Time-Dependent Wall Shear Stress in a Duct with Arbitrary Cross-Section*

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The wall shear stress in an unsteady duct flow is determined by equalizing the pressure gradient calculated from the momentum integral equation to that from the energy equation. This method is applicable to an unsteady flow subjected to a relatively low acceleration in a duct with arbitrary cross-section. Consideration on unsteady laminar flows in a pipe and rectangular ducts with various aspect ratios reveals that the present representation for time-dependent wall shear stress is superior to the quasi-steady representation commonly used so far. This fact suggests that the present representation is more useful than the quasi-steady one in the turbulent regime too. Accordingly, the representation for unsteady turbulent flow in the above mentioned two kinds of ducts are derived.

Key Words : Unsteady Flow, Pulsating Flow, Accelerating Flow, Wall Shear Stress, Circular Pipe, Rectangular Duct

1. Introduction

It is well known that the instantaneous velocity distribution of an unsteady pipe flow which varies its motion very slowly with time is closely similar to that of steady-state one both in laminar and turbulent cases⁽¹⁾⁻⁽³⁾. In such a case, the pressure gradient $\Delta p/l$ is commonly evaluated by substituting the quasi-steady representation for the wall shear stress τ_w into the momentum integral equation. Meanwhile, the value of $\Delta p/l$ derived by applying the same representation for τ_w to the energy equation does not agree with the above mentioned one. The application of the quasi-steady representation for τ_w brings this disagreement. In this paper, a new representation for τ_w is determined by equalizing the pressure gradient calculated from the momentum integral equation to that from the energy equation. This method needs only the knowledge of the effective inertia K .

At the first place, the representation for unsteady laminar flows in a pipe and rectangular ducts are formulated. The comparison of these representations with the analytical ones makes clear that the present representation gives more excellent values than that derived from the well-known quasi-steady representation over the range of a relatively high acceleration region. On the basis of this result, it may possibly be considered that the present

representation is also superior to the quasi-steady one in the turbulent regime. Therefore, we derive the representation for unsteady turbulent flows in a circular pipe and rectangular ducts, and examine them.

The present method has the advantage of being applicable both to laminar and turbulent flows, and provides the available clues to formulate the more precise representation for turbulent flow in a relatively high acceleration region.

2. Nomenclature

- A : cross-sectional area of duct
- a, b : half width and half height of duct
- a_1, b_1, c : coefficients and exponent of Eq. (7)
- D : diameter of circular pipe
- D_h : hydraulic diameter = $4A/(\text{wetted perimeter}) = 4ab/(a+b)$ for rectangular duct
- K : effective inertia
- $k = u/u_m$
- l : length of test section
- n : exponent in Eq. (29)
- p : pressure
- $\Delta p/l$: pressure gradient
- Re : instantaneous Reynolds number $\equiv u_m D_h / \nu$
- r : radial distance
- $r' = r/R$
- t : time
- u_m : cross-sectional mean velocity
- u_{cl} : center-line velocity
- u^* : friction velocity
- (x, y, z) : rectangular coordinates shown in Fig. 2
- γ : aspect ratio = a/b
- ε : eddy viscosity
- η : distance from duct wall
- κ : modified effective inertia = $K-1$
- λ_q : quasi-steady friction factor

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- ν : kinematic viscosity
 $\nu_x = \nu + \varepsilon$
 ρ : density
 τ_w : wall shear stress
 Q : dimensionless quantity defined by Eqs. (53) and (54)
 ω : angular frequency
 ω' : dimensionless frequency $\equiv R^2 \omega / \nu$
 ω^* : dimensionless frequency $\equiv b^2 \omega / \nu$
 Subscripts and others
 1: fundamental wave (except for a_1, b_1)
 os, ta: oscillating component and time-averaged value
 |, $\angle$: amplitude and phase angle

3. Derivation of the representation for τ_w including the term proportional to acceleration

To facilitate the analysis, we first deal with the circular pipe flow. The equation of motion for an incompressible unsteady turbulent flow can be written as⁽⁴⁾

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \nu_x \frac{\partial \bar{u}}{\partial r} \right) \quad (1)$$

in which the modified eddy viscosity ν_x is defined as $\nu_x = \nu + \varepsilon$. The overscore " - " which denotes the ensemble averaged value is hereafter omitted for brevity as far as no confusion may occur. And for a laminar flow, as a matter of course, " - " can be omitted and eddy viscosity ε must be zero.

After replacing the pressure gradient $-\partial \bar{p} / \partial x$ in Eq. (1) by $\Delta p / l$ for an incompressible flow, integration of Eq. (1) over the cross-section gives the following momentum integral equation (2).

$$\frac{\Delta p}{l} = \rho \frac{du_m}{dt} + \frac{4}{D} \tau_w \quad (2)$$

Next, an energy equation may be constructed by multiplying the both sides of Eq. (1) by $\bar{u}$. The energy equation is then integrated over the cross-section, and there results

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^R \pi r \bar{u}^2 dr &= -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \pi R^2 \bar{u}_m \\ &- \int_0^R 2\pi r \nu_x \left(\frac{\partial \bar{u}}{\partial r} \right)^2 dr \quad (3) \end{aligned}$$

When a flow varies its motion very slowly with time, the instantaneous velocity distribution can be described by the laws of steady flow. This has already been confirmed for the laminar flows by the theoretical analysis and experiments and for the turbulent flows by the experiments⁽¹⁾. It has been also ascertained that the distribution of ν_x is the same as that of the steady-state⁽⁵⁾⁽⁶⁾.

Therefore, we introduce the expression for the velocity u as

$$u = k(r') u_m(t) \quad (4)$$

then the left hand side of Eq. (3) can be written in the following form

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^R \pi r \bar{u}^2 dr \\ = u_m \frac{du_m}{dt} \int_0^R 2\pi r k^2(r') dr \quad (5) \end{aligned}$$

Here, the function $k(r')$ should be expressed as $k[r', R_e(t)]$ for a turbulent flow in an exact sense. But the dependence of $k[r', R_e(t)]$ on $R_e(t)$ may reasonably be neglected within the limited range of Reynolds number. This gets support by Figs. 4 and 5 shown later.

Since the distributions of $\bar{u}$ and ν_x are both supposed the same as the steady-state ones respectively, the second term on the right hand side of Eq. (3) which represents the viscous dissipation becomes

$$\int_0^R 2\pi r \nu_x \left(\frac{\partial \bar{u}}{\partial r} \right)^2 dr = \pi R^2 \frac{\lambda_q}{2D} u_m^2 \quad (6)$$

where λ_q is the quasi-steady friction factor and is evaluated by the following relations.

$$\lambda_q = \begin{cases} c/R_e & (\text{laminar flow}) \\ a_1/R_e^{b_1} & (\text{turbulent flow}) \end{cases} \quad (7)$$

For turbulent flow, the most proper empirical formula for λ_q must be adopted corresponding to the range of R_e .

Substitutions of Eqs. (5) and (6) into Eq. (3) and letting $-\partial \bar{p} / \partial x$ be $\Delta p / l$ give

$$\frac{\Delta p}{l} = K \rho \frac{du_m}{dt} + \frac{\lambda_q}{2D} \rho u_m^2 \quad (8)$$

$$K = \int_0^1 2r' k^2(r') dr' \quad (9)$$

It is convenient to rewrite Eq. (8) as

$$\frac{\Delta p}{l} = \rho \frac{du_m}{dt} + \kappa \rho \frac{du_m}{dt} + \frac{\lambda_q}{2D} \rho u_m^2 \quad (10)$$

$$\kappa = K - 1 \quad (11)$$

K is known as the effective inertia and κ is called the modified effective inertia in this paper.

Upon substitution of Eq. (10) into Eq. (2), we finally get

$$\tau_w = \frac{\lambda_q}{8} \rho u_m^2 + \frac{\kappa \rho D}{4} \frac{du_m}{dt} \quad (12)$$

After the comparison of Eq. (12) with the well-known quasi-steady representation, i.e.,

$$\tau_w = \lambda_q \rho u_m^2 / 8 \quad (13)$$

it is easily found that the present representation differs from Eq. (13) by magnitude of $(\kappa \rho D / 4) du_m / dt$. Estimation of Eq. (12) for a flow in ducts of arbitrary cross-section can be done by replacing D in Eq. (12) by the hydraulic diameter D_h and calculating K from the following equation.

$$K = \int_A (u/u_m)^2 dA / A \quad (14)$$

4. Comparison of the analytical and the present representation for wall shear stress τ_w in laminar unsteady flow

It will be shown that when the acceleration of the flow du_m/dt is relatively small, the values of τ_w calculated from the present representation defined by Eq.(12) agree with those from the published analytical ones for unsteady laminar flow in a pipe and rectangular ducts. And the range of acceleration valid for Eq.(12) is examined.

4.1 Circular pipe

The velocity profile for a steady laminar flow is the well-known Poiseuille parabola

$$k(r') = 2(1 - r'^2) \quad (15)$$

Evaluation of Eqs.(9) and (11) with Eq.(15) gives

$$K = 4/3, \kappa = 1/3 \quad (16)$$

and Eq.(12) can be written as

$$\tau_w = \frac{8\mu}{D} u_m + \frac{\rho D}{12} \frac{du_m}{dt} \quad (17)$$

For example, an oscillating flow is considered. The analytical representation for wall shear stress τ_w in such a flow is previously reported by the authors in the form⁽³⁾

$$\tau_w = \frac{8\mu}{D} \left[\eta_1(\sqrt{\omega'}) u_m + \frac{\xi_1(\sqrt{\omega'})}{\omega'} \frac{R^2}{\nu} \frac{du_m}{dt} \right] \quad (18)$$

$$\left. \begin{aligned} \eta_1(\sqrt{\omega'}) &= \frac{\sqrt{\omega'}}{4} \frac{M_1(\sqrt{\omega'})}{M_2(\sqrt{\omega'})} \\ &\times \cos[\theta_1(\sqrt{\omega'}) - \theta_2(\sqrt{\omega'}) + 3\pi/4] \\ \xi_1(\sqrt{\omega'}) &= \frac{\sqrt{\omega'}}{4} \frac{M_1(\sqrt{\omega'})}{M_2(\sqrt{\omega'})} \\ &\times \sin[\theta_1(\sqrt{\omega'}) - \theta_2(\sqrt{\omega'}) + 3\pi/4] \end{aligned} \right\} \quad (19)$$

In the limit when ω' approaches 0, we get

$$\lim_{\omega' \rightarrow 0} \eta_1(\sqrt{\omega'}) \rightarrow 1, \lim_{\omega' \rightarrow 0} \frac{\xi_1(\sqrt{\omega'})}{\omega'} \rightarrow \frac{1}{24} \quad (20)$$

Therefore, Eq.(17) agrees with Eq.(18).

The results of $\eta_1(\sqrt{\omega'})$ and $\xi_1(\sqrt{\omega'})/\omega'$ calculated from Eq.(19) and those from Eq.(17) [that is, $\eta_1(\sqrt{\omega'}) = \lambda_q R_e / 64 = 1$, $\xi_1(\sqrt{\omega'})/\omega' = 1/24$] are shown in Fig.1 by the solid lines and the broken lines respectively. And the substitution of $\lambda_q = 64/R_e$ into the quasi-steady representation defined by Eq.(13) yields $\eta_1(\sqrt{\omega'}) = 1$, $\xi_1(\sqrt{\omega'})/\omega' = 0$. These relations are also shown by the broken lines in Fig.1(a) and by the dot-dash lines in Fig.1(b). In Fig. 1, the results for rectangular ducts discussed later are also shown, though in this case ω' must be replaced by ω^* . As shown in the figure, close agreement can be seen between Eq.(17) and Eq.(18) in the range of $\sqrt{\omega'} \leq 1$. In Fig.1 (b), the

(a) values of $\eta_1(\sqrt{\omega'}), \xi_1(\sqrt{\omega'})$

(b) values of $\xi_1(\sqrt{\omega'})/\omega', \xi_1(\sqrt{\omega^*})/\omega^*$

Fig.1 Comparison of the analytical, present and quasi-steady representation for wall shear stress

values of $\xi_1(\sqrt{\omega'})/\omega'$ evaluated from the analytical representation accord with that from the quasi-steady one in the limit when ω', ω^* approach ∞ , but the values of $\xi_1(\sqrt{\omega'})$ from the former representation approaches infinite and that from the latter is always zero.

It has been already verified⁽⁷⁾ that for a more general unsteady flow in a pipe, the wall shear stress τ_w can be evaluated by Eq.(17) when the acceleration of flow is relatively small. However, the applicable range of Eq.(17) cannot be determined unless the time-dependent history of u_m is known.

4.2 Rectangular duct

With the coordinate system shown in Fig.2, $k(y', z')$ is given by the following equation⁽⁸⁾

Fig.2 Coordinate system for rectangular duct

$$k(y', z') = \frac{1 - z'^2 - 4 \sum_{n=0}^{\infty} \frac{\sin N_n}{N_n^3} \frac{\cosh N_n \gamma y'}{\cosh N_n \gamma} \cos N_n z'}{\frac{2}{3} - \frac{4}{\gamma} \sum_{n=0}^{\infty} \frac{\tanh N_n \gamma}{N_n^5}} \quad (21)$$

where

$$\left. \begin{aligned} y' &= y/a, z' = z/b, \\ N_n &= (2n+1)\pi/2, \gamma = a/b \end{aligned} \right\} \quad (22)$$

Hence the effective inertia K is defined as

$$K = \int_0^1 \int_0^1 k^2(y', z') dy' dz' \quad (23)$$

Substituting Eq.(21) into Eq.(23) and after carrying out the integration, we get

$$\kappa = \frac{1}{g^2} \left(\frac{8}{15} - \frac{12}{\gamma} \sum_{n=0}^{\infty} \frac{\tanh N_n \gamma}{N_n^5} + 4 \sum_{n=0}^{\infty} \frac{1}{N_n^6 \cosh^2 N_n \gamma} \right) - 1 \quad (24)$$

$$g = \frac{2}{3} - \frac{4}{\gamma} \sum_{n=0}^{\infty} \frac{\tanh N_n \gamma}{N_n^5} \quad (25)$$

The variation of κ -value with aspect ratio γ is shown in Fig.3.

In the same way as in the previous section, we consider an oscillating flow for comparison. Equation (12) can be transformed into the following form for such a flow

$$\tau_w = \frac{8\mu}{D_h} \left[\frac{u_m}{\phi(\gamma)} + \frac{\kappa}{2(1+1/\gamma)^2} \frac{b^2}{\nu} \frac{du_m}{dt} \right] \quad (26)$$

$$\phi(\gamma) = (1+1/\gamma)^2 g(\gamma) \quad (27)$$

Meanwhile the analytical representation is given by⁽⁹⁾

$$\tau_w = \frac{8\mu}{D_h} \left[\eta_1(\sqrt{\omega^+}) u_m + \frac{\xi_1(\sqrt{\omega^+})}{\omega^+} \frac{b^2}{\nu} \frac{du_m}{dt} \right] \quad (28)$$

though the representation for $\eta_1(\sqrt{\omega^+})$ and $\xi_1(\sqrt{\omega^+})$ are omitted as they are very

Fig.3 κ -value for laminar steady flow in rectangular duct

lengthy. In Eq.(28), $\eta_1(\sqrt{\omega^+})$ and $\xi_1(\sqrt{\omega^+})/\omega^+$ reduce to $1/\phi(\gamma)$ and $\kappa/[2(1+1/\gamma)^2]$ respectively, when $\omega^+ = b^2\omega/\nu$ approaches 0. So, Eq.(28) is identical to Eq.(26) in this limit. For a more general unsteady flow in rectangular ducts, it has been verified numerically that τ_w calculated from Eq.(26) agrees with that from the analytical one. From the results shown in Fig.1, it is clear that Eq.(26) approximates Eq.(28) with good accuracy in the range of $\sqrt{\omega^+} \lesssim 1$. Further, from Fig.1(b), it is apparent that Eq.(26) is superior to the quasi-steady representation.

5. Representation for τ_w in turbulent flows < Evaluation of modified effective inertia κ >

As is analogized from the foregoing results for laminar flow, it is reasonably be considered that the present representation is superior to the quasi-steady one also in turbulent regime. Therefore, we derive the effective inertia K and the modified effective inertia κ for steady turbulent flows in a pipe and rectangular ducts.

5.1 Circular pipe

5.1.1 $1/n$ -th power law

If we assume that the instantaneous velocity distribution can be described by the following $1/n$ -th power law

$$u = u_{cl}(1-r')^{1/n} \quad (29)$$

then we get

$$k(r') = \frac{(n+1)(2n+1)}{2n^2} (1-r')^{1/n} \quad (30)$$

Substitution of Eq.(30) into Eq.(9) and comparison with Eq.(11) yield

$$K = \frac{(n+1)(2n+1)^2}{4n^2(n+2)} \quad (31)$$

$$\kappa = \frac{5n+1}{4n^2(n+2)} \quad (32)$$

Here the exponent n in the above equations is calculated from the following relation⁽¹⁰⁾

$$n = 2 \log_{10}(Re/10) \quad (33)$$

Additionally, in case of conventionally used $n=7$, we get

$$K = 50/49, \kappa = 1/49 \quad (34)$$

5.1.2 Logarithmic law

Applying the universal velocity distribution law⁽¹¹⁾

$$u^+ = \begin{cases} \eta^+ & (\eta^+ \leq 5) \\ -3.05 + 5 \ln \eta^+ & (5 < \eta^+ < 30) \\ 5.5 + 2.5 \ln \eta^+ & (\eta^+ \geq 30) \end{cases} \quad (35)$$

into Eq.(9), we obtain

$$K = (8/R_+^2) [3.125 R_+^{*2} (\ln R_+^*)^2 + 4.375 R_+^{*2} \ln R_+^* + 5.4375 R_+^{*2}]$$

$$-1084.1R^+ + 12020] \dots\dots\dots (36)$$

$$R_e = 4[R^+(0.875 + 1.25 \ln R^+)$$

$$-63.896 + 573.20/R^+] \dots\dots\dots (37)$$

in which $\eta^+ = u^* \eta / \nu$, $R^+ = u^* R / \nu$ and η denotes the distance measured from the wall.

The κ -values calculated from the foregoing two kinds of velocity distribution laws are shown in Fig.4. To make a comparison, the experimental results which were obtained under the condition that turbulent transition occurred at $R_e \approx 2000$, and the calculated ones after Brown et al.⁽¹²⁾ are also shown in the same figure. The experimental data shown in Fig.4 are the results calculated from the measured velocity distribution presented in references (12), (14) and (15) by the authors. Brown et al.⁽¹²⁾ had calculated the values of K by applying the following velocity distribution law

$$u^+ = \begin{cases} \eta^+ & (\eta^+ \leq \eta_i^+) \\ 5.5 + 2.5 \ln \eta^+ & (\eta^+ > \eta_i^+) \end{cases} \dots\dots\dots (38)$$

in which η_i^+ denotes the point of intersection of the above two formulae. As the κ -value derived by subtracting 1 from K calculated from Eq.(36) agrees well with the experimental ones in the range of $R_e \approx 3000$, we recommend Eq.(36). In the region of $R_e = 2000 \sim 3000$, κ may be approximated by the straight line connecting the two points, $\kappa = 1/3$ at $R_e = 2000$, $\kappa = 0.1089$ at $R_e = 3000$ in the figure.

5.2 Rectangular duct

It must be difficult to evaluate the κ -values precisely for turbulent flows in rectangular ducts. In addition, there may be no experimental data to compare. So, in this paper, we propose the κ -value for steady turbulent flows in parallel-plates channel on the basis of the comparisons of the experimental results. For flows in rectangular ducts, we only present the estimated values.

5.2.1 Parallel-plates channel

(a) $1/n$ -th power law Using the foregoing Eq.(29), then we get

Fig.4 κ -value for turbulent steady flow in circular pipe

$$k(z') = (n+1)/n(1-z')^{1/n} \dots\dots\dots (39)$$

$$K = (n+1)^2 / [n(n+2)] \dots\dots\dots (40)$$

$$\kappa = 1 / [n(n+2)] \dots\dots\dots (41)$$

Now, we introduce the following empirical relation between u_{cl}/u_m and R_e after Dean⁽¹⁶⁾

$$\left. \begin{aligned} u_{cl}/u_m &= 1.29R_e^{-0.0116} \\ 12000 < R_e < 1200000 \end{aligned} \right\} \dots\dots\dots (42)$$

Combination of an expression obtained by substituting $z'=0$ into Eq.(39), i.e.,

$$u_{cl}/u_m = (n+1)/n \dots\dots\dots (43)$$

and Eq.(42) gives

$$n = 1 / (1.29R_e^{-0.0116} - 1) \dots\dots\dots (44)$$

(b) Logarithmic law It makes sure by the results presented as Fig.4 in reference (16) that the coefficient 2.5 and constant 5.5 in the last formula in Eq.(35) are also applicable for steady turbulent flows in parallel-plates channel. Accordingly, we assume the same velocity distribution as described by Eq.(35), then K is calculated as

$$K = (16b^+ / R_e^2) [6.25b^+ (\ln b^+)^2 + 15b^+ \ln b^+ + 15.25b^+ - 1084.1] \dots\dots\dots (45)$$

$$R_e = 4[b^+(3 + 2.5 \ln b^+) - 63.896] \dots\dots\dots (46)$$

where $b^+ = u^* b / \nu$

In Fig. 5, the κ -values calculated from the above two kinds of velocity distribution laws are shown to compare with the experimental ones⁽¹⁴⁾⁽¹⁷⁾⁽¹⁸⁾. As is analogized from the circular pipe flows, it may be reasonable to evaluate the κ -value from Eqs.(45) and (11) in the region $R_e \approx 3000$, though the comparison cannot be made in a larger Reynolds number range.

5.2.2 Square duct

We assume that the velocity distribution in each $1/8$ -th portion of square cross-section shown in Fig. 6 can be described by Eq.(35), and friction velocity u^* is constant at all the peripheral walls. Since Hartnett et al.⁽¹⁹⁾ had shown the satisfactory agreements between the measured velocity distributions in square and

Fig.5 κ -value for turbulent steady flow in parallel-plates channel

rectangular ducts after Nikuradse⁽²⁰⁾ and that calculated from Eq.(35), the former assumption is almost reasonable. Referring to the experimental results after Fujita⁽²¹⁾ and Leutheusser⁽²²⁾, the latter assumption for friction velocity may contain the errors of about $-20\% \sim +10\%$ for square duct ($\gamma=1$), and about $\pm 20\%$ for rectangular duct with $\gamma=3$.

Figure 7 shows that the results of calculation based on the foregoing assumptions do not differ greatly from the experimental ones⁽²³⁾. In this case, K and R_e are calculated from the equations obtained by replacing R^+ in Eqs.(36) and (37) by b^+ .

5.2.3 Rectangular duct

As being a bold attempt, we tentatively divide the rectangular cross-section into two regions, I, II, as shown in Fig.8, and friction velocity is assumed constant at all the peripheral walls. Further, we assume that the flows in the region I and II are considered to be the flows in square and parallel-plates channel respectively. On the basis of these assumptions, the calculation is carried out for the velocity distribution in duct with $\gamma=3$. This case is in the range of γ where the present model is a fairly poor approximation. From Fig.9, a good agreement between the calculated and the observed results can be seen in the neighbourhood of z' -axis, but in the regions adjacent to the side walls, the calculated results are inaccurate. As mentioned in 5.2.2, an error due to the assumption for friction velocity is about $\pm 20\%$. Therefore it should be noted that the x -values shown later have nearly the same extent of error in case of $\gamma=3$. The effective inertia K and the Reynolds number R_e can be calculated as follows.

$$K = \frac{1}{R_e^2(1+1/\gamma)^2} \left\{ \frac{32/\gamma[5.4375b^{*2} + 4.375b^{*2} \ln b^* + 3.125b^{*2}(\ln b^*)^2]}{-1084.1b^* + 12020} + 16(1-1/\gamma)b^*[6.25b^*(\ln b^*)^2 + 15b^* \ln b^*] + 15.25b^* - 1084.1 \right\} \quad (47)$$

$$R_e = \frac{8}{(1+\gamma)b^*} \left[\frac{b^{*2}(0.875 + 1.25 \ln b^*)}{-63.896b^* + 573.20} \right] + \frac{4(\gamma-1)}{\gamma+1} \left[\frac{b^*(3 + 2.5 \ln b^*)}{-63.896} \right] \quad (48)$$

The x -values calculated for the cases of $\gamma=1, 2, 5, \infty$ and a circular pipe are listed in Table 1. Only the slight differences can be seen among them. The x -values presented here may be enough for a practical use, although we cannot conclude this definitely for rectangular duct flows, as there is a lack of experimental confirmation.

Fig.8 Two regions in rectangular cross-section

Fig.6 Coordinate system for a square duct

Fig.7 Velocity distribution in turbulent square duct flow (the solid lines were drawn by freehand writing in the original paper⁽²³⁾)

Fig.9 Velocity distribution in turbulent rectangular duct flow ($\gamma=3$)

6. The applicable range of the present representation for τ_w in turbulent unsteady flow

The applicable range of Eq.(12) for turbulent unsteady flow should be determined only by the experiments. As being available for this, we can find only the measured results of the four characteristic parameters which describe the quantitative relations between inertia, viscous and pressure gradient terms of the momentum integral equation for pulsating flow. But the measurements were performed only in circular pipes⁽²⁾ and rectangular ducts⁽²⁴⁾, and the range of Reynolds number was narrow and limited as $10^4 \leq R_e \leq 10^5$. In this range of Reynolds number, we obtain $\kappa \leq 0.05$ from Table 1. To disclose the influence of κ -value to the characteristic parameters, we calculate them by using Eq. (12) in the same manner as described in reference (2). The results are as follows.

$$\begin{aligned} \Phi_{t,1} &= \rho \omega |u_{m,os,1}| / (|\Delta p_{os,1}|/l) \\ &= \frac{1}{1+\kappa} \left[1 + \frac{1}{(1+\kappa)^2 Q^2} \right]^{-1/2} \dots\dots\dots (49) \end{aligned}$$

$$\begin{aligned} \Phi_{z,1} &= 4|\tau_{w,os,1}|/D_h / (|\Delta p_{os,1}|/l) \\ &= \kappa \Phi_{t,1} \left(1 + \frac{1}{\kappa^2 Q^2} \right)^{1/2} \dots\dots\dots (50) \end{aligned}$$

$$\begin{aligned} \alpha_{t,1} &= \angle(\Delta p_{os,1}/l) - \angle u_{m,os,1} \\ &= \tan^{-1}[(1+\kappa)Q] \dots\dots\dots (51) \end{aligned}$$

$$\begin{aligned} \alpha_{z,1} &= \angle(\Delta p_{os,1}/l) - \angle \tau_{w,os,1} \\ &= \alpha_{t,1} - \tan^{-1}(\kappa Q) \dots\dots\dots (52) \end{aligned}$$

where Q for turbulent flow is defined as

$$Q = \begin{cases} \frac{8\omega'}{a_1(2-b_1)R_{eia}^{1-b_1}} & (\text{circular pipe}) \\ \frac{32(1+1/\gamma)^{-2}\omega^+}{a_1(2-b_1)R_{eia}^{1-b_1}} & (\text{rectangular duct}) \end{cases} \dots\dots\dots (53)$$

In addition, Q for laminar flow can be evaluated as

$$Q = \begin{cases} 8\omega'/c & (\text{circular pipe}) \\ \frac{32}{c} \left(\frac{1}{1+1/\gamma} \right)^2 \omega^+ & (\text{rectangular duct}) \end{cases} \dots\dots\dots (54)$$

Table 1 κ -values for steady turbulent flows

$R_e \times 10^{-3}$	κ				
	Circular pipe	Rectangular duct			
		$\gamma=1$	$\gamma=2$	$\gamma=5$	$\gamma=\infty$
2.5	0.1209	0.1209	0.1186	0.1153	0.1128
3	0.1089	0.1089	0.1068	0.1038	0.1014
4	0.0922	0.0922	0.0904	0.0877	0.0855
10	0.0552	0.0552	0.0539	0.0518	0.0501
40	0.0288	0.0288	0.0277	0.0263	0.0250
100	0.0207	0.0207	0.0198	0.0187	0.0178
400	0.0141	0.0141	0.0135	0.0128	0.0118
1000	0.0116	0.0116	0.0111	0.0103	0.0096
4000	0.0090	0.0090	0.0086	0.0080	0.0075
10000	0.0078	0.0078	0.0075	0.0069	0.0064

Fig.10 Effect of κ -values on $\Phi_{t,1}$

Fig.11 Effect of κ -values on $\Phi_{z,1}$

Fig.12 Effect of κ -values on $\alpha_{t,1}$

Fig.13 Effect of κ -values on $\alpha_{z,1}$

The calculated results from Eqs.(49)~(52) are shown in Figs.10~13. When $\kappa = 0$, the above equations are reduced to the quasi-steady representations. We compare the calculated results for $\kappa = 0$ and $\kappa = 0.04$ with the experimental ones which were obtained for turbulent pulsating pipe flows and presented in Figs.8~11 of reference (2). Although the results are not shown in this paper, it may be concluded that the both calculated results seem to be applicable in the range of $\sqrt{Q} \leq 1$. But as the scatter of data is relatively wide, it cannot be determined how Eq.(12) is superior to Eq.(13) in the range of $10^4 \leq Re \leq 10^5$.

To determine the applicable range of Eq.(12), and further, to derive a more general representation for τ_w in turbulent flow, the precise measurements of τ_w are demanded for accelerating and decelerating flows as well as for pulsating flows, especially in the range of Reynolds number $2 \times 10^3 \leq Re \leq 10^4$ where the effects of unsteadiness appear markedly.

7. Conclusions

(1) A new representation for wall shear stress τ_w in an unsteady duct flow is determined by equalizing the pressure gradient $\Delta p/l$ calculated from the momentum integral equation to that from energy equation. The application of this method to an unsteady flow in a duct with arbitrary cross-section needs only the knowledge of steady-state velocity distribution in such a duct. As defined by Eq.(12), the present representation for τ_w is constructed by adding the term proportional to acceleration to the quasi-steady representation defined by Eq.(13). In this paper, the proportional constant κ is called as the modified effective inertia.

(2) For laminar flows in a pipe and rectangular ducts, the values of modified effective inertia κ are calculated and shown in Fig.3. Comparing the present representation with analytical ones, we conclude that the present representations give more excellent values than those from commonly used quasi-steady representations over the range of relatively high acceleration region. The range of acceleration valid for the present representation can be considered as $\sqrt{\omega'}/\sqrt{\omega''} \leq 1$ for laminar oscillating flows in a pipe and rectangular ducts. However, for a more general unsteady flow except these, the applicable range of acceleration cannot be discussed unless time-dependent history of u_m is given.

(3) For turbulent flows in a pipe and rectangular ducts, the κ -values are calculated and listed in Table 1. On the basis of the comparison of the calculated and observed results for the characteristic parameters, it is disclosed for turbulent pulsating flows in a pipe and rectangular ducts that the applicable range of acceleration valid for Eq.(12) is $\sqrt{Q} \leq 1$ in the range of $10^4 \leq Re \leq 10^5$. For another type of unsteady flow, it is impossible to determine the applicable range of Eq.(12),

as there may be no experimental data to compare at the present.

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