A Method for Calculation of Stress Intensity Factor

by Finite Difference Method *

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A numerical method of calculation by the finite difference method to solve an elastic stress analysis problem of rectangular domain is presented, and a scheme for treating the neighborhood of a crack tip is proposed. By using this method, the stress intensity factors of rectangular plates and strips with one or two cracks in mode I and mode II are calculated, and these results are compared with results by some other methods presented before to show the validity of this method.

Key Words : Elasticity, Numerical Analysis, Finite Difference Method, Crack, Stress Intensity Factor, J-Integral

1. Introduction

The finite difference method is seldom used at present in the fields of elasticity and plasticity, since it is inadequate for treatment of the irregularly shaped boundary of the region. On the other hand it seems that this method is more suitable to modern high speed computers than other numerical methods for the reason that its theoretical formulation is very simple and only a repetition of many simple calculations is necessary.

In this paper, a method to solve a two-dimensional elasticity problem for rectangular regions by using the finite difference method is established, then a scheme to treat the neighborhood of a crack tip is proposed. By the above-mentioned method and scheme the stress intensity factors in mode I and mode II are calculated for some rectangular plates or strips with one or two cracks. The calculated results are compared with the results obtained by some other methods to show the validity of our method.

2. Fundamental equations

In this study a plane stress problem is considered. As well known, the treatments of the plane strain problem are

quite similar to those of the plane stress problem.

The fundamental equations of the plane stress problem for the orthogonal Cartesian coordinates are expressed with commonly used symbols as follows ⁽¹⁾;

Equilibrium equations ;

$$\left. \begin{aligned} \partial \sigma_x / \partial x + \partial \tau_{xy} / \partial y + X &= 0 \\ \partial \tau_{xy} / \partial x + \partial \sigma_y / \partial y + Y &= 0 \end{aligned} \right\} \dots\dots\dots (1)$$

Stress-strain relations ;

$$\left. \begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} \epsilon_x + \frac{\nu E}{1-\nu^2} \epsilon_y \\ \sigma_y &= \frac{\nu E}{1-\nu^2} \epsilon_x + \frac{E}{1-\nu^2} \epsilon_y \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy} \end{aligned} \right\} \dots\dots\dots (2)$$

Displacement-strain relations ;

$$\left. \begin{aligned} \epsilon_x &= \partial u / \partial x \\ \epsilon_y &= \partial v / \partial y \\ \gamma_{xy} &= \partial v / \partial x + \partial u / \partial y \end{aligned} \right\} \dots\dots\dots (3)$$

In Eq.(1) the terms of the body force, X and Y , are neglected, then from Eq.(1) ~ Eq.(3) the equilibrium equations expressed in terms of displacements are obtained as follows ;

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{1-\nu}{1+\nu} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= 0 \\ \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{1-\nu}{1+\nu} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) &= 0 \end{aligned} \right\} \dots\dots (4)$$

Also from Eqs.(2) and (3) the stress-displacement relations are obtained as follows ;

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$$\left. \begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} \frac{\partial u}{\partial x} + \frac{\nu E}{1-\nu^2} \frac{\partial v}{\partial y}, & \sigma_y &= \frac{\nu E}{1-\nu^2} \frac{\partial u}{\partial x} + \frac{E}{1-\nu^2} \frac{\partial v}{\partial y} \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \right\} \dots\dots\dots (5)$$

Eq.(5) is used in describing the stress boundary conditions and in calculating the stress field from the solved displacement field.

Now the region to be treated is divided by the unequally spaced rectangular lattice and each lattice point is numbered as shown in Fig.1. Then, by the finite difference approximation Eqs.(4) and (5) are transformed approximately into algebraic equations. The descriptions of these algebraic equations are omitted here to save the space, and some typical forms of the finite difference approximation are given as follows ;

$$\left. \begin{aligned} \frac{\partial u(i, j)}{\partial x} &= \frac{(\Delta x_{i-1})^2 \{u(i+1, j) - u(i, j)\} + (\Delta x_i)^2 \{u(i, j) - u(i-1, j)\}}{\Delta x_{i-1} \Delta x_i (\Delta x_{i-1} + \Delta x_i)} \\ \frac{\partial^2 u(i, j)}{\partial x^2} &= \frac{2\Delta x_{i-1} \{u(i+1, j) - u(i, j)\} - 2\Delta x_i \{u(i, j) - u(i-1, j)\}}{\Delta x_{i-1} \Delta x_i (\Delta x_{i-1} + \Delta x_i)} \end{aligned} \right\} \dots\dots\dots (6)$$

Eq.(6) is the "central finite difference approximation", but as described afterwards, the "backward or forward finite difference approximation" is needed for the first order derivative of the variables in order to express the boundary conditions at the corner point of a rectangular region. An example of those difference approximations is given as follows ;

$$\frac{\partial u(i, j)}{\partial x} = \frac{(\Delta x_{i-1})^2 \{u(i-2, j) - u(i, j)\} - (\Delta x_{i-2} + \Delta x_{i-1})^2 \{u(i-1, j) - u(i, j)\}}{\Delta x_{i-2} \Delta x_{i-1} (\Delta x_{i-2} + \Delta x_{i-1})} \dots\dots\dots (7)$$

3. Method of solution

As an example a rectangular plate with a crack at its center, which is subjected to uniform tension, is considered as shown in Fig.2. The symbols in this figure are used in the following. On account of symmetry of the problem, a quarter part covered with inclined fine lines in Fig.2 is treated and it is divided into many small rectangular domains by the lattice as shown in Fig.3. The dotted lines in it indicate the imaginary lattice points.

In this method, an iterative method, the so-called SOR method⁽²⁾ is adopted. For this purpose, the displacement u or v at each lattice point is expressed by v or u at the same point, respectively, together with u and v at the neighboring lattice points. Such equation of u or v is called "u's equation" or "v's equation", respectively. The algebraic equation obtained from the first or second equation of Eq.(4) is used as "u's or v's equation", respectively, for all lattice points inside the considered rectangular domain.

For the lattice points on the boundary of the considered domain, the following treatments are adopted.

- (a) Case of the stress boundary condition (for edges HA, EF and FG in Fig.2)

The imaginary lattice points are arranged outside the edges HA, EF and FG as shown in Fig.3, and the boundary conditions on these edges are expressed by the stress-displacement relations. From these equations of conditions "u's and v's equations" are derived for the imaginary lattice points, thereby, the equation of condition for the stress acting in x -direction or y -direction is used as "u's equation" or "v's equation", respectively. For the displacements at the lattice points on the boundaries, the algebraic equations from Eq.(4) are adopted.

- (b) Case of the symmetry condition (for edges AE and GH in Fig.2)

As in the case of (a), the imaginary lattice points are arranged and the displacements at these points are expressed by those at the points inside the domain by using the symmetry conditions. For the displacements at the lattice points on the boundaries, Eq.(4) is used. But when it is selfevident that u or v vanishes on the boundaries, this fact is used as the boundary condition.

Fig. 1 Unequally spaced rectangular lattice

Fig. 2 Uniform tension of a rectangular plate with a crack at its center

- (c) Case of the corner condition
(for point F in Fig.2)

For the boundary condition at the corner point, various methods of treatments were tried and it was found that the following method gave the best results.

The lattice points near the corner point F are numbered as shown in Fig.4. For point F Eq.(4) is not adopted but by using Eq.(5) and the backward finite difference formula the equations of u_s and v_s are derived from the boundary conditions of σ_{xF} and σ_{yF} , respectively. In this way, u_s and v_s become unnecessary. The condition of τ_{xyF} is used for expressing v_s or u_s by adopting the method stated in (a). In this example, it is used in the equation of v_s , then u_s , u_s and v_s are related with u and v at some lattice points inside the domain by quadratic curves. In Fig.2 the points E, G and F are also corner points, but for these points the conditions of symmetry are adopted.

- (d) Case of the crack-tip condition
(for point A in Fig.2)

As shown in Fig.5, two lattice points are arranged at the tip of the crack, and two imaginary lattice points for them are also arranged. For one of these lattice points at the tip the condition of the free surface is used and for the other lattice point the condition of symmetry is used. In Eq.(4) of the lattice point (IC, 3) in Fig.5, u of the point at the free surface and v of the point of symmetry for the tip point are used. This treatment is the result of the trial and error method. In the example of a pure shear problem, the treatment is reversed.

Fig. 3 Lattice diagram for problem of Fig. 2

Fig. 4 Lattice points at a corner of rectangular plate

- (e) Case of the displacement condition
(not found in Fig.2)

When the displacements of the lattice points on the boundary are given, their values are used directly. Accordingly, no equation is necessary for these points.

Now, as stated above, from the finite difference equations corresponding to Eq. (4) and some boundary conditions, two algebraic equations are obtained corresponding to each lattice point, i.e. u 's equations and v 's equations. All of these equations compose a set of simultaneous linear equations with the unknowns $u_{i,j}$ and $v_{i,j}$. To solve these equations, the so-called SOR method⁽²⁾ is adopted, where all equations are divided into two groups of u 's and v 's equations, and first giving some constant values to all of v 's in u 's equations, the solution of u 's are obtained by the SOR method. Then, substituting the obtained values of u 's in v 's equations, the values of v 's are obtained by the same method. These procedures are repeated until all values of u 's and v 's converge within some prescribed limit.

After the values of u 's and v 's for all lattice points are found, the stress distribution can be calculated by the finite difference equations corresponding to Eq.(5).

4. Calculations of stress intensity factors

In order to obtain the stress intensity factors from the values of u 's and v 's calculated by the method stated in the preceding chapter, the displacement method, the stress method and the J-integral method are adopted. These three methods are outlined in the following.⁽³⁾

4.1 Displacement method and stress method

From the distributions of the displacements v or u and the stresses σ_v or τ_{xy} on the x -axis in Fig.6, the approximate value of the stress intensity factor K_I^* or K_{II}^* , respectively, is calculated by the following equations;

Displacement method;

$$\left. \begin{aligned} K_I^* &= \frac{2G}{\kappa+1} \left[\sqrt{\frac{2\pi}{r}} (v)_{\theta=\pi} \right] \\ K_{II}^* &= \frac{2G}{\kappa+1} \left[\sqrt{\frac{2\pi}{r}} (u)_{\theta=\pi} \right] \end{aligned} \right\} \dots\dots\dots (8)$$

Fig. 5 Lattice points near crack tip

where

$$\kappa = \frac{3-\nu}{1+\nu} \quad (9)$$

in the case of the plane stress.
Stress method ;

$$\left. \begin{aligned} K_I^* &= \sqrt{2\pi r} (\sigma_y)_{\theta=0} \\ K_{II}^* &= \sqrt{2\pi r} (\tau_{xy})_{\theta=0} \end{aligned} \right\} \quad (10)$$

In the above equations, the suffices I and II indicate mode I and mode II, respectively. By using the above equations, the values of K_I^* or K_{II}^* are plotted against the distance of any point on the x axis from the crack tip, r , and the limit values of K_I^* and K_{II}^* when r tends to zero are estimated. These limit values are those to be obtained.

4.2 J-integral method

As shown in Fig. 7, an integral along an arbitrary contour from an arbitrary point on the under surface of the crack to an arbitrary point on its upper surface is defined as follows ;

$$J = \int_r \left\{ W - \left(\sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial v}{\partial x} \right) \right\} dy + \int_r \left(\tau_{xy} \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial x} \right) dx \quad (11)$$

This is the so-called J-integral, where w is given by

$$W = \frac{1}{2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) \quad (12)$$

for linear elasticity as in this case. By using the J-integral, the stress intensity factor is given by the following equation;

$$K_I = \sqrt{\frac{8GJ_I}{\kappa+1}}, \quad K_{II} = \sqrt{\frac{8GJ_{II}}{\kappa+1}} \quad (13)$$

Fig. 6 Stress field near crack tip

Fig. 7 J-integral

where κ is given in Eq. (9). The components of the stress and the strain are decomposed into symmetric and asymmetric components. J_I is the value of J where only symmetric components of the displacement and the stress are used in Eq. (11). In the same way, J_{II} is one for asymmetric components of the displacement and the stress.⁽⁴⁾ In this study the calculations of the J-integral is carried out by the trapezoidal formula, and the stress intensity factor is obtained by the formulas in Eq. (13).

Fig. 8 illustrates the results obtained by the three methods stated above for the case of Fig. 2. The coordinate L means the total length of the integral contours in the J-integral, and the coordinate r means the distance of any point from the crack tip in the displacement and stress methods. A chain line in Fig. 8 indicates Ishida's value⁽³⁾ obtained by the collocation method. It is found from Fig. 8 that the values by the displacement method and the J-integral method coincide well with Ishida's values. Also it is found that the state of variations of the values by the J-integral is almost stable except in the cases of very short integral contours. In this study, therefore, the values of the J-integral for several different integral contours are calculated and their mean value is substituted into Eq. (13). The validity of the value of the stress intensity factor thus obtained is afterwards verified by the displacement method or the stress method.

5. Numerical examples

Numerical calculations are carried out for four cases. Poisson's ratio is always assumed as $\nu = 0.3$, and the stress intensity factor is expressed by the following nondimensional parameters ;

$$\left. \begin{aligned} F_I &= K_I / \sigma_0 \sqrt{\pi a} \\ F_{II} &= K_{II} / \tau_0 \sqrt{\pi a} \end{aligned} \right\} \quad (14)$$

where σ_0 and τ_0 are the given loading stresses.

Fig. 8 Calculations of stress intensity factors for case of Fig. 2

5.1 Uniform tension of a rectangular plate with one crack in its center

The analyzed model is shown in Fig.2. The states of change of the calculated F_I against the change of the length of the crack or of the length of the rectangular plate are shown in Fig.9. It is found that the calculated results coincide well with Ishida's solutions.

5.2 Pure shear of a rectangular plate with one crack in its center

The analyzed model is shown in Fig.10. Due to the anti-symmetry of this problem, a quarter part covered with fine inclined lines is considered. The calculated results are shown in Fig.11. From Fig.11(a) it is found that the value of F_{II} tends to unity as the length of the crack tends to zero. Also it is found from Fig.11(b) that as the length of the rectangular plate becomes larger, the value of F_{II} tends to 1.17 which is considered a value for the infinitely long strip with a crack. Generally, similar trends are recognized in the cases of uniform tension (Fig.9) and pure shear (Fig.11).

(a) Effect of change of crack length

(b) Effect of change of plate length

Fig. 9 Stress intensity factors for case of Fig. 2

For this problem L.Chow and K.L.Lau⁽⁵⁾ calculated the stress intensity factor by the finite element method. But their values do not tend to 1.0 as in Fig.11(a) but to about 0.8. It is considered, therefore, that their results are less accurate than those by the present method.

5.3 Uniform tension of a rectangular plate with two collinear cracks of equal length in symmetric position on the center section

The analyzed model is shown in Fig.12. The outside and inside tips of the cracks are named points A and B, respectively. The calculated results are shown in Fig.13. From Figs.13(a) and (b) it is found that the values of F_I for A and B tend to unity as the length of the cracks becomes smaller. Also it is found from four figures in Fig.13 that as the interval between two cracks or between the edge of the plate and the tip of the crack (point A) becomes smaller, the value of F_I becomes larger. In Fig.13 Ishida & Terada's value⁽⁶⁾ of F_I by the collocation method are indicated. The differences between their results and the results by the present method become larger when the interval mentioned above is very small, although the maximum difference is less than 5%.

5.4 Pure shear of a rectangular plate with two collinear cracks of equal length in symmetric position on the center section

The analyzed model is shown in Fig.14, and the calculated results are shown in Fig.15. This problem has never been treated and the data to be compared with our results are not available, but some strong similarities are recognized between Fig.15 and Fig.13.

All numerical calculations in this study were carried out by the computer system ACOS 1000 in the Large Computer Center of Osaka University. In the example of Fig.2, the finite difference lattice 19×16 was used as shown in Fig.3. The computing time in this case was about 15 seconds. In the example of Fig.12, the lattice 39×27 was used and the computing time was about 2~3 minutes. In the problems of mode II, almost the same lattices were used as in the case of mode I and the computing times were almost the same.

To check the accuracy of our solutions, the numbers of the lattice points in the numerical examples stated above were changed more or less, and the states

Fig. 10 Pure shear of a rectangular plate with a crack at its center

of variations of the values of F_I or F_{II} were carefully checked. Table 1 and Table 2 are two examples of the results of such examinations. It is found from these Tables that the numbers of the lattice points (i. e. the number of meshes) with * mark give the values of F_I (or F_{II}) accurate enough for practical purposes. In all numerical examples stated above, the numbers of the lattice points were almost the same as those with * mark in these two Tables. From these Tables and also by comparing our results with those by other researchers as

in Fig.8, Fig.9 and Fig.13, it is considered that our solutions of the stress intensity factor have accuracies within about 1 % except some cases with very inadequate conditions.

6. Conclusions

In this study, a numerical method to solve the plane elastic problem of some rectangular domains by the finite difference method was presented, and a scheme for treating the neighborhood of a crack tip

Fig. 11 Stress intensity factors for case of Fig. 10

Fig. 12 Uniform tension of a rectangular plate with two collinear cracks of equal length in symmetric position on the center section

Fig. 13 Stress intensity factors for case of Fig. 12

contained in the rectangular domain was proposed. By these methods, rectangular plates or strips with one or two cracks were treated and the stress intensity factors were calculated. Some of these stress intensity factors were found to coincide well with those by other researchers, and some new data were obtained.

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Table 1 Relation between number of lattice points and value of stress intensity factor (case of Fig. 9)

Number of Mesh (Horizontal x Vertical)	Value of F_I
19 x 16*	1.331
31 x 27	1.329

Table 2 Relation between number of lattice points and value of stress intensity factor (case of Fig. 13)

Number of Mesh (Horizontal x Vertical)	Value of F_I	
	Point A	Point B
25 x 18	1.083	1.084
39 x 27*	1.088	1.088
49 x 36	1.088	1.088

Fig. 14 Pure shear of a rectangular plate with two collinear cracks of equal length in symmetric position on the center section

(a) Effect of change of crack length (point A)

(b) Effect of change of crack length (point B)

(c) Effect of change of interval between cracks (point A)

(d) Effect of change of interval between cracks (point B)

Fig. 15 Stress intensity factors for case of Fig. 14