Torsion of a Conical-form Rigid Cylinder Embedded in an Infinite Elastic Thick Plate*

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The problem of torsion of a conical-form rigid cylinder embedded in an elastic half space has been investigated in a previous paper. The problem may be regarded as an analytical model of stress analysis in drilling. The principal object of this paper is an application of the problem of a half space to one of an infinite thick plate.

The method of solution used here is to distribute torsional body forces in a thick plate so as to satisfy the boundary conditions of a conical hole of an infinite elastic thick plate. For this purpose, we apply Green's functions for torsional body force problems of an infinite elastic thick plate. Stresses and displacements of the thick plate with a conical hole and torques necessary to twist the conical-form rigid cylinder are shown graphically and compared with the results about an elastic half space.

Key Words: Elasticity, Torsion, Thick Plate, Conical Hole, Unpenetrated Hole, Body Force, Green's Function, Stress Analysis, Numerical Analysis

1. Introduction

This paper concerns a three dimensional elasticity problem of a thick plate subjected to torsional displacements by a conical-form rigid cylinder embedded into a conical hole. The problem may be regarded as an analytical model of stress analysis in drilling. Here we consider only the torsional stresses of the problem because the stresses due to an indentation force and the stresses due to a torsional force can be treated separately in axisymmetric deformations of the linear theory of elasticity.

The fundamental principle of the method of solution used here is to distribute torsional body forces in the interior of a thick plate with no hole and to determine the magnitudes of the body forces distributed so as to satisfy the boundary conditions at a conical hole imagined in the thick plate. There are no body forces in the thick plate with a conical hole because we distribute body forces only in a region which is removed from the thick plate as a conical hole. In order to obtain an integral equation for the problem, we apply an analytical solution (Green's functions for torsional body force problems of a thick plate)⁽¹⁾ for a thick plate subjected to a torsional body force acting on a circle. That is, the present problem considered here is formulated by an integral equation with an unknown function corresponding to magnitudes of the body forces distributed. The method of solution is based in principle on the so-called body force method⁽²⁾ and also

may be regarded as a kind of the so-called indirect boundary element method with fictitious boundary.⁽³⁾⁽⁴⁾

Numerical calculations are performed. The distributions of stresses and displacements of the thick plate with a conical hole and torques necessary to twist the conical-form rigid cylinder are shown graphically and compared with the results about an elastic half space.

2. Boundary Conditions

Consider an infinite elastic thick plate with the thickness H subjected to torsional displacements by a conical form rigid cylinder embedded into a conical hole as shown in Fig.1. Here we used cylindrical coordinates (r, θ, z) and notations C , b and 2α are the maximum radius, the depth and the taper angle of a conical hole of the thick plate, respectively. It is assumed that the conical form rigid cylinder is inserted into the bottom of the conical hole and perfectly bonded to the surface of the conical hole and that both the surfaces of the thick plate are free of applied surface forces. When the conical form rigid cylinder is rotated through an angle ω about the z -axis and the thick

Fig.1 A thick plate subjected to torsion by a conical form rigid cylinder.

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plate is rigidly fixed at infinity ($r \rightarrow \infty$), the boundary conditions of the problem are expressed as follows :

- (i) surface of a conical hole
 $(r=r_1, z=z_1); u_2=\omega r$,
 (ii) surfaces of the plate
 $(z=0 \text{ and } z=H); \tau_{z\theta}=\tau_{zr}=\sigma_z=0$,
 (iii) $r \rightarrow \infty; u_2=0$ (1)

where u_2 is torsional displacement about the z -axis. It must be noted that the displacements and stresses of the problem are always $u_1=u_3=0, \sigma_r=\sigma_\theta=\sigma_z=\tau_{rz}=0$, and the nonvanishing components of stresses and displacements are $\tau_{r\theta}, \tau_{z\theta}$, and u_2 . The relations between the coordinates (r_1, z_1) of a point on the surface of a conical hole and the distance s of the point (r_1, z_1) from the apex of a conical hole are expressed as follows :

$$s = \sqrt{r_1^2 + [z_1 - (H-b)]^2}$$

$$r_1 = [z_1 - (H-b)] \tan \alpha \quad \text{..... (2)}$$

3. Method of Solution

In order to obtain a solution to the problem mentioned above, we shall apply Green's functions for torsional body force problems of a thick plate. Green's functions used here are defined as a solution to the problem of a thick plate satisfying the boundary conditions

$$z=0 \text{ and } +H, 0 \leq r < \infty;$$

$$\sigma_z=\tau_{zr}=\tau_{z\theta}=0 \quad \text{..... (3)}$$

and subjected to the torsional body force

$$F_2 = \frac{1}{2\pi r} \delta(r-a) \delta(z-h) \quad \text{..... (4)}$$

acting on a circle ($r=a, z=h$) in the interior of the thick plate as shown in Figs.2 and 3, and are expressed as follows ⁽¹⁾ :

$$\left. \begin{aligned} u_2' &= \frac{1}{4\pi^2 \mu \sqrt{ar}} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} Q_{1/2}(x_k) \\ \tau_{r\theta}' &= \frac{1}{8\pi^2 r \sqrt{ar}} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{x_k^2 - 1} [x_k Q_{1/2}(x_k) \right. \\ &\quad \left. - Q_{-1/2}(x_k)] \left(\frac{r}{a} - x_k \right) - 3Q_{1/2}(x_k) \right\} \\ \tau_{z\theta}' &= \frac{1}{8\pi^2 ar \sqrt{ar}} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{z_k}{x_k^2 - 1} [x_k Q_{1/2}(x_k) \\ &\quad - Q_{-1/2}(x_k)] \end{aligned} \right\} \quad \text{..... (5)}$$

where $\delta()$ is a Dirac delta function, μ is the shear modulus of elasticity, $Q_\nu(x_k)$ is a Legendre function of the second kind of order ν , and x_k represents

Fig. 2 A thick plate subjected to a torsional body force acting on a circle.

$$x_k = \frac{r^2 + a^2 + z_k^2}{2ar} \quad \text{..... (6)}$$

provided that

$$\left. \begin{aligned} z_1 &= z - 2(m-1)H + (-1)^m h \\ z_2 &= z + 2mH - (-1)^m h \\ z_3 &= z - 2mH + (-1)^m h \\ z_4 &= z + 2(m-1)H - (-1)^m h \end{aligned} \right\} \quad \text{..... (7)}$$

Figure 4 shows a thick plate with no hole. In this figure, the two-dot-chain line represents a plane which is to become a surface of a conical hole, the dotted line is a plane of distribution of body forces, and the notation ε is a distance between these two planes. When the torsional body forces of Eq.(4) are distributed on the dotted line, the stresses and displacements due to the body forces distributed are expressed in terms of Green's functions of Eq.(5) as follows :

$$\left\{ \begin{aligned} u_2 \\ \tau_{r\theta} \\ \tau_{z\theta} \end{aligned} \right\} = \int_0^{s'} f(\xi) \left\{ \begin{aligned} u_2' \\ \tau_{r\theta}' \\ \tau_{z\theta}' \end{aligned} \right\} d\xi \quad a=a(\xi), h=h(\xi) \quad \text{..... (8)}$$

where $f(\xi)$ is an unknown function corresponding to the magnitudes of the body forces, ξ , ($0 \leq \xi \leq s'$), is a distance from the apex ($r=0, z=H-b+\varepsilon/\sin\alpha$) to a point on the conical plane (dotted line), s' is a length of the dotted line and $a(\xi)$ and $h(\xi)$ represent

$$\left. \begin{aligned} a(\xi) &= \xi \sin \alpha \\ h(\xi) &= H - b + \varepsilon/\sin \alpha + \xi \cos \alpha \end{aligned} \right\} \quad \text{..... (9)}$$

From the definition of Green's functions, Eq.(8) satisfies the boundary conditions (ii) and (iii) in Eq.(1) regardless of

Fig.3 A torsional force acting on a circle.

Fig.4 A thick plate with a conical hole and the plane of distribution of body forces.

the value of $f(\xi)$. Therefore, if we choose $f(\xi)$ so as to satisfy the boundary conditions (i) in Eq.(1), then Eq.(8) satisfies all the conditions of Eq.(1). Application of the boundary condition (i) of Eq.(1) to Eq.(8) yields

$$\omega r_i = \int_0^{s'} f(\xi) (u_2')_{a=a(\xi), h=h(\xi)} d\xi \quad (r=r_i, z=z_i) \quad (10)$$

Expression (10) is an integral equation with an unknown function $f(\xi)$. To reduce the integral equation to a set of linear algebraic equations, we perform appropriate discretizations of the distributed body forces $f(\xi)$. For this purpose, we divide the plane ($0 \leq \xi \leq s'$) of distribution of body forces into N intervals

$$\Delta \xi_n = \xi_n - \xi_{n-1} \quad (0 \leq \xi_n \leq s'; n=1 \sim N, \xi_0=0, \xi_N=s') \quad (11)$$

and assume that the magnitudes of the body forces distributed are constants in all intervals divided. That is, we assume that the body forces are N piecewise constant forces as follows:

$$f(\xi) = \sum_{n=1}^N f_n [S_t(\xi - \xi_{n-1}) - S_t(\xi - \xi_n)] \quad (12)$$

where $S_t(\cdot)$ is Heaviside's step function and f_n stand for N unknown constants corresponding to the magnitudes of the body forces distributed.

Substitution of $f(\xi)$ of Eq.(12) into Eq.(8) yields

$$\omega r_i = \sum_{n=1}^N f_n \int_{\xi_{n-1}}^{\xi_n} (u_2')_{a=a(\xi), h=h(\xi)} d\xi \quad (r=r_i, z=z_i) \quad (13)$$

Since there are N unknown constants f_n in Eq.(13), we shall consider N points ($r=r_i, z=z_i; i=1 \sim N$) on the plane (two-dot-chain line in Fig.4) which is to become a surface of a conical hole, and to determine the values of f_n so as to satisfy the relation of Eq.(13) at these selected points

$$r = r_i = [z_i - (H-b)] \tan \alpha, \quad z = z_i, \quad (i=1 \sim N; 0 < r_i < C, H-b < z_i < H) \quad (14)$$

It follows from the above that we can construct a set of simultaneous linear algebraic equations. Solving the equations and substituting their solutions f_n into the expressions

$$\begin{Bmatrix} u_2 \\ \tau_{r\theta} \\ \tau_{z\theta} \end{Bmatrix} = \sum_{n=1}^N f_n \begin{Bmatrix} (u_2')^n \\ (\tau_{r\theta}')^n \\ (\tau_{z\theta}')^n \end{Bmatrix} \quad (r=r_i, z=z_i) \quad (15)$$

then stresses and displacements at any point of the thick plate with a conical hole can be obtained. In Eq.(15), $(u_2')^n$ and so on represent displacements and stresses at a

point (r, z) due to body forces acting on the interval $\Delta \xi_n$ and are expressed as follows:

$$\begin{Bmatrix} (u_2')^n \\ (\tau_{r\theta}')^n \\ (\tau_{z\theta}')^n \end{Bmatrix} = \int_{\xi_{n-1}}^{\xi_n} \begin{Bmatrix} u_2' \\ \tau_{r\theta}' \\ \tau_{z\theta}' \end{Bmatrix}_{a=a(\xi), h=h(\xi)} d\xi \quad (16)$$

The torque T necessary to twist the conical form rigid cylinder through an angle ω about the z -axis is obtained from the expression

$$T = \sum_{n=1}^N \bar{f}_n a_n \quad (17)$$

provided that

$$\left. \begin{aligned} \bar{f}_n &= f_n \Delta \xi_n \\ a_n &= (\xi_n + \xi_{n-1}) \sin \alpha / 2 \end{aligned} \right\} \quad (18)$$

where $\bar{f}_n$ is a torsional force acting on the interval $\Delta \xi_n$ and a_n is a distance from the z -axis to a midpoint $(\xi_n + \xi_{n-1})/2$ on the interval $\Delta \xi_n$. A proof of the relation (17) is given in Appendix of the paper.

4. Numerical Results

Based on the method of solution described above, numerical calculations were performed for several values of H/c under the assumption of $\alpha=60^\circ$. In the numerical calculations, we used the values of $N=60$ and $\xi/c=0.0002$. The integrals in Eq.(16) were calculated by Simpson's rules of integration. The results are shown in Figs.5-9. The blank circles in these figures represent the results⁽⁵⁾ for the case of an elastic half space ($H/c \rightarrow \infty$).

Figure 5 shows the distributions of torsional stresses $\tau_{\alpha\theta}$ transmitted to the surface of a conical hole from a conical form rigid cylinder. The notation s in abscissa represents a point (r, z) on the surface of a conical hole defined by Eq.(2). From the figure, we see the following facts:

(1) The torsional stresses $\tau_{\alpha\theta}$ become larger as H is larger, but the stresses for the case of $H/c > 2.0$ have almost same distributions regardless of the value H and agree with the results about a half space. That is,

Fig.5 Distributions of contact stresses $\tau_{\alpha\theta}$ transmitted to the surface of a conical hole.

the stresses obtained by the assumption of a half space represent an upper limit to the stresses of a thick plate.

(2) The stresses $\tau_{\alpha\theta}$ have a singularity at the vicinity of the edge ($z=H$, $r=c$) of a conical hole.

Figure 6 shows the distributions of displacements u_2 for the cases of $H/c=2.0$ (solid line), 1.0(dotted line) and 0.6(dot-chain-line). The curved lines of $z/H=0$ and 1.0 are distributions on the surfaces of a thick plate and the curved lines of $z/H=1.0-1.0/\sqrt{3.0}$ are distributions on the plane $z=H-b$ in a thick plate. The displacements u_2 become larger as the thickness H decreases, and approach the results about a half space with an increase of H . That is, the displacements u_2 for a half space represent a lower limit to u_2 of a thick plate.

Figure 7 shows the distributions of stresses $\tau_{r\theta}$ for the cases of $H/c=2.0$ (solid line), 1.0(dotted line) and 0.6(dot-chain-line). It must be said that the distributions of $\tau_{r\theta}$ on the planes $z/H=0$ and $1.0-1.0/\sqrt{3.0}$ for the case of $H/c=0.6$ coincide with each other. From the figure, we see the following facts :

(1) The stresses $\tau_{r\theta}$ have a singularity at the vicinity of the edge ($z=H$, $r=c$) of a conical hole.

(2) Roughly speaking, the stresses $\tau_{r\theta}$ become larger and approach a uniform distribution in the z -direction as the thickness H decreases. In the region of $r/c>3.0$ the

stresses $\tau_{r\theta}$ are almost uniformly distributed in the z -direction.

(3) The stresses $\tau_{r\theta}$ are ordinarily larger than those of a half space, but it seems that in the case of $H/c>2.0$ the stresses for a thick plate and a half space coincide with each other. That is, the stresses for a half space represent a lower limit to the stresses for a thick plate.

Figure 8 shows the distributions of stresses $\tau_{z\theta}$ on the plane $z=H-b$ for the cases of $H/c=2.0$, 1.0 and 0.6. The stresses become larger and approach the results about a half space as H increases. That is, the stresses for a half space represent the upper limit to the stresses for a thick plate. The stresses decrease rapidly as H decreases, and they are almost equal to 0 regardless of H in region of $r/c>3.0$.

Figure 9 shows the relations between the torque T required to twist the rigid cylinder and the thickness H of a thick plate. The dotted line in the figure represents the results about a half space. The torque T is smaller than the torque T_∞ for a half space and approach T_∞ with an increase of H . In the cases of $H/c>2.5$, T and T_∞ are almost equal to each other.

Fig.6 Distributions of displacements u_2 .

Fig.7 Distributions of stresses $\tau_{r\theta}$.

Fig.8 Distributions of stresses $\tau_{z\theta}$ on the plane $z=H-b$.

Fig.9 Torques T to twist a conical form rigid cylinder.

5. Conclusions

The principal results obtained in this paper are summarized as follows :

(1) A method of solution is proposed for the problem of an elastic thick plate subjected to torsional displacements by a conical form rigid cylinder embedded into a conical hole, applying Green's functions for torsional body force problems of an elastic thick plate.

(2) The distributions of the stresses and displacements in the thick plate with a conical hole were shown and compared with the results about an elastic half space.

(3) The results about an elastic half space are the upper limits to (i) the contact stresses transmitted from a rigid cylinder to the surface of a conical hole, (ii) the stresses $\tau_{z\theta}$ and (iii) the torques T required to twist a rigid cylinder, and also the lower limits for (i) the displacements u_z and (ii) the stresses $\tau_{r\theta}$.

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Appendix

Here we shall show a proof of Eq.(17). For any value of R under the condition $c < R$, the torque T is expressed by

$$T = 2\pi R^2 \int_0^H (\tau_{r\theta})_{r=R} dz \quad (19)$$

By the way, when $x \rightarrow \infty$ the following relations hold :

$$Q_{1/2}(x) = \frac{\pi}{4x\sqrt{2x}}, \quad Q_{-1/2}(x) = \frac{\pi}{\sqrt{2x}} \quad (20)$$

Therefore, when $R \rightarrow \infty$, $\tau_{r\theta}'$ in Eq.(5) becomes

$$\begin{aligned} (\tau_{r\theta}')_{r=R} = & -\frac{3}{32\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{R^2 + a^2 + z_n^2}} \right. \\ & \times \left[\frac{R-a}{(R-a)^2 + z_n^2} - \frac{R+a}{(R+a)^2 + z_n^2} \right. \\ & \left. \left. + \frac{2a}{R^2 + a^2 + z_n^2} \right] \right\} \quad (21) \end{aligned}$$

Substituting Eq.(21) into Eq.(19) and integrating Eq.(19) with respect to z , we can obtain⁽⁶⁾

$$T = \sum_{n=1}^N f_n \int_{\xi_{n-1}}^{\xi_n} [G(H) - G(0)] d\xi \quad (22)$$

where

$$\begin{aligned} G(z) = & -\frac{3}{16} R^2 \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{2aR}} \tan^{-1} \right. \\ & \left(\frac{z_n}{R-a} \sqrt{\frac{2aR}{R^2 + a^2 + z_n^2}} \right) - \frac{1}{2\sqrt{aR}} \log \left| \frac{1+y}{1-y} \right| \\ & \left. + \frac{2az_n}{(R^2 + a^2)\sqrt{R^2 + a^2 + z_n^2}} \right\} \quad (23) \end{aligned}$$

$$y = \frac{z_n \sqrt{2aR}}{(R+a)\sqrt{R^2 + a^2 + z_n^2}} \quad (24)$$

When $|\beta| \ll 1.0$, we have

$$\left. \begin{aligned} \tan^{-1} \beta &= \beta - \frac{1}{3} \beta^3 \\ \log \frac{1+\beta}{1-\beta} &= 2 \left(\beta + \frac{1}{3} \beta^3 \right) \end{aligned} \right\} \quad (25)$$

Therefore, when $R \rightarrow \infty$, $G(z)$ of Eq.(23) becomes

$$\begin{aligned} G(z) = & -\frac{3}{4} a \sum_{n=1}^{\infty} \left[\frac{z_n}{\sqrt{R^2 + a^2 + z_n^2}} \right. \\ & \left. \times \left(1 - \frac{1}{3} \frac{1}{1 + z_n^2/r^2} \right) \right] \quad (26) \end{aligned}$$

On the other hand, the infinite series

$$\left. \begin{aligned} \sum_{n=1}^{\infty} \left(\frac{z_n}{\sqrt{R^2 + a^2 + z_n^2}} \right) \\ \sum_{n=1}^{\infty} \left(\frac{z_n}{\sqrt{R^2 + a^2 + z_n^2}} \frac{1}{1 + z_n^2/r^2} \right) \end{aligned} \right\} \quad (27)$$

converge to 2.0 for $z=H$ and to 0 for $z=0$ respectively. Hence, we get

$$G(H) = -a, \quad G(0) = 0 \quad (28)$$

Substituting Eq.(28) into Eq.(22) and integrating Eq.(22) with respect to ξ with the aid of Eq.(9), we can obtain

$$T = -\frac{\sin \alpha}{2} \sum_{n=1}^N f_n (\xi_n^2 - \xi_{n-1}^2) \quad (29)$$

It is easily seen that T of Eq.(29) coincides with T of Eq.(17).

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