(13) showed the validity of the Oseen drag instead of the Stokes drag at very low Reynolds numbers, where the steady flow around a sphere was considered to occur within the so-called Stokes region. In these analytical and experimental works it may be considered the main goal to determine the sphere drag accurately. From the viewpoint of the flow field around a sphere only qualitative discussions seem to have been made(4)(5)(9)(13), and few quantitative discussions seem to have been made.

The numerical analysis of the Navier-Stokes equation for the steady incompressible flow past a sphere was initiated considerably later than the analytical and experimental works to determine the sphere drag, even at low Reynolds numbers. The previous numerical analyses of the flow past a sphere at low Reynolds numbers are summarized in Table 2. The main purposes of the previous numerical analyses may be considered to be a determination of the sphere drag at low and intermediate Reynolds numbers as well as a determination of the Reynolds number at which the flow begins to separate from the
sphere surface. Jenson (1959) made calculations in the form of the stream function and vorticity to obtain the drag coefficients of the sphere drag and the flow field for Re=5, 10, 20, 40 by applying relaxation methods directly to the continuity and Navier-Stokes equations in the form of finite differences. Here, Re is the Reynolds number related to the sphere diameter. Jenson's procedure using relaxation methods has laid the foundation to the succeeding numerical analyses. However, as is well known, relaxation methods are generally suitable to a desk-calculation, but not to a computer-calculation. For instance our preliminary computer-calculations applying directly Jenson's procedure showed that Jenson's solution satisfied the Navier-Stokes equation to an order of 10^{-5}. Numerical solutions for the sphere drag and vorticity distributions were given by LeClair, Hämäläinen & Pruppacher (1970) and so on, using successive-over-relaxation methods. Dennis & Walker (1971) obtained more accurate solutions for the sphere drag, vorticity and stream function over the range of Re=0.1 to 40 by applying a series-truncation method to a semi-analytical formulation, whereby the variables were expanded as a series of Legendre functions.

Our latest measurements of the sphere drag (19) showed that the Oseen formula for the sphere drag was most accurate for theoretically - predicted drags at the Reynolds numbers below 0.4. In addition our application of a new flow-visualization technique to the flow field around a sphere (15) revealed that the streamlines drawn by using the first-order solution to Oseen's equation represented the actual streamlines most accurately in the same range of the Reynolds numbers. However since whether the Stokes approximation is applicable or not has significant effect on the low-Reynolds-number hydrodynamics, we feel that the flow field around a sphere at low Reynolds numbers should be discussed furthermore from the viewpoint of a numerical analysis of the Navier-Stokes equation. For this reason the continuity and Navier-Stokes equations in the form of finite differences are solved numerically by using successive-over-relaxation methods for the steady flow past a sphere at Re=0.01, 0.1, 0.2, 0.4, 0.6, 0.8, 1.0 in the present paper. The numerically-obtained streamlines are compared with the previous visualized streamlines at the Reynolds numbers below 0.4. The analytical solutions, including the second-order solution to Oseen's equation determined in the present paper, are discussed in detail quantitatively.

2. Nomenclature

- $a$: lattice spacing in $z$-direction
- $b$: lattice spacing in $\theta$-direction
- $A_n, B_n$: constants in general solution to Oseen's equation
- $D_s$: hydrodynamic drag of sphere
- $F_s$: Stokes drag
- $\gamma = \xi e^{\sin \theta}$
- $\rho$: pressure
- $R$: radius of sphere
- $Re$: Reynolds number related to sphere diameter
- $R_u$: residuals of continuity equation
- $S_u$: residuals of Navier-Stokes equation

Table 1 Chronological table of authors.

<table>
<thead>
<tr>
<th>Year</th>
<th>General and Analytical</th>
<th>Numerical</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>1800</td>
<td>1823 Navier, C.L.M.H.</td>
<td>1845 Stokoe, G. (H-S eq.)</td>
<td></td>
</tr>
<tr>
<td>1851</td>
<td>Stokes, G.C.</td>
<td>1889 Whitehead, A.N.</td>
<td>1900 Allen, H.S.</td>
</tr>
<tr>
<td>1901</td>
<td></td>
<td></td>
<td>1911 Arnold, H.B.</td>
</tr>
<tr>
<td>1910</td>
<td></td>
<td></td>
<td>1925 Castleman, R.A.</td>
</tr>
<tr>
<td>1911</td>
<td>Rybczynski, W.</td>
<td>1932 Lamb, H. (Textbook)</td>
<td>1959 Jenson, V.O.</td>
</tr>
</tbody>
</table>

Table 2 Variables in previous calculations.

<table>
<thead>
<tr>
<th>Method</th>
<th>$a$</th>
<th>$b$</th>
<th>$n$</th>
<th>$r_u$</th>
<th>$Re$</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jenson</td>
<td>Relaxation</td>
<td>0.2</td>
<td>$\xi/15$</td>
<td>1.8</td>
<td>6</td>
<td>$5, 10, 20, 40$</td>
</tr>
<tr>
<td>Rimon et al</td>
<td>SOR</td>
<td>0.05</td>
<td>$\xi/30$</td>
<td>3.05</td>
<td>21</td>
<td>1$\times1000$</td>
</tr>
<tr>
<td>LeClair et al</td>
<td>SOR</td>
<td>0.025$\times0.05$</td>
<td>$\xi/60$</td>
<td>6.91</td>
<td>1000</td>
<td>0.01, 400</td>
</tr>
<tr>
<td>Dennis et al</td>
<td>Series-truncation</td>
<td>0.0245</td>
<td>—</td>
<td>5.0</td>
<td>148</td>
<td>0.1$\times40$</td>
</tr>
</tbody>
</table>
\[ U: \text{velocity of uniform flow} \]
\[ v: \text{velocity vector} \]
\[ u_r: \text{r-component of velocity} \]
\[ u_\theta: \text{\(\theta\)-component of velocity} \]
\[ (x, y): \text{cylindrical coordinates non-dimensionalized by} \ R \]
\[ z: \text{z at infinity, corresponding to} \ r = \infty \]
\[ \zeta: \text{\(\theta\)-component of vorticity} \]
\[ \xi: \text{vorticity vector} \]
\[ v: \text{kinematic viscosity of fluid} \]
\[ \rho: \text{density of fluid} \]
\[ \psi: \text{stream function for axisymmetric flow} \]
\[ \phi: \text{stream function of Chester-Breach's solution} \]
\[ \phi_n: \text{stream function of basic solution to Oseen's equation} \]
\[ \phi_i: \text{stream function of first-order solution to Oseen's equation} \]
\[ \phi_2: \text{stream function of second-order solution to Oseen's equation} \]
\[ \alpha: \text{acceleration factor to the Navier-Stokes equation} \]
\[ \omega: \text{acceleration factor to the continuity equation} \]

Subscripts and Superscripts

\[ \hat{\cdot}: \text{lattice number in} \ z\text{-direction} \]
\[ \vec{\cdot}: \text{lattice number in} \ \theta\text{-direction} \]
\[ \kappa: \text{k-th approximations in successive calculations} \]

3. Formulation

The continuity and Navier-Stokes equations for the incompressible viscous flow are represented as

\[ \text{div} \ v = 0 \] \hspace{1cm} (1)

and

\[ \frac{\partial v_r}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p + \nu \nabla^2 v \] \hspace{1cm} (2)

respectively. When we introduce the vorticity

\[ \zeta = \text{rot} \ \psi \] \hspace{1cm} (3)

Eq. (2) is represented as

\[ \frac{\partial}{\partial t} \zeta - \text{rot} (\zeta \times \xi) = -\nu \text{rot rot} \ \zeta \] \hspace{1cm} (4)

For an axisymmetrical flow in spherical coordinates the stream function \( \psi \) may be introduced as

\[ u_r = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} \]
\[ u_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} \] \hspace{1cm} (5)

Substitution of Eq. (5) into Eqs. (3) and (4) yields

\[ \zeta = D^2 \psi \sin \theta \] \hspace{1cm} (6)

\[ \frac{\partial^2 \psi}{\partial t^2} + \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial \psi}{\partial r} \right) \right) + 2 \nu \sin \theta \frac{\partial}{\partial \theta} \left( \frac{\partial \psi}{\partial \theta} \right) + \frac{2}{r} \frac{\partial \psi}{\partial r} \] \hspace{1cm} (7)

where

\[ D = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial}{\partial r} \right) \right) \]

\[ D^2 = D^2 \psi \sin \theta \]
\[ \psi = \frac{\partial \psi}{\partial r} \]
\[ \phi = \frac{\partial \phi}{\partial r} \]

For the steady flow around a sphere all quantities are made dimensionless by putting

\[ r = R, \quad \psi = \frac{\psi R}{U}, \quad \zeta = \frac{\zeta R}{U}, \quad R = 2RU \]

For simplicity all quantities are written omitting the superscript ' in the following descriptions. Eqs.(6) and (7) are represented as

\[ D^2 \psi = \zeta r \sin \theta \] \hspace{1cm} (8)

\[ \frac{R}{2r^2 \sin \theta} \left( \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial \theta} \left( 2\psi \cot \theta \right) \right) \]

\[ -\frac{2}{r} \frac{\partial \psi}{\partial r} \] \hspace{1cm} (9)

in the nondimensionalized form. The boundary conditions for a uniform flow past a sphere are represented as

\[ \psi = r^2 \sin \theta / 2 \] \hspace{1cm} (10)

\[ \psi(1, \theta) = 0 \] \hspace{1cm} (11)

Since Jensen's procedure for relaxation methods may be applicable to finite-difference approximations,

\[ \frac{R}{2r^2 \sin \theta} \left( \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial}{\partial \theta} \left( 2\psi \cot \theta \right) \right) \]

\[ -\frac{2}{r} \frac{\partial \psi}{\partial r} \] \hspace{1cm} (12)

is used instead of Eq. (9). Thus the governing equations for \( \psi \) and \( \zeta \) become Eqs.(6) and (12). Since \( \psi \) and \( \zeta \) are both expected to vary most rapidly near the sphere surface, it is convenient to replace the radial coordinate \( r \) by \( z \) through the transformation

\[ r = \exp z \] \hspace{1cm} (13)

In the \( z-\theta \) plane the difference increments will be assigned equal values. This transformation yields the following equations:

\[ e^{2z} D^2 \psi - e^{2z} \sin \theta \psi = 0 \] \hspace{1cm} (14)

\[ R^2 \left( \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) - 2 \psi \frac{\partial}{\partial z} \right) e^{2z} \sin \theta \]

\[ -e^{2z} D^2 g = 0 \] \hspace{1cm} (15)

where

\[ f = \psi / e^{2z} \sin \theta \]
\[ g = \zeta e^{2z} \sin \theta \] \hspace{1cm} (16)

Let \( a \) and \( b \) lattice spacings in the \( z \)- and \( \theta \)-directions respectively. Then a lattice

\[ \psi = 0 \]
\[ \zeta = 0 \]

Fig.1 Field configuration and boundary conditions.
point may be given by
\[ z = ia, \quad (i = 0 - m, \quad m = z_a) \]
\[ \theta = jb, \quad (j = 0 - n, \quad n = \pi) \]
\[ (17) \]
\[ (18) \]
where \( z_a \) represents \( z \) at infinity assumed for the present calculations. The application of central-difference approximations to the derivatives in Eqs. (14) and (15) yields
\[ \psi_{i-1} - \frac{2a}{2a^2} + \psi_{i-1} - \frac{2a}{2a^2} + \psi_{i} - \frac{2a}{2a^2} + \psi_{i+1} - \frac{2a}{2a^2} - \psi_{i} - \frac{2a}{2a^2} - \psi_{i+1} - \frac{2a}{2a^2} - \frac{2b}{2b} \sin ja \]
\[ = R_i \]
\[ (19) \]
\[ g_{i-1} - \frac{2a}{2a^2} + g_{i-1} - \frac{2a}{2a^2} + g_{i} - \frac{2a}{2a^2} + g_{i+1} - \frac{2a}{2a^2} - g_{i} - \frac{2a}{2a^2} - \frac{2b}{2b} \cos ja \]
\[ + \frac{R_i a}{4} \sin ja \left( \frac{\partial^2 \psi}{\partial x^2} \right)_{i} \cdot \frac{f_{i+1} - f_{i-1}}{b} \cdot \frac{\partial \psi}{\partial x} \cdot \frac{f_{i+1} - f_{i-1}}{b} = S_{ij} \]
\[ (20) \]
where
\[ \left[ \frac{\partial^2 \psi}{\partial x^2} \right]_{i} = \psi_{i+1} - \psi_{i-1} \]
\[ \frac{2a}{2a^2} \cdot \frac{\partial \psi}{\partial x} \cdot \frac{2b}{2b} \cdot \frac{\partial^2 \psi}{\partial x^2} \]
\[ \frac{2a}{2a^2} \cdot \frac{2b}{2b} \cdot \frac{\partial^2 \psi}{\partial x^2} \]
\[ (21) \]
where \( R_i \) and \( S_{ij} \) represent residuals at a point \( z = ia, \theta = ja \) for calculations using directly relaxation methods. The boundary conditions in the form of finite differences are represented as
\[ v = 0, \quad \psi = e^{iax} \sin ja \]
\[ (22) \]
from Eqs. (10) and (11). However the boundary conditions for \( \zeta \) can not be defined a priori.

When
\[ \psi = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k \]
is assumed at the neighbourhood of \( z = 0 \) and \( C_k \) is determined for the boundary conditions of \( v \),
\[ \zeta_i = \frac{8\zeta_{i-1} - 4\zeta_{i-2} + \zeta_{i-3}}{2a^2} \]
\[ (23) \]
may be calculated by extrapolating the numerical values of \( \psi \) and \( \psi \) at \( z = ia, \theta = ja \). At \( z = z_a, \theta = \pi \) it is obvious. Since both of \( c \) and \( \zeta \) approach 0 at \( z = 0 \) and \( \theta = \pi \), \( f \) can not be given. But \( f \) may be approximated as
\[ f_{i,0} = \lim_{\theta \to \pi} \frac{\zeta_{i,0}}{e^{iax}} \sin \theta \approx \frac{1}{e^{iax}} \]
at \( \theta = 0 \) and \( \theta = \pi \). Therefore all boundary conditions for \( \psi \) and \( \zeta \) may be represented in the form of finite differences as
\[ \zeta_{i,0} = \frac{8\zeta_{i-1} - 4\zeta_{i-2} + \zeta_{i-3}}{2a^2}, \quad g_{i,0} = \frac{8\zeta_{i-1} - 4\zeta_{i-2} + \zeta_{i-3}}{2a^2} \sin j \theta \]
\[ (24) \]
\[ (25) \]
These conditions are similar to those of Jansen, but in the form of finite differences in the present paper. Fig. 1 illustrates the field configuration and boundary conditions in the \( z - \theta \) plane.

In the successive-over-relaxation method, \((k+1)\)th values for \( \zeta \) and \( \psi \) are represented as
\[ \zeta^{(k+1)} = \zeta^{(k)} + a \cdot b \left( \frac{\zeta^{(k)} - 2 - a}{2a^2} \sin ja + \zeta^{(k)} + 2 - a}{2a^2} \sin ja \right) \]
\[ + c_1^{(k)} \cdot 2 - a \cot ja \cdot e^{ia} \sin(j + 1)b + \zeta^{(k)} + 2 - a \cdot 2a^2 \cdot e^{ia} \sin(j - 1)b \]
\[ + \frac{R_{i,j}}{4a^2} \cdot e^{ia} \sin ja \left( \psi^{(k)} - \psi^{(k)} - 2a^2 + \psi^{(k)} + 2a^2 \cdot 2a^2 \right) \cdot e^{ia} \sin(j - 1)b \]
\[ - (\psi^{(k)} - \psi^{(k)} - 2a^2 + \psi^{(k)} + 2a^2 \cdot 2a^2 \cdot e^{ia} \sin(j - 1)b) \]
\[ (24) \]
\[ \psi^{(k+1)} = \psi^{(k)} + a \cdot b \left( \frac{\psi^{(k)} - 2 - a}{2a^2} \sin ja + \psi^{(k)} + 2 - a}{2a^2} \sin ja \right) \]
\[ + c_1^{(k)} \cdot 2 - a \cot ja \cdot e^{ia} \sin(j + 1)b + \psi^{(k)} + 2 - a \cdot 2a^2 \cdot e^{ia} \sin(j - 1)b \]
\[ + \frac{R_{i,j}}{4a^2} \cdot e^{ia} \sin ja \left( \psi^{(k)} - \psi^{(k)} - 2a^2 + \psi^{(k)} + 2a^2 \cdot 2a^2 \right) \cdot e^{ia} \sin(j - 1)b \]
\[ (25) \]
where \( a \) and \( 2b \) are acceleration factors. The optimal values for \( a \) and \( 2b \) should be determined in the preliminary calculations.
4. Numerical Calculations

Numerical solutions are obtained by using Eqs. (24) and (25) for the boundary conditions given by Eq. (23). The first-order solutions to Oseen's equation are adopted as the initial values for \( \phi \) and \( \xi \), that is, \( \phi_0^0 \) and \( \xi_0^0 \). The variables in the calculations are tabulated in Table 3. IBM 4341 (1.2MIPS) and a double-precision scheme in FORTRAN are used. The solutions are assumed to converge when all of \( |R_d| \), \( |S_d| \), \( |\phi^{n-1} - \phi^n| \) and \( |\xi^{n-1} - \xi^n| \) become less than \( 10^{-4} \) at all lattice points.

The accuracies in the present calculations may be considered to be dependent on the lattice spacings and the distance from the sphere surface to the outer boundary. The truncation errors, which are caused by using finite-difference approximations for first- and second-order derivatives at a lattice spacing \( a \), may be estimated at an order of \( O(a^7) \). If Stokes' solution is used to estimate the truncation errors, 0.42 % and 0.1 % are estimated for the cases of \( a=0.1 \) and \( a=0.05 \) respectively. Since numerical solutions are expected not to deviate so far from Stokes' solution, these truncation errors may be considered sufficiently small. Therefore \( a=0.1 \) and \( a=0.05 \) are used as lattice spacings in the present calculations. \( z_a = 4.6(r_a=100) \) is adopted as the outer boundary, because little difference was recognized between streamlines obtained for the cases of \( z_a = 4.6 \) and those for \( z_a = 6.91(r_a=100) \) at \( Re=0.1 \) in the preliminary calculations.

The effects of acceleration factors \( \omega_a \) and \( \omega_b \) on the convergences of the solutions for 50 iterations are shown in Fig. 2. From this figure the optimum values of \( \omega_a \) and \( \omega_b \) are determined as shown in Table 3.

5. Results and Discussions

In this chapter, the second-order solution to Oseen's equation will be determined first. Then the problem, how accurately Stokes', Oseen's and Chester's solutions satisfy the Navier-Stokes equation, is considered quantitatively. Finally the streamlines and the sphere drag, predicted by these analytical solutions, are compared with those obtained in the present numerical solutions and in the previous results of flow visualization.

5.1 Higher-order solution to Oseen's equation

The application of the Oseen approximation to the Navier-Stokes equations for the axisymmetric flow Eq. (26) yields

\[
\frac{R_c}{2} \left( \cos \theta \frac{\partial \phi}{\partial r} - r \sin \theta \frac{\partial \phi}{\partial \theta} \right) = D^2 \phi = D^2 \psi
\]

where \( \psi \) denotes the stream function of general solution to Oseen's equation. Israel (1) introduced a general solution to Eq. (26) as

\[
\phi = \frac{r^2}{2} \sin \theta + m_s(1 - \cos \theta) + r^2 \sin \theta \left( B_1 + B_2 \frac{\partial}{\partial r} + \cdots \right) \frac{1}{r} + \left( A_0 + A_1 \frac{\partial}{\partial r} + A_2 \frac{\partial^2}{\partial r^2} \right) + \cdots
\]

where the first term of the right-hand side represents a uniform flow, the second term the source with the intensity \( m_s \), the third term a doublet and its derivatives, and the fourth term the Oseenlet and its derivatives respectively. For the most simple case, such as for \( m_s = 0 \), \( B_1 = B_2 = \cdots = 0 \), and \( A_0 = A_1 = A_2 = \cdots = 0 \), the results are easily determined with the boundary conditions Eqs. (10) and (11). This yields the famous solution derived by Oseen. The famous Oseen solution is represented as

\[
\phi = -\frac{1}{4} \left( r^2 + \frac{1}{r} \right) \sin \theta - \frac{3}{R_e} \left( 1 + \cos \theta \right) \left( 1 - \exp \left( -\frac{R_e}{4} \frac{1}{r} (1 - \cos \theta) \right) \right)
\]

\( \phi_b \) is the basic solution to Oseen's equation in the present paper. At the sphere surface \( m_s = 3R_e \sin \theta (1 - \cos \theta) \left[ (1 - \cos \theta) \frac{1}{6} - \cdots \right] / \delta (R_e) \).

Thus it should be noted that \( \phi_b \) does not satisfy the surface condition to first order of \( Re \). This fact leads to a misunderstanding that the Oseen approximation breaks down near the sphere surface. However when \( A_0 \) and \( B_0 \) are determined to the \( n \)-th order for the boundary conditions, the surface conditions can be improved up to \( (n+1) \)-th order of \( Re \). These solutions to Oseen's
These solutions are named the first-, second-, ..., and n-th-order solution to Oseen's equation according to the order described in the above. At the sphere surface, these solutions take the values equivalent to the following order of \( \text{Re} \):

\[
\begin{align*}
[\psi_1]_{r=R} &= O(R_0^0), \\
[\psi_2]_{r=R} &= O(R_0^2), \\
&\quad \ldots, \\
[\psi_n]_{r=R} &= O(R_0^n). \\
\end{align*}
\]

It should be noted that \( \psi_0 \) represents a solution which gives the sphere drag derived by Goldstein(1929)\(^{(s)} \) and revised by Shank(1955)\(^{(s)} \).

5.2 Residuals of the Navier-Stokes equation

Stokes' solution is represented as

\[
\psi_0 = (2r^2 - 3r^3 + 1/r^2) \sin \theta / 4
\]

as is well known. Chester-Breath's solution is obtained by using the procedure of matched-asymptotic-expansions for Stokes' and Oseen's solutions is represented as

\[
\psi = \psi_0 + R_0 \psi_1 + R_0^2 \psi_1 \log(R_0^2/2) + R_0^3 \psi_2 + R_0^4 \psi_2 + R_0^5 \psi_2 \log(R_0^2/2) + \ldots
\]

Fig.3 shows the residuals of the Navier-Stokes equation for these analytical solutions at \( \text{Re}=0.4 \). About the characteristics of Stokes' solution, it has been explained by many authors, for instance by Van Dyke\(^{(s)} \), that the substitution of Stokes' solution into the Navier-Stokes equation yields the ratio of the inertial term to the viscous term represented as

\[
\text{Inertial term / viscous term} = O(\text{Re}^{1/2} / \text{Re}) = O(\text{Re}^{-1/2})
\]

in an order-of-magnitude sense, and that the Stokes approximation breaks down under the condition of \( \text{Re}=O(1) \). However, the direct substitution of Stokes' solution into the Navier-Stokes equation yields the residual represented as

\[
\text{Ss} = -\frac{3}{8} \text{Re} \left( \frac{2r^2 - 3r^3 + 1}{r^2} \right) \sin \theta \cos \theta.
\]

The profile of residuals in Fig.3-(a) shows that the residuals for Stokes' solution take their maximum value at each four points as \((1.91, 54.7^\circ), (1.92, 125.3^\circ), (1.91, 254.7^\circ)\) and \((1.91, 305.3^\circ)\). This means that the Stokes approximation actually breaks down first at these four points. The residuals of the Navier-Stokes equation for Chester-Breath's solution are much smaller than those for Stokes' and Oseen's solutions as shown in Fig.3-(c), but it should be noted that Chester-Breath's solution does not completely satisfy the Navier-Stokes equation. The residuals of the Navier-Stokes equation for n-th order solution to Oseen's equation show good agreements with those for the first-order solution.

5.3 Flow field around a sphere

Fig.4 shows the streamlines of a uniform flow past a sphere. Since the basic solution to Oseen's equation disagrees with the numerical solutions near the sphere and the n-th order solution shows good agreement with the first-order solution to Oseen's equation, the Oseen streamlines are drawn by...
using the first-order solution to Oseen's equation. The streamlines of a uniform flow past a sphere are shown in Fig.4. The streamlines, which are drawn based on the numerical calculations in the present paper, agree quite well with the previously visualized streamlines at the Reynolds numbers below 0.4. In the upstream the Stokes streamlines deviate outwards from the numerical streamlines as they approach the sphere from infinity, while in the downstream the Stokes streamlines agree with the numerical streamlines. The Oseen streamlines agree with the numerical streamlines in the upstream, while in the downstream the Oseen streamlines deviate outwards from the numerical streamlines as they passing over the sphere. These characteristics become more remarkable as Re increases to 0.4. The Chester-Breach streamlines show the characteristics similar to those of the Stokes streamlines. However the deviations of the Chester-Breach streamlines from the numerical streamlines are approximately equivalent to those of the Oseen streamlines. Fig.5 shows these characteristics more clearly in the form of deviations of the analytical stream functions from the numerical stream functions for the case of Re=0.4.

5.4 Sphere drag

The sphere drag is computed numerically by using Jensen's procedure. The sphere drag, non-dimensionalized with the Stokes drag, is plotted in Fig.6 as D/Do vs. Reynolds number related to the sphere diameter. The present numerical results agree quite well with the previous results obtained semi-analytically by Dennis-Walker at the Reynolds

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Fig.4 Streamlines around a sphere:
- numerical and visualized;
- - - - Stokes' solution;
- - - - 1st-order solution to Oseen's equation;
- - - - Chester-Breach's solution.

Fig.5 (|\psi_{\text{num}} - \psi_{\text{analyt.}}|)/\psi_{\text{num.}} \times 10^3 at Re=0.4.
numbers below 1.0. The present results agree well with the previous measurements(11). The Oseen and Chester-Breach drag deviate from the sphere drag, which has been determined numerically and experimentally, only within the measuring error of ±1.2 percent at the Reynolds numbers below 0.4. From Fig.6 the Oseen formula for the sphere drag, which is derived by using the first-order solution to Oseen’s equation, is found to be the most simple and suitable formula of the sphere drag among the formulas predicted analytically at the Reynolds numbers below 0.4.

6. Conclusions

The numerically-obtained streamlines agree well with the visualized streamlines at the Reynolds numbers below 0.4, where the Reynolds number is related to the sphere diameter. These numerical streamlines are well approximated by the Oseen streamlines, which are based on the first-order solution to Oseen’s equation, as well as by the Chester-Breach streamlines at the Reynolds numbers below 0.4. The sphere drag, determined numerically based on the Navier-Stokes equation, agrees well with the previous measurements and is well approximated by the Oseen formula of the sphere drag.

It is concluded that the Oseen approximation represents the actual flow around a sphere at the Reynolds numbers below 0.4 as accurately as Chester-Breach’s solution. For more complicated cases of creeping flows, for instance Brownian motions or diffusions of small particles in fluid, the solutions to Oseen’s equation are likely to be a more simple and suitable model than the other analytical solutions.

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