Transient Responses of Fluid Lines* (Step Responses of Single Pipeline and Series Pipelines)

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Transient responses to sudden changes in pressure or flow rate in both single pipeline and series pipelines are theoretically studied under open-end and closed-end condition.

A fundamental equation used in this analysis is the one for fluid line model in which the frequency-dependent effect of viscosity is taken into consideration. It is confirmed that the theoretical results well agree with experimental ones obtained under various system conditions and parameters.

Key words: Fluid Power System, Fluid Transient, Step Response, Single Pipeline, Series Pipeline, Analytical Solution

1. Introduction

It is well known that the dynamic characteristics, such as frequency- or transient-response, of various fluid line systems can be accurately evaluated in terms of the transfer matrix in which the frequency-dependent effect of viscosity is taken into consideration. So long as the frequency response is concerned, its analysis can be made with considerable easiness⁽¹⁾ merely by substituting $j\omega$ for s in the matrix equation (where s : Laplace operator, ω : circular frequency). The transient response analysis made in time domain, on the other hand, will be very difficult, because the inverse Laplace transforms of the matrix have to be taken. The process of taking the inverse Laplace transforms in some analytical methods, without introducing any approximation into the matrix, can be very complicated even in the simplest case such as a single and uniform pipeline system. Hence some numerical methods of analyses⁽²⁾⁽³⁾, such as the method of characteristics, are necessarily employed for calculating the time-domain response within a certain degree of tolerated accuracy.

A theoretical method to derive the time-domain response solution, however, has been worked out by Brown⁽⁴⁾. He proposed an approximated relation to the propagation constant in the matrix and, by means of it, obtained a transient response solution. But this solution is the one for an infinitely lengthened single pipeline system⁽⁴⁾⁽⁵⁾. An application of Brown's method to a system with a finite length

was carried out by Holmboe⁽⁶⁾, who compared the experimental results with the derived solution. It should be noted, however, that the step response solution derived by Holmboe is valid only in the time interval smaller than the first cycle of the response.

In view of the situation described above concerning the transient response in a single pipeline, the intention of the present study is, in the first place, to obtain the step response solutions which are valid over wider time-intervals than those in Holmboe's and also which are to be formulated as simpler and more reasonable relations. In the second place, as an application of the above method of analysis to other pipeline systems, the step response solutions will be derived for a system consisting of two uniform pipelines with different cross sections, that is a series pipelines.

It is supposed in the analysis that the pipe-end (receiving-end) condition is either closed-end or open-end. And the input variable to the pipe systems is considered to be either the pressure or the flow-rate at the sending-end of pipe (named the pressure input or the flow-rate input respectively), and to have a step-wise shape. Under these conditions of pipe systems, the pressure response solutions will be derived as a function of pipe coordinate position. The validity of thus obtained solutions is further investigated by comparing them with experimental results.

2. Theoretical Analysis

Two types of pipeline systems, a single pipe and series pipes shown in Fig.1, are analyzed in this study. An analysis to obtain the step response solutions for each pipe system is made in the following, where either the pressure input or flow-rate input is varied.

Using the notations Q_0 for flow-rate

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and P_0 for pressure as indicated in Fig.1(b), the transfer matrix equations for the series pipeline system can be expressed by Eqs.(1) and (2), when a fluid-line model is employed in which the frequency-dependent effect of viscosity is taken into consideration.⁽¹⁾

$$\begin{bmatrix} Q_{i1} \\ P_{i1} \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_i & Z_{ci}^{-1} \sinh \Gamma_i \\ Z_{ci} \sinh \Gamma_i & \cosh \Gamma_i \end{bmatrix} \begin{bmatrix} Q_{i2} \\ P_{i2} \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} Q_{ix} \\ P_{ix} \end{bmatrix} = \begin{bmatrix} \cosh \Gamma_{ix} & -Z_{ci}^{-1} \sinh \Gamma_{ix} \\ -Z_{ci} \sinh \Gamma_{ix} & \cosh \Gamma_{ix} \end{bmatrix} \begin{bmatrix} Q_{i1} \\ P_{i1} \end{bmatrix} \quad (2)$$

$$\left. \begin{aligned} Z_{ci} &= \gamma_i \rho c / a_i, \quad \gamma_i = \sqrt{I_0(\chi_i) / I_2(\chi_i)}, \quad \chi_i = r_i \sqrt{s / \nu} = \sqrt{S / D_i} \\ \Gamma_i &= \gamma_i S L_i, \quad \Gamma_{ix} = \gamma_i S X, \quad \Gamma_{2x} = \gamma_2 S (X - L_1), \quad D_i = \nu t_0 / r_i^2 \\ L_i &= l_i / l, \quad X = x / l, \quad S = l s / c, \quad t_0 = l / c, \quad l = l_1 + l_2 \end{aligned} \right\} \quad (3)$$

where, $i = 1, 2$ and ρ : mass density of fluid, ν : kinematic viscosity, c : wave propagation velocity, a_i : cross sectional area of pipe i , s : Laplace operator, I_0 , I_2 : modified Bessel functions.

The relation of γ_i , propagation constant, in Eq.(3) can be approximated in the form,⁽⁴⁾

$$\gamma_i \approx 1 + 1/\chi_i + 1/\chi_i^2 = 1 + \sqrt{D_i/S} + D_i/S \quad (4)$$

This can be derived under the assumption that $|\chi_i|$ is large, and it is widely known as the approximated relation introduced by Brown (henceforth, Brown's approximation).

2.1 Step response for single pipeline

The relevant equations to the series pipelines, Eqs.(1) to (4), become useful as the ones for the single pipe by eliminating the subscript i assigned to each variable in the equations. We can then derive the transfer functions as represented in Eqs.(5) and (6), which relate either the input variable P_i or Q_i with the output variable P_x (the pressure at the pipe coordinate x), respectively.

$$\frac{P_x}{P_i} = \frac{Z_c \sinh [(1-X)\Gamma] + Z_L \cosh [(1-X)\Gamma]}{Z_c \sinh \Gamma + Z_L \cosh \Gamma} \quad (5)$$

$$\frac{P_x}{Q_i} = \frac{Z_c \sinh [(1-X)\Gamma] + Z_L \cosh [(1-X)\Gamma]}{\cosh \Gamma + Z_L Z_c^{-1} \sinh \Gamma} \quad (6)$$

where, Z_L is the load impedance defined by $Z_L = P_2 / Q_2$. The wave reflection number defined by

$$R_L = (Z_L - Z_c) / (Z_L + Z_c) \quad (7)$$

is now introduced for the pipe-end posi-

tion ($X=1$). By using Eq.(7), the transfer functions of Eqs.(5) and (6) can be rewritten into simpler forms and expanded into infinite series.

$$\frac{P_x}{P_i} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k+l} R_L^{k+l-2} e^{-\lambda \tau} \quad (8)$$

$$\frac{P_x}{Q_i} = Z_c \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} R_L^{k+l-2} e^{-\lambda \tau} \quad (9)$$

where

$$\lambda = 2(k+l-2) + (-1)^{k+l} X \quad (10)$$

Introducing Brown's approximation Eq.(4) into Eqs.(8) and (9), we thus obtain the next equations.

$$\frac{P_x}{P_i} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k+l} R_L^{k+l-2} e^{-\lambda S} e^{-\lambda \sqrt{D_i/S}} e^{-\lambda D} \quad (11)$$

$$\frac{P_x}{Q_i} = \left(1 + \sqrt{\frac{D}{S}} + \frac{D}{S}\right) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} R_L^{k+l-2} e^{-\lambda S} e^{-\lambda \sqrt{D_i/S}} e^{-\lambda D} \quad (12)$$

where $\bar{Q}_i = (\rho c / a) Q_i$.

In Eqs.(11) and (12) each input variable of P_i and $\bar{Q}_i$ is taken equal to $1/S$, for deriving the step responses. If the values of R_L in these equations are real numbers such as 1 or -1, which corresponds with the pipe-end conditions of closed- or open-end respectively, it is possible to take the inverse Laplace transforms of Eq.(11) and (12) in which $P_i = 1/S$ or $\bar{Q}_i = 1/S$ is taken, in terms of the Laplace transform pairs generally used. As the result, the step response solutions P_x are obtained for each input variable of pressure and flow-rate.

(1) Step response solution for pressure input

$$P_x(\tau) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k+l} R_L^{k+l-2} e^{-\lambda D} \operatorname{erfc} \left(\frac{\lambda}{2} \sqrt{\frac{D}{\tau}} \right) u(\tau) \quad (13)$$

(2) Step response solution for flow-rate input

$$P_x(\tau) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} R_L^{k+l-2} e^{-\lambda D} \left\{ (2 - \lambda D) \sqrt{\frac{D\tau}{\pi}} \exp \left(-\frac{\lambda^2 D}{4\tau} \right) + \left[1 + D \left(\tau - \lambda + \frac{\lambda^2 D}{2} \right) \right] \operatorname{erfc} \left(\frac{\lambda}{2} \sqrt{\frac{D}{\tau}} \right) \right\} u(\tau) \quad (14)$$

(a) Single pipeline

(b) Series pipeline

Fig.1 Pipeline systems

where $\tau = T - \lambda$, $T = t/t_0$, t : time, $u(\tau)$: unit step function, erfc : complementary error function.

With respect to the solution represented by Eq.(14), Holmboe has also derived a similar one to Eq.(14) under the specific condition of $R_L = -1$, though it has a certain limitation with regard to the region of validity. Namely the response calculation is possible only within the first cycle of response, in other words, in the region of $T \leq 4$.

Compared to Holmboe's, the present solution has the feature that it is formulated as a relation without any limitation to which the Holmboe's is subjected.

In the following, the results of response curves calculated by Eqs.(13) and (14) are shown and compared with the ones obtained from the other method of calculation or experiment, in order to confirm their validity.

Firstly, in Fig.2, the response curve calculated from Eq.(13) is indicated (the solid line), and compared with the one calculated by the method of characteristics⁽²⁾ (the broken line). These calculations were made under the conditions of pressure input and of closed-end-pipe ($R_L = 1$).

In Fig.3, the calculated curve from Eq.(14) is indicated (broken line), compared with both the experimental result (solid line) and the calculated one given by Holmboe⁽⁹⁾ (a-dotted-chain line).

The corresponding condition for these is that of flow-input and of open-end-pipe. In the figure, the broken line curve is not found in the region $T \leq 4$,

because it almost coincides with the a-dotted-chain line curve. It is observed in the figure that the calculated value from Eq.(14) agrees well with the experimental one obtained by Holmboe, even in a fairly wide region beyond $T=4$, which is the least upper bound in Holmboe's solution.

From the investigation made so far, we can conclude that the present solutions, Eqs.(13) and (14) are valid well over a considerably wider time region, owing to the appropriateness of Brown's approximation Eq.(4) when the present solution is applied to the transient response analysis.

2.2 Step response for series pipelines

The relevant equations to the series pipelines shown in Fig.1(b) are Eqs.(1) to (3) and the relations,

$$Q_{12} = Q_{21}, \quad P_{12} = P_{21} \quad (15)$$

The above two equations have to be satisfied at the position at which the two pipes are connected to construct a series system.

In order to derive a transfer function in the relation written by P_{1x}/P_{11} , for example, as is seen in the preceding section, the two transfer matrix equations given as Eq.(1) taking $i=1$ and 2, (which corresponds to the matrices for the pipe l_1 and l_2 in Fig.1(b), respectively), are firstly combined with each other to satisfy the relation of Eq.(15). The next equation is obtained by using Eq.(2),

$$\frac{P_{1x}}{P_{11}} = \left\{ (Z_{c1} + Z_{c2}) \sinh \Gamma_{ax} + (Z_{c1} - Z_{c2}) \sinh \Gamma_{bx} + Z_L \left[\left(1 + \frac{Z_{c1}}{Z_{c2}} \right) \cosh \Gamma_{ax} + \left(1 - \frac{Z_{c1}}{Z_{c2}} \right) \cosh \Gamma_{bx} \right] \right\} / \left\{ (Z_{c1} + Z_{c2}) \sinh \Gamma_a + (Z_{c1} - Z_{c2}) \sinh \Gamma_b + Z_L \left[\left(1 + \frac{Z_{c1}}{Z_{c2}} \right) \cosh \Gamma_a + \left(1 - \frac{Z_{c1}}{Z_{c2}} \right) \cosh \Gamma_b \right] \right\} \quad (16)$$

$$\Gamma_a = \Gamma_1 + \Gamma_2, \quad \Gamma_b = \Gamma_1 - \Gamma_2, \quad \Gamma_{ax} = \Gamma_a - \Gamma_{1x}, \quad \Gamma_{bx} = \Gamma_b - \Gamma_{1x}, \quad Z_L = P_{22}/Q_{22} \quad (17)$$

The other transfer functions of P_{2x}/P_{11} , P_{1x}/Q_{11} and P_{2x}/Q_{11} can be derived as well and written by similar expressions to Eq.(16). Based upon thus obtained transfer functions, a discussion is made later about the series system with the pressure input and flow-rate input given.

2.2.1 Response for pressure input

The step response solution is derived in this section for the series system to which the pressure input P_{11} is given. As to the output variables for the system, the series pipeline has to be considered such that it has two kinds of pressures P_{1x} and P_{2x} , since the output position x can exist for $0 \leq x \leq l_1$ or $l_1 < x \leq l$. In the following, the response solutions are investigated based upon the transfer functions P_{1x}/P_{11} for $i=1$ and $i=2$, although the illustration is made mainly about the case of $i=1$.

Fig.2 Comparison of the analytical solution with the results of calculation by the method of characteristics

Fig.3 Comparison of the analytical solution with the results of calculation and measurement by Holmboe

Substitution of $Z_i = \infty$ into Eq.(16) yields the following transfer function for the closed pipe-end condition.

$$\frac{P_{1x}}{P_{11}} = \frac{(1+Z_{c1}/Z_{c2}) \cosh \Gamma_{ax} + (1-Z_{c1}/Z_{c2}) \cosh \Gamma_{bx}}{(1+Z_{c1}/Z_{c2}) \cosh \Gamma_a + (1-Z_{c1}/Z_{c2}) \cosh \Gamma_b} \quad (18)$$

The above equation expressed in Laplace S -domain has to be inverse Laplace transformed. The transformation can be made by changing Eq.(18) as follows.

Firstly, Eq.(18) can be rewritten into the next form by defining ϕ and ψ as the first and second terms in the denominator of Eq.(18), respectively, and then by expanding the term of $(\phi+\psi)^{-1}$ into the infinite series.

$$\frac{P_{1x}}{P_{11}} = \sum_{n=0}^{\infty} \sum_{k=1}^2 \sum_{l=1}^2 (-1)^{n-1} \frac{(e^{\Gamma_a} + e^{-\Gamma_a})^{n-1}}{(e^{\Gamma_a} + e^{-\Gamma_a})^n} \frac{(1-Z_{c1}/Z_{c2})^{n+k-2}}{(1+Z_{c1}/Z_{c2})^{n+k-2}} e^{(-1)^k [\Gamma_{1x} - \Gamma_1 + (-1)^k \Gamma_1]} \quad (19)$$

Further the two terms given in fractions in Eq.(19) are also expanded into series by following each step below.

With Y_1 defined as the first fraction term, that is

$$Y_1 = \frac{(e^{\Gamma_a} + e^{-\Gamma_a})^{n-1}}{(e^{\Gamma_a} + e^{-\Gamma_a})^n} \quad (20)$$

Y_1 can be expanded into the following expression.

$$Y_1 = \sum_{m=1}^n \sum_{l=1}^2 (-1)^{l-1} \frac{(n+l-2)!}{(m-1)!(n-m)!(l-1)!} e^{-[(2m+2l-3)\Gamma_1 + (2n-2m+2l-1)\Gamma_2]} \quad (21)$$

For the convenience of later discussion, the second term Y_2 is defined by introducing the integers ξ_i and η_i as

$$Y_2 = \frac{(1-Z_{c1}/Z_{c2})^{\xi_i}}{(1+Z_{c1}/Z_{c2})^{\eta_i}} = \frac{(1-A_r \gamma_1/\gamma_2)^{\xi_i}}{(1+A_r \gamma_1/\gamma_2)^{\eta_i}} \quad (22)$$

where A_r is the ratio of cross sectional areas denoted by $A_r = a_2/a_1$. The integers ξ_i and η_i for the present case ($i=1$) take the next values (c.f., (Eq.19)).

$$\xi_1 = \eta_1 = n+k-2 \quad (23)$$

Expanding Eq.(22) into series after substituting Eq.(4) for γ_1, γ_2 into Eq.(22), and neglecting the smaller terms of order than $1/S$, we have

$$Y_2 \approx U_i - V_i \sqrt{S} - W_i/S \quad (24)$$

where

$$\left. \begin{aligned} U_i &= E^{\eta_i}/F^{\eta_i}, \quad V_i = G(\xi_i F + \eta_i E) E^{\eta_i-1}/F^{\eta_i+1} \\ W_i &= \left\{ \xi_i F^2 \left[EH - \frac{G^2}{2} (\xi_i - 1) \right] - \xi_i \eta_i EFG^2 + \eta_i E^2 \left[FH - \frac{G^2}{2} (\eta_i + 1) \right] \right\} \frac{E^{\eta_i-2}}{F^{\eta_i+2}} \\ E &= 1 - A_r, \quad F = 1 + A_r, \quad G = A_r(\sqrt{D_1} - \sqrt{D_2}), \quad H = A_r(D_1 - \sqrt{D_1 D_2}) \end{aligned} \right\} \quad (25)$$

With Y_1 defined as the product of the two terms written as exponential functions in Eqs.(19) and (21), Y_1 can be approximated as follows by introducing Eq.(4) and also by applying the same approximation method as the one for the derivation of Eq.(24).

$$Y_1 \approx e^{(-1)^k [\Gamma_{1x} - \Gamma_1 + (-1)^k \Gamma_1]} e^{-(2m+2l-3)\Gamma_1 + (2n-2m+2l-1)\Gamma_2} \approx e^{-(f_i S + g_i \sqrt{S} + h_i)} \quad (26)$$

where

$$f_i = M_i + N_i, \quad g_i = M_i \sqrt{D_1} + N_i \sqrt{D_2}, \quad h_i = M_i D_1 + N_i D_2 \quad (27)$$

and the relations of M_i, N_i for $i=1$ are given by

$$M_1 = [2(l+m)-3+(-1)^k]L_1 - (-1)^k X, \quad N_1 = [2(l-m+n)-1-(-1)^{k+k}]L_2 \quad (28)$$

Introducing the relations so far obtained, Eqs.(21), (24) and (26) into Eq.(19), we finally get the resultant equation from Eq.(19), written as,

$$\frac{P_{1x}}{P_{11}} = \sigma_i \sum_{n=1}^{\infty} \sum_{k=1}^2 \sum_{l=1}^2 \sum_{m=1}^n \sum_{l=1}^2 (-1)^{\eta_i} \frac{(n+l-2)!}{(m-1)!(n-m)!(l-1)!} \left(U_i - \frac{V_i}{\sqrt{S}} - \frac{W_i}{S} \right) \times \exp[-(f_i S + g_i \sqrt{S} + h_i)] \quad (29)$$

where σ_i, ζ_i , for $i=1$ are given by

$$\sigma_1 = 1 \dots\dots\dots (30)$$

$$\zeta_1 = n+1 \dots\dots\dots (31)$$

The above method to obtain Eq.(29), that is the case of $i=1$, is also applicable to the corresponding analysis in the case of $i=2$, and the resultant equations can be given as illustrated in the following.

In the case of $i=2$ the corresponding equation to Eq.(29) can be written as an equation in which the symbol $\sum_{n=1}^{\infty}$ is omitted from Eq.(29). For the convenience of later discussion, the resultant equation for $i=2$ is thus assumed to coincide with Eq.(29), supposing that the symbol $\sum_{n=1}^{\infty}$ in Eq.(29) is omitted when i is 2. In regard to the variables M_i, N_i and so on for that equation, they differ from the foregoing ones. They are given as

$$\left. \begin{aligned} M_2 &= [2(l+m)-3]L_1 \\ N_2 &= [2(l-m+n)-1]L_2 + (-1)^K(1-X) \\ \xi_2 &= n-1, \eta_2 = n, \sigma_2 = 2 \\ \zeta_2 &= n+1 \end{aligned} \right\} \dots\dots\dots (32)$$

$$\dots\dots\dots (33)$$

The analytical method used in the preceding under the condition of $Z_L = \infty$ (closed-end pipe) is also applicable to analyses under the condition of $Z_L = 0$ (open-end pipe). Here, the corresponding result which is different from the preceding ones is presented only by the next relations.

$$\zeta_1 = k+K+m-1, \zeta_2 = K+m \dots\dots\dots (34)$$

It has thus been made clear that Eq.(29) is just the relation that meets satisfactorily all the conditions treated so far, irrespective of $i=1$ or 2 and also the pipe-end condition of $Z_L=0$ or $Z_L=\infty$.

Comparing Eq.(29) with Eq.(12), we recognize that they are represented fundamentally by the same relations, as far as the terms comprising Laplace variable S are concerned. Thus the equation, Eq.(29), can similarly be inverse Laplace transformed under the condition of $P_1 = 1/S$ and consequently it leads to a step response solution as follows.

$$P_{ix}(\tau_i) = \sigma_i \sum_{n=1}^{\infty} \sum_{k=1}^2 \sum_{m=1}^2 \sum_{l=1}^2 (-1)^{n+l} \frac{(n+l-2)!}{(m-1)!(n-m)!(l-1)!} \frac{E^m}{F^m} e^{-\sigma_i \tau_i} \times \left[b_i \sqrt{\frac{D_1 \tau_i}{\pi}} \exp\left(-\frac{D_1 \beta_i^2}{4\tau_i}\right) + c_i \operatorname{erfc}\left(\frac{\beta_i}{2} \sqrt{\frac{D_1}{\tau_i}}\right) \right] u(\tau_i) \dots\dots\dots (35)$$

$$\left. \begin{aligned} w_i &= (\xi_i F^2 - \eta_i E^2) - (\xi_i F + \eta_i E)[(\xi_i F + \eta_i E) + 2B] \\ \tau_i &= T - (M_i + N_i), \quad a_i = M_i + N_i/A_r, \quad \beta_i = M_i + N_i/\sqrt{A_r} \\ B &= (1 + \sqrt{A_r})(\sqrt{A_r} + 1/\sqrt{A_r}), \quad E = 1 - A_r, \quad F = 1 + A_r \end{aligned} \right\} \dots\dots\dots (36)$$

$$\left. \begin{aligned} b_i &= [2(\xi_i F + \eta_i E) + D_1 \beta_i w_i / (2B)] / B \\ c_i &= 1 - D_1 B_i (\xi_i F + \eta_i E) / B - D_1 w_i (\tau_i + D_1 \beta_i^2 / 2) / (2B^2) \end{aligned} \right\} \dots\dots\dots (37)$$

It has to be noted that the symbol $\sum_{n=1}^{\infty}$ in Eq.(35) is omitted when i is 2, in accordance with the assumptions made in the present analysis.

2.2.2 Response for flow-rate input

In this section, a similar investigation to the preceding one is made about the flow-rate input case under the pipe-end condition of $Z_L = 0$ only, because of its practical importance. The transfer function for it can be derived as follows, for the case of $i=1$, in the same way as when obtaining Eq.(18).

$$\frac{P_{ix}}{Q_{11}} = \frac{Z_{c1}[(1+Z_{c1}/Z_{c2}) \sinh \Gamma_{ax} - (1-Z_{c1}/Z_{c2}) \sinh \Gamma_{bx}]}{(1+Z_{c1}/Z_{c2}) \cosh \Gamma_a - (1-Z_{c1}/Z_{c2}) \cosh \Gamma_b} \dots\dots\dots (38)$$

It is known from the comparison of Eq.(38) with Eq.(18) that the most significant difference between them is the existence of Z_{c1} appearing in the numerator of Eq.(38). It can be proved that the above difference produces Eq.(29) when we merely replace the coefficients U_i, V_i and W_i with somewhat different expressions. The corresponding new coefficients $\bar{U}_i, \bar{V}_i$ and $\bar{W}_i$ for Eq.(38) can be written as

$$\left. \begin{aligned} \bar{U}_i &= U_i, \quad \bar{V}_i = V_i - \sqrt{D_1} U_i, \\ \bar{W}_i &= W_i + \sqrt{D_1} V_i - D_1 U_i \end{aligned} \right\} \dots\dots\dots (39)$$

For this reason, we can regard the equation for pressure-input Eq.(29) as also the one for the flow-rate input, replacing the coefficients $\bar{U}_i, \bar{V}_i, \bar{W}_i$ in Eq.(29) with ones in Eq.(39) and simulta-

neously replacing the left-hand side in Eq.(29), P_{ix}/P_{ii} , with P_{ix}/Q_{ii} [where $Q_{ii} = (Q_{ii}/\rho c/a_i)$]. It is certain now that the step response solution for flow-rate input of $Q_{ii}=1/S$ is also given by Eq.(35), by the inverse Laplace transformation of Eq.(29). All of the previously defined parameters or coefficients can be commonly used as the ones for Eq.(35), except the quantities shown in the next.

$$\left. \begin{aligned} b_i &= \{D_i \beta_i [w_i / (2B) - v_i] + 2v_i\} / B \\ c_i &= 1 - D_i \{B_i v_i + [w_i / (2B) - v_i] (\tau_i + D_i \beta_i^2 / 2)\} / B \\ v_i &= \xi_i F + \eta_i E + B, \quad \xi_i = k + K + I - 1, \quad \eta_i = K + I \end{aligned} \right\} \dots\dots\dots (40)$$

At this stage with all the solutions obtained, it may be worthwhile to summarize the results, after rearranging them in accordance with each pipe-end condition of $Z_i = 0$ or $Z_i = \infty$, and each input variable of P_{ii} or Q_{ii} .

1. Equation (35) is the response solution for all of the pipe-end conditions and input variables treated here, when it is supposed that the symbol $\sum_i$ in the equation is omitted when i is 2. Similarly Eqs.(23),(28),(30),(32),(36) are the relations for every condition of Z_i and input variable.

2. In addition, the relations shown in the following item of either (2.a) or (2.b) are necessary for the pressure-input case or for the flow-rate, respectively.

(2.a). Additional equations in pressure-input case

Eqs.(31),(33),(37); when $Z_i = \infty$

Eqs.(34),(37) ; when $Z_i = 0$

(2.b). Additional equation in flow-rate input case

Eq.(40); when $Z_i = 0$

3. Comparison with Experiment

3.1 The method of experiment

The arrangement of experimental apparatus is schematically shown in Fig.4, corresponding to the case when the pressure-input is chosen for the input. A step wise change in input pressure or volumetric flow rate was produced by controlling the quick response directional control valve mounted at the sending- or the receiving-end of the test pipe, corresponding to the response test for pressure-input or flow-rate input, respectively. For the measurement of responses to the above inputs, the pressures at arbitral pipe positions were picked up by

the pressure transducers and recorded on an oscillograph film after once being memorized in transient memory. The experimental conditions and the dimensions of test pipes (single- and series-pipes) are summarized in Table 1. An example of oscillograms recorded at the run-1 experiment (i.e., pressure input case, Table 1) is shown in Fig.5. The oscillograph record appearing in the lower side of Fig.2 indicates the pressure measured at the sending-end of pipe and shows that the input wave-form represents a fairly good step-wise change. The record in the upper indicates the pressure response caused by the input change. By using such measured response curve, comparisons with the theoretical ones are made and the results are presented in the next section.

3.2 Result of experiment

The results to be compared are shown in the following in accordance with the measurements in the run-1 to 6, Table 1. Firstly, three comparisons under the same pressure-input condition are made in Figs.6 (a) to (c). The value of $A_i (= a_2/a_1)$, the ratio of pipe cross-sectional areas, is varied in three levels. The solid line and the broken line in the figures represent the measured and calculated values, respectively. Secondly,

Fig.4 Experimental apparatus (pressure input case)

Fig.5 An example of oscillogram record

Table 1

Input and pipe-end	Magnitude of input	Radius cm		$Ar = a_2/a_1$	length m			$Lr = l_2/l_1$	v m ² /s	c m/s	$D = \nu t_0/r_1^2$	$X = x/l$	No
		r_1	r_2		l_1	l_2	l						
Pressure input, closed-end	$P_{11}=0.25$ MPa	0.46	0.63	1.90	4.36	6.30	10.7	1.45	0.61×10^{-4}	1300	0.024	0.779	1
	$P_{11}=0.25$	0.46		1			11.0		0.48	1300	0.019	1.00	2
	$P_{11}=0.27$	0.63	0.46	0.53	6.31	4.35	10.7	0.689	0.53	1200	0.012	0.219	3
Volumetric flow input, open-end	$q_{11}=1.60$ L/min	0.46	0.63	1.90	4.37	6.83	11.2	1.56	0.54	1290	0.022	0.745	4
	$q_{11}=1.11$	0.46		1			11.0		0.38	1290	0.015	0.50	5
	$q_{11}=1.50$	0.63	0.46	0.53	6.85	4.35	11.2	0.635	0.64	1390	0.013	0.257	6

the comparison made under the same flow-rate input condition are made in Figs.7 (a) to (c).

From the observation of Figs.6 and 7, it is clear that the experimental value in each run well agrees with the calculated one, within the range of values A_r investigated in this experiment, i.e., about $0.5 < A_r < 2$. The validity of the theoretical analysis, therefore, has been confirmed, at least within this range of A_r .

4. Concluding remarks

Each pressure response caused by step input of flow-rate or pressure was investigated both theoretically and experimentally using two types of pipe systems, a single pipeline and a series one. It is verified that the response solutions derived by introducing Brown's approximation well agree with the experiment.

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(a) $A_r=1.9$

(b) $A_r=1$

(c) $A_r=0.53$

Fig.6 Comparisons in pressure input case

(a) $A_r=1.9$

(b) $A_r=1.0$

(c) $A_r=0.53$

Fig.7 Comparison in flow-rate input case