Coupled Vibrations between Railway Vehicle Wheels and a Rail^{*} (2nd Report; Axial Steady-state Vibration of a Rotating Wheel)

By Takashi AYABE^{**}, Atsuo SUEOKA^{***} and Hideyuki TAMURA^{***}

Rolling noise arises from the vibration due to the interaction between wheels and rail through the contact points. The authors treated the vibration of a rail in the horizontal and vertical directions in the previous papers. In this report, the axial vibration of a rotating wheel, at the circumference of which shear force and spin moment from Kalker's linear creep theory act simultaneously, is formulated in order to consider the coupled vibration between wheels and rail in the horizontal direction. It is shown that the response of the steady-state vibration is obtained by solving the simultaneous algebraic equations with respect to the contact forces. As a verification of the present analytical method, the theoretical and experimental impedance characteristics of a stationary wheel suspended with springs at a point on the outer circumference are compared.

Key Words : Vibration of Continuous System, Railway, Wheels and Rail, Creep Force, Coupled Vibration

1. Introduction

It is important to elucidate the contribution of the respective vibrations of the wheels and the rail to the rolling noise which occurs in the travelling motion of railway vehicles on a straight track like the Shinkansen. Arai et al.⁽¹⁾⁽²⁾ pointed out from the velocity spectrum in the vertical direction of the rail measured in the Shinkansen that the noise is caused by the vibration of the rail rather than that of the wheels. However the individual effect of the vertical and the horizontal vibrations of the rail upon the noise is not clear. The noise can not be explained by only the individual vibration characteristics of wheel and rail but it arises from the complicated interaction of the vibration characteristics between rail and wheels through the points of contact. It is necessary to compare the vibration levels of the coupled vibrations in the horizontal and the vertical directions in order both to elucidate the origin of this noise theoretically and to substantiate the measured data of the noise and the vibration. Few investigations are known which have treated the coupled vibrations regarding the rail and wheel as elastic bodies, except for the lateral free vibration analysis by Matsuhisa et al.⁽³⁾ of a system of a wheel combined with a short unconstrained beam through only one linear spring. However their analytical model itself presents some problems in representing a real wheel/rail system.

The authors, in the previous reports

⁽³⁾⁽⁴⁾, analyzed the vibration in the vertical direction and the vibration in the horizontal direction of coupled bending and twisting, of an infinitely long and straight rail, and they showed, among other things, that the impedance at the head of the rail in the horizontal direction is fairly low in comparison with that in the vertical direction. In this report, as part of the theoretical investigations of the coupled vibration between rail and wheels and as supplement to the equations of motion of the rail treated in the previous reports, the authors formulate the axial vibration of a rotating wheel in the case that the contact forces, including the contact moment, act upon a point of contact between a rail and a rotating wheel. Moreover they apply Kalker's creep theory⁽⁵⁾ to the contact forces and describe a method of analysis which produces the steady-state solutions of the coupled vibration in the horizontal direction. To establish the validity of the analysis, the theoretical mechanical impedance of a stationary wheel is computed and is compared with the experimental one for the case that a point at the outer circumference of the wheel is elastically supported by two concentrated springs.

2. Fundamental Theory

2.1 Equations of motion Centrifugal and Coriolis forces act upon a rotating wheel, but since the rotating velocity of the wheel is fairly low in comparison with the natural frequencies of the wheel in the axial direction even in the Shinkansen, in this report the authors neglect these two effects and analyse the coupled lateral vibration between a rail and a rotating wheel. Hence the analysis of the free vibration of the rotating wheel is the same as that of the stationary wheel.

The wheel, which is symmetrical with respect to the axial centre plane but has a certain change in thickness, can be treated

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** Research Assistant, Faculty of Engineering, Kyushu University, Hakozaki, Higashi-ku, FUKUOKA.

*** Professor, Faculty of Engineering, Kyushu University, Hakozaki, Higashi-ku, FUKUOKA.

as an assembly of rings with uniform thickness by suitable division of the wheel. To the out-of-plane motion of the respective rings of the wheel, Mindlin's theory of circular plates⁽⁶⁾ is applied. We consider the rotatory coordinate-system (r, θ) as shown in Fig.1. It is assumed that the contact shearing force q_r and moments m_r and $m_{r\theta}$ act simultaneously at a point of contact between a rail and a wheel. The equations of motion of the j -th uniform ring are represented by the following equations, in which subscript and superscript j refer to j -th ring from the inside ($j=1, \dots, N$):

$$\frac{\partial M_r^j}{\partial r} + \frac{1}{r} \frac{\partial M_{r\theta}^j}{\partial \theta} + \frac{M_r^j - M_{r\theta}^j}{r} - Q_r^j = \rho H_j \frac{\partial^2 \psi_r^j}{\partial t^2} - \frac{m_r}{r} \delta(R, -Q\theta - \gamma) \quad (1\cdot a)$$

$$\frac{\partial M_{r\theta}^j}{\partial r} + \frac{1}{r} \frac{\partial M_r^j}{\partial \theta} + \frac{2M_{r\theta}^j}{r} - Q_{\theta}^j = \rho H_j \frac{\partial^2 \psi_{\theta}^j}{\partial t^2} - \frac{m_{r\theta}}{r} \delta(R, -Q\theta - \gamma) \quad (1\cdot b)$$

$$\frac{\partial Q_r^j}{\partial r} + \frac{1}{r} \frac{\partial Q_{\theta}^j}{\partial \theta} + \frac{Q_r^j}{r} = \rho h_j \frac{\partial^2 w^j}{\partial t^2} - \frac{q_r}{r} \delta(R, -Q\theta - \gamma) \quad (1\cdot c)$$

where M_r , $M_{r\theta}$ and $M_{\theta r}$ are moments and Q_r and Q_{θ} are shearing forces in the corresponding direction respectively, and ψ_r , ψ_{θ} and w are angular displacements about the θ - and the r -axis and axial displacement respectively [cf. Fig.1], $H = h^3/12$, ρ is density, h is thickness of plate, t is time, $\delta(R, -Q\theta - \gamma) = \delta(r-R)\delta(\theta+Q\theta+\gamma)$, $\delta(\cdot)$ is Dirac delta function, R is outer radius of wheel, Q is rotating angular velocity and γ is an initial angular coordinate of the point of contact.

The relationships between forces and displacements are given by the following expressions⁽⁶⁾:

$$\frac{M_r^j}{D_j} = \frac{\partial \psi_r^j}{\partial r} + \frac{\nu}{r} \left(\psi_r^j + \frac{\partial \psi_{\theta}^j}{\partial \theta} \right) \quad (2\cdot a)$$

$$\frac{M_{r\theta}^j}{D_j} = \frac{1}{r} \left(\psi_r^j + \frac{\partial \psi_{\theta}^j}{\partial \theta} \right) + \nu \frac{\partial \psi_r^j}{\partial r} \quad (2\cdot b)$$

$$\frac{2M_{\theta r}^j}{(1-\nu)D_j} = \frac{1}{r} \left(\frac{\partial \psi_r^j}{\partial \theta} - \psi_{\theta}^j \right) + \frac{\partial \psi_{\theta}^j}{\partial r} \quad (2\cdot c)$$

$$Q_r^j = \frac{\pi^2 G}{12} h_j \left(\psi_r^j + \frac{\partial w^j}{\partial r} \right) \quad (2\cdot d)$$

$$Q_{\theta}^j = \frac{\pi^2 G}{12} h_j \left(\psi_{\theta}^j + \frac{1}{r} \frac{\partial w^j}{\partial \theta} \right) \quad (2\cdot e)$$

where $D_j = Eh_j^3/12(1-\nu^2)$, E and G are the respective moduli of elasticity and rigidity of the wheel and ν is Poisson's ratio.

2.2 Symmetric and anti-symmetric modes

The boundary conditions of the inner circumference, $r=r_0$, which is fixed, and the outer circumference, $r=R$, which is free, are considered. The contact force and moments acting upon the point of contact are regarded as external force and moments respectively. The physical quantities ψ_r , ψ_{θ} , w , M_r , $M_{r\theta}$ and Q_r are continuous at a radial coordinate $r=r_j$ ($j=1, \dots, N-1$).

We obtain the solutions of the equations of free vibration without external forces derived from Eq.(1) according to Mindlin⁽⁶⁾ and apply the boundary conditions and the conditions of the continuity describ-

ed above to the solutions. Then the natural circular frequencies ω_{kl} are computed, where k is the number of diametral nodal lines ($k=0, 1, \dots$) and l is the number of circular nodal lines ($l=0, 1, \dots$). Defining the characteristic functions of ψ_r^j , ψ_{θ}^j and w^j corresponding to ω_{kl} by $\bar{\psi}_{r,kl}^j$, $\bar{\psi}_{\theta,kl}^j$ and $\bar{w}_{kl}^j$ respectively, one set of the characteristic functions are represented as follows⁽⁶⁾:

$$\bar{\psi}_{r,kl}^j(r, \theta) = \bar{\psi}_{r,kl}^j(r) \cos k\theta \quad (3\cdot a)$$

$$\bar{\psi}_{\theta,kl}^j(r, \theta) = \bar{\psi}_{\theta,kl}^j(r) \sin k\theta \quad (3\cdot b)$$

$$\bar{w}_{kl}^j(r, \theta) = \bar{w}_{kl}^j(r) \cos k\theta \quad (3\cdot c)$$

where $r_{j-1} \leq r \leq r_j$. $\bar{\psi}_{r,kl}^j(r)$ etc. are shape functions of the characteristic mode and involve Bessel functions and 6 or 4 integral constants. From the continuity of the displacements at the boundary between the adjacent rings, the phase angles with respect to θ are equal in every ring. The mode of Eq.(3) is called the symmetric mode, since the axial displacement is symmetric with respect to the diametral axis $\theta=0$. Incidentally, the following expressions represent the other set of characteristic functions which show the same shape of mode as Eq.(3) but differ from Eq.(3) only in the phase angle:

$$\bar{\psi}_{r,kl}^j(r, \theta) = \bar{\psi}_{r,kl}^j(r) \sin k\theta \quad (4\cdot a)$$

$$\bar{\psi}_{\theta,kl}^j(r, \theta) = -\bar{\psi}_{\theta,kl}^j(r) \cos k\theta \quad (4\cdot b)$$

$$\bar{w}_{kl}^j(r, \theta) = \bar{w}_{kl}^j(r) \sin k\theta \quad (4\cdot c)$$

We call the mode of Eq.(4) the anti-symmetric mode. This mode does not exist in the axially symmetric vibration. Substituting Eqs.(3) and (4) into Eq.(2) and applying the manner of representation of Eqs.(3) and (4), the characteristic functions of the moments and shearing forces, namely $\bar{M}_{r,kl}^j(r)$, $\bar{Q}_{r,kl}^j(r)$ etc. are produced. Since these characteristic functions satisfy the equations of free vibration without external forces derived from Eq.(1), the following relations hold good⁽⁷⁾:

$$\frac{d\bar{M}_{r,kl}^j}{dr} + \frac{k\bar{M}_{r\theta,kl}^j}{r} + \frac{\bar{M}_{r,kl}^j - \bar{M}_{\theta,kl}^j}{r} - \bar{Q}_{r,kl}^j = -\rho H_j \omega_{kl}^2 \bar{\psi}_{r,kl}^j \quad (5\cdot a)$$

$$\frac{d\bar{M}_{r\theta,kl}^j}{dr} - \frac{k\bar{M}_{\theta,kl}^j}{r} + \frac{2\bar{M}_{r\theta,kl}^j}{r} - \bar{Q}_{\theta,kl}^j = -\rho H_j \omega_{kl}^2 \bar{\psi}_{\theta,kl}^j \quad (5\cdot b)$$

$$\frac{d\bar{Q}_{r,kl}^j}{dr} + \frac{k\bar{Q}_{\theta,kl}^j}{r} + \frac{\bar{Q}_{r,kl}^j}{r} = -\rho h_j \omega_{kl}^2 \bar{w}_{kl}^j \quad (5\cdot c)$$

Furthermore, the characteristic functions of Eqs.(3) and (4) satisfy the following

Fig.1 Coordinates of wheel and etc.

orthogonal relationship for $(m, s) \neq (k, l)$:

$$\sum_{j=1}^N \int_0^{2\pi} \int_{r_{j-1}}^{r_j} \{(\psi_{r,ms}^j \psi_{r,kl}^j + \psi_{\theta,ms}^j \psi_{\theta,kl}^j) \cdot H_j + h_j w_{,ms}^j w_{,kl}^j\} r dr d\theta = 0 \quad (6)$$

3. Formulation of Wheel Response

3.1 Forced vibrations We assume the steady-state solutions of Eq.(1) as follows, in order to consider the change of phase with respect to θ .

$$\left. \begin{aligned} \psi_r^j &= \sum'' \bar{\psi}_{r,ms}^j (C_{ms} \cos m\theta + D_{ms} \sin m\theta) \\ \psi_\theta^j &= \sum'' \bar{\psi}_{\theta,ms}^j (C_{ms} \sin m\theta - D_{ms} \cos m\theta) \\ w^j &= \sum'' \bar{w}_{,ms}^j (C_{ms} \cos m\theta + D_{ms} \sin m\theta) \end{aligned} \right\} \quad (7)$$

where C_{ms} and D_{ms} are both unknown functions of time, and the symbol $\sum''$ means a double summation from zero to ∞ with respect to the subscripts m and s . When the coupled vibration between a rail and a wheel is considered, the solutions must be assumed to be linear combinations of the symmetric and the anti-symmetric modes because not only q_r and m_r but also spin moment m_{rs} acts simultaneously upon the point of contact. The behaviour

of the vibration mode for the rotating wheel is also obtained accurately under the assumption of Eq.(7).

Substituting Eq.(7) into Eqs.(1) and (2) and applying the relationship of Eq.(5) yields

$$\rho H_j \sum'' \bar{\psi}_{r,ms}^j (a_{ms} \cos m\theta + \beta_{ms} \sin m\theta) = m_r \delta(R, -Q\theta - \gamma)/r \quad (8\cdot a)$$

$$\rho H_j \sum'' \bar{\psi}_{\theta,ms}^j (a_{ms} \sin m\theta - \beta_{ms} \cos m\theta) = m_{rs} \delta(R, -Q\theta - \gamma)/r \quad (8\cdot b)$$

$$\rho h_j \sum'' \bar{w}_{,ms}^j (a_{ms} \cos m\theta + \beta_{ms} \sin m\theta) = q_r \delta(R, -Q\theta - \gamma)/r \quad (8\cdot c)$$

where $\begin{bmatrix} a_{ms} \\ \beta_{ms} \end{bmatrix} = \left(\frac{d^2}{dt^2} + \omega_{ms}^2 \right) \begin{bmatrix} C_{ms} \\ D_{ms} \end{bmatrix}$

We multiply both sides of Eqs.(8.a), (8.b) and (8.c) by Eqs.(3.a), (3.b) and (3.c) respectively, add the resulting equations on both sides and integrate the resultant relations over the total region of the wheel with the help of the orthogonal relationship of Eq. (6). We repeat similarly the above process by using Eqs.(4.a), (4.b) and (4.c) in place of Eqs.(3.a), (3.b) and (3.c) respectively. After tedious calculations, the following equation with subscripts m and s is obtained:

$$\left\{ \frac{d^2}{dt^2} + \omega_{ms}^2 (1 + i\eta) \right\} \begin{bmatrix} C_{ms} \\ D_{ms} \end{bmatrix} = \frac{1}{\rho l_{ms}} \left\{ m_r \bar{\psi}_{r,ms}^N(R) + q_r \bar{w}_{,ms}^N(R) \right\} \begin{bmatrix} \cos m(Q\theta + \gamma) \\ -\sin m(Q\theta + \gamma) \end{bmatrix} - m_{rs} \bar{\psi}_{\theta,ms}^N(R) \begin{bmatrix} \sin m(Q\theta + \gamma) \\ \cos m(Q\theta + \gamma) \end{bmatrix} \quad (9)$$

where $l_{ms} = (1 + \delta_{m0}) \pi \sum_{j=1}^N \int_{r_{j-1}}^{r_j} [H_j \{(\bar{\psi}_{r,ms}^j)^2 + (\bar{\psi}_{\theta,ms}^j)^2\} + h_j (\bar{w}_{,ms}^j)^2] r dr$,

δ_{m0} is Kronecker's delta and $\bar{\psi}_{\theta,ms}^j \equiv 0$ for the axially symmetric modes. In the second term on the left hand side of Eq.(9), the effect of structural damping is included, where η is a loss factor and $i = \sqrt{-1}$.

3.2 Steady-state vibrations The contact force and moments are periodic in the steady-state vibration. Defining the circular frequency by ω , we set

$$q_r = \bar{q}_r e^{i\omega t}, m_r = \bar{m}_r e^{i\omega t}, m_{rs} = \bar{m}_{rs} e^{i\omega t} \quad (10)$$

where $\bar{q}_r$, $\bar{m}_r$ and $\bar{m}_{rs}$ are unknown complex amplitudes. Correspondingly the following complex representation of Eq.(9) makes the analytical treatment easy.

$$\left\{ \frac{d^2}{dt^2} + \omega_{ms}^2 (1 + i\eta) \right\} \begin{bmatrix} A_{ms} \\ B_{ms} \end{bmatrix} = \frac{1}{\rho l_{ms}} \left\{ m_r \bar{\psi}_{r,ms}(R) + q_r \bar{w}_{,ms}(R) \right\} \begin{bmatrix} e^{-im(Q\theta + \gamma)} \\ e^{im(Q\theta + \gamma)} \end{bmatrix} - i m_{rs} \bar{\psi}_{\theta,ms}(R) \begin{bmatrix} e^{-im(Q\theta + \gamma)} \\ -e^{im(Q\theta + \gamma)} \end{bmatrix} \quad (11)$$

where $A_{ms} = C_{ms} + iD_{ms}$, $B_{ms} = C_{ms} - iD_{ms}$. Hereafter superscripts j and N showing the ring number are omitted for simplicity. The following relations hold good:

$$\left. \begin{aligned} C_{ms} \cos m\theta + D_{ms} \sin m\theta &= (1/2)(A_{ms} e^{-im\theta} + B_{ms} e^{im\theta}) \\ C_{ms} \sin m\theta - D_{ms} \cos m\theta &= (i/2)(A_{ms} e^{-im\theta} - B_{ms} e^{im\theta}) \end{aligned} \right\} \quad (12)$$

Substituting Eq.(10) into Eq.(11), the solutions of Eq.(11) become as follows:

$$\begin{bmatrix} A_{ms}/\bar{A}_{ms} \\ B_{ms}/\bar{B}_{ms} \end{bmatrix} = e^{i(\omega \mp mQ)(t \mp m\gamma)}, \quad \begin{bmatrix} \bar{A}_{ms} \\ \bar{B}_{ms} \end{bmatrix} = \frac{\bar{m}_r \bar{\psi}_{r,ms}(R) + \bar{q}_r \bar{w}_{,ms}(R) \mp i \bar{m}_{rs} \bar{\psi}_{\theta,ms}(R)}{\rho l_{ms} (\omega_{ms}^2 (1 + i\eta) - (\omega \mp mQ)^2)} \quad (13\cdot a)$$

where the dual signs are in the same order. Using the relations of Eq.(12), substitution of the above equation into Eq.(7) gives the following results for the displacements:

$$\begin{bmatrix} \psi_r \\ \psi_\theta \\ w \end{bmatrix} = \frac{1}{2} \sum'' \left\{ \begin{bmatrix} \bar{\psi}_{r,ms}(r) \\ i \bar{\psi}_{\theta,ms}(r) \\ \bar{w}_{,ms}(r) \end{bmatrix} \bar{A}_{ms} e^{-im\varphi} + \begin{bmatrix} \bar{\psi}_{r,ms}(r) \\ -i \bar{\psi}_{\theta,ms}(r) \\ \bar{w}_{,ms}(r) \end{bmatrix} \bar{B}_{ms} e^{im\varphi} \right\} e^{i\omega t} \quad (13\cdot b)$$

where $\varphi = \theta + Q\theta + \gamma$ (cf. Fig.1).

Equations (13.a,b) are the fundamental equations of the rotating wheel which give the theoretical steady-state response of the wheel in the coupled vibration. If the contact force and moments are formulated, the vibration of the wheel is obtained from Eqs.(13). Here φ is the so-called Euler coordinate in which the point of contact is a starting point. In the formulation of the rotating wheel, this coordinate φ is introduced instead of the rotatory coordinate θ .

3.3 Force and moment acting upon a point of contact In the case of wheels rolling on a rail, the creep forces are considered the contact forces. In this report, the contact forces are formulated by applying Kalker's linear creep theory⁽⁸⁾. The following assumptions are made (cf. Fig.2):

- (1) The slope of wheel tyre can be neglected (1/40 in Shinkansen) and the point of contact is at the head of the rail.
- (2) The wheel is an elastic body. The inner circumference is fixed and the wheel shaft points always in the y -direction. Hence the circumferential velocity RQ points always in the $-x$ -direction.

Fig.2 Force and moments acting at a point of contact on a wheel

The shearing force q_r and the spin moment $m_{r\theta}$ act simultaneously upon the point of contact. Under the assumption described above, these relate to the relative velocity Δw and the relative angular velocity $\Delta\psi$ (cf. Fig.2) in the horizontal and the vertical directions respectively as follows:

$$\left. \begin{aligned} q_r &= -f_{22}\Delta w/v - f_{23}\Delta\psi/v \\ m_{r\theta} &= f_{23}\Delta w/v - f_{33}\Delta\psi/v \\ \Delta w &= \dot{w}(R, \varphi, t)|_{\varphi=0} - \dot{y}_T(x, t)|_{x=0} \\ \Delta\psi &= \dot{\psi}_\theta(R, \varphi, t)|_{\varphi=0} - \dot{\psi}(x, t)|_{x=0} \end{aligned} \right\} \dots\dots\dots (14)$$

where f_{22} , f_{23} and f_{33} are Kalker's creep coefficients⁽⁸⁾, v is a travelling speed $= RQ$, y_T and ψ are a horizontal displacement and a slope of bending at the head of the rail respectively⁽⁹⁾ and x is a moving coordinate whose origin is at the point of contact and which moves at speed in the longitudinal direction of the rail and so is an Euler coordinate. Furthermore notation " $\dot{}$ " means a material derivative, that is, $\partial/\partial t + Q\partial/\partial\varphi$ for the displacement of the wheel and $\partial/\partial t - v\partial/\partial x$ for the displacement of the rail.

m_r will be given as an excited moment arising from the movement of a point of contact subject to the load on a wheel by virtue of the lateral random irregularity on the head of the rail.

In parallel with the formulation of the steady-state response of the wheel in the previous section, the steady-state response of the rail in the horizontal direction has already been formulated in the previous reports⁽³⁾⁽⁴⁾ for the case that a periodic force of Eq.(10) moves at constant speed, considering the coupling between lateral deflection and torsion of the rail. The displacements of the rail were represented by $\bar{q}_r$, $\bar{m}_r$ and $\bar{m}_{r\theta}$ in the same manner as in Eq.(13). Therefore substitution of these expressions of displacement and Eq.(10) into Eq.(14) gives complex simultaneous equations with respect to the amplitudes of the contact forces $\bar{q}_r$ and $\bar{m}_{r\theta}$. When the solution to the equations is again substituted into the expressions for the displacements of wheel and rail, the responses of one wheel and the rail in the coupled vibration are decided. The present analytical method can be applied to the coupled vibration between a rail and many wheels. In such a case, the simultaneous equations of order $2 \times$ (number of wheels) with respect to $\bar{m}_{r\theta}$

Fig.3 Dimensions of wheel (mm)

and $\bar{q}_r$ acting upon every wheel are obtained. Further details will be given together with the computational results in a later report.

The creep theory has been applied to the movement in a zigzag direction of a rigid wheel shaft which has a low frequency. We can not find any reports in which the applicability of the creep theory, the dependence of frequency and the nonlinearity of the creep forces are discussed in detail, though it is said that the creep forces act upon the point of contact between a rail and wheels even in the vibration with such high frequency as in this report. Therefore, as a first step we apply Kalker's linear creep theory to the coupled vibration between a rail and a wheel in the horizontal direction and consider the vibration characteristics theoretically. This is of significant importance in comparing the theory with the actual phenomena. It may be the subject for a future study to examine experimentally the creep characteristic in the high frequency range and the effect of the roughness of a surface, based on the results of the comparison.

On the other hand, in the coupled vibration in the vertical direction, it is necessary to consider the Hertzian contact restoring force between a rail and a wheel as a contact force and the roughness of the head of the rail in the vertical direction as an external excitation.

4. Characteristic of a Stationary Wheel

In order to verify the formulation in the previous chapter, we examine the vibration characteristic of an individual stationary wheel. With the purpose of establishing the validity of the solution of the coupled vibration described in the section 3.3, the characteristic of the wheel which is supported by two springs in two directions at a point on the outer circumference is also considered.

4.1 Steady-state response In the stationary state, the response of the wheel to an external force $p e^{i\omega t}$ which acts at a point $(\bar{r}, \bar{\theta})$ on the side of the wheel is discussed. The external force has the same direction as that of q_r and has a forced circular frequency ω . In order to summarize the vibration characteristic, we introduce the point impedance Z_p which is given by dividing the external force by the velocity in the axial direction at the external point. Equation (13) is rewritten by replacing $-\gamma$ with θ_0 as follows:

$$\begin{bmatrix} w \\ \psi_\theta \end{bmatrix} = e^{i\omega t} \sum_n \frac{1}{\mu_{ns}} \left[\bar{m}_{r\theta} \bar{\psi}_{\theta,ms}(R) \begin{bmatrix} -\bar{w}_{ms}(r) \sin m\theta_0 \\ \bar{\psi}_{\theta,ms}(r) \cos m\theta_0 \end{bmatrix} + p \bar{w}_{ms}(\bar{r}) \begin{bmatrix} \bar{w}_{ms}(r) \cos m\bar{\theta} \\ \bar{\psi}_{\theta,ms}(r) \sin m\bar{\theta} \end{bmatrix} \right]$$

Fig.4 Comparison of the theoretical and experimental results of the wheel

$$+ \{ \bar{q}_r \bar{w}_{ms}(R) + \bar{m}_r \bar{\psi}_{r,ms}(R) \} \begin{bmatrix} \bar{w}_{ms}(r) \cos m\theta_0 \\ \bar{\psi}_{\theta,ms}(r) \sin m\theta_0 \end{bmatrix} \dots\dots\dots (15)$$

where

$$\mu_{ms} = \rho l_{ms} \{ \omega_{ms}^2 (1 + i\eta) - \omega^2 \}, \quad \theta_0 = \theta - \theta_0, \quad \bar{\theta} = \theta - \bar{\theta}.$$

If $\bar{w}_{ms}(r)$ in the first equation of Eq.(15) is replaced by $\bar{\psi}_{r,ms}(r)$, then $\bar{\psi}_r$ is obtained. In the case of the individual wheel, that is, $(\bar{m}_r = \bar{q}_r = \bar{m}_{r0} = 0)$, the axial displacement is symmetric with respect to the diametral axis $\theta = \bar{\theta}$ from Eq.(15), and there always exists automatically a loop on the diameter that passes through the excited point selected arbitrarily.

In the case that one point (R, θ_0) on the outer circumference of the wheel is supported with a concentrated linear spring k_y in the y -direction and a concentrated rotational spring k_ϕ around the z -axis (cf. Fig.1), the contact force and moments become as follows:

$$q_r = -k_y w(R, \theta_0, t), \quad m_{r0} = -k_\phi \psi_\theta(R, \theta_0, t), \quad m_r = 0 \dots\dots\dots (16)$$

Substitution of Eqs.(10) and (15) into Eq.(16) yields the following equations with respect to the amplitudes of the contact forces $\bar{q}_r$ and $\bar{m}_{r0}$:

$$\begin{aligned} [1 + k_y \sum'' \{ \bar{w}_{ms}(R) \}^2 / \mu_{ms}] \bar{q}_r &= -k_y \bar{p} \sum'' \bar{w}_{ms}(R) \bar{w}_{ms}(\bar{r}) \cos m\Phi / \mu_{ms} \\ [1 + k_\phi \sum'' \{ \bar{\psi}_{\theta,ms}(R) \}^2 / \mu_{ms}] \bar{m}_{r0} &= -k_\phi \bar{p} \sum'' \bar{\psi}_{\theta,ms}(R) \bar{w}_{ms}(\bar{r}) \sin m\Phi / \mu_{ms} \end{aligned} \dots\dots\dots (17)$$

where $\Phi = \theta_0 - \bar{\theta}$. The response of the wheel having the characteristic of the contact forces of Eq.(16) is decided by substituting $\bar{q}_r$ and $\bar{m}_{r0}$ obtained from Eq.(17) into Eq.(15). According to a detailed examination of Eqs.(15) and (17), when $\eta = 0$, $\sin m\Phi \neq 0$ and $\cos m\Phi \neq 0$, the resonances do occur not at $\omega = \omega_{ms}$ but at the circular frequencies ω satisfying the transcendental equations

$$\begin{aligned} 1 + k_y \sum'' \{ \bar{w}_{ms}(R) \}^2 / \mu_{ms}^0 &= 0 \\ 1 + k_\phi \sum'' \{ \bar{\psi}_{\theta,ms}(R) \}^2 / \mu_{ms}^0 &= 0 \end{aligned} \dots\dots\dots (18)$$

where $\mu_{ms}^0 = \rho l_{ms} (\omega_{ms}^2 - \omega^2)$. Defining the roots of the first and the second equations of Eq.(18) by ω_y and ω_ϕ respectively, these roots represent the natural circular frequencies of the wheel caused by the springs k_y and k_ϕ respectively and the resonances due to them are here called the resonances of the k_y and k_ϕ series. In the limit of k_y and $k_\phi \rightarrow 0$, ω approaches ω_{ms} from Eq.(18). Hence the frequencies ω_y and ω_ϕ bifurcate from ω_{ms} when the springs k_y and k_ϕ exist.

4.2 Numerical computational results and

considerations

A 1/5 scale model of the standard wheel in the Tokaido Shinkansen (Number of rings $N=4$, the material is S45C) as shown in Fig.3 was used to compute the mechanical impedance and make an experiment using continuous excitations. In computation, we used 53 natural frequencies in the range of $f_{ms} = \omega_{ms} / 2\pi \leq 60$ kHz in turn and the corresponding characteristic modes, with assumed values $\nu = 0.3$ and $\eta = 0.01$. In order to fix the inner circumference of the wheel as rigidly as possible, a single block structure of the axle and wheel was adopted in the experiment and its axle part was rigidly fixed on a surface plate. As a supporting spring, an M5 steel bolt of 33mm length was connected between the surface plate and the outer circumference of the model wheel. By direct measurement, we obtained $k_y = 7.92 \times 10^7$ N/m and $k_\phi = 30$ Nm/rad. By substituting these values into Eq.(18), the natural frequencies $f_y = \omega_y / 2\pi$ and $f_\phi = \omega_\phi / 2\pi$ were computed.

Figure 4 indicates the comparison

Table 1 Theoretical natural frequencies f_{ms} , f_v of the wheel (Hz)

(m,s)	(1,0)		(0,0)		(2,0)		(3,0)	
Freq.	f_{ms}	f_v	f_{ms}	f_v	f_{ms}	f_v	f_{ms}	f_v
Fig.4(a)	1143	—	1498	—	2103	—	5403	—
Fig.4(b)	—	1346	—	1732	—	2862	—	5655
Fig.4(c)	1143	1346	—	1732	2103	2862	5403	5655
No.	1	1'	2	2'	3	3'	4	4'

of the numerical and experimental values of the point mechanical impedance, where (a) is a case without spring support, (b) is a case of the excited point being supported with the springs and (c) is a case of the elastically supported point being displaced 160° from the excited point. The abscissa is the external frequency (units in Hz) and the standard value 0dB of the ordinate is common to both the theoretical and experimental results.

In Table 1 are shown the theoretical natural frequencies corresponding to the resonance points indicated by the same number in Fig.4. Figure 5 shows the theoretical characteristic modes at $\theta_0 - \theta = 160^\circ$ corresponding to the resonance points of the k_v series which are represented by the number with notation " ' ". The results are summarized as follows:

(1) When the excited point is supported elastically [Fig.4(b)], only the resonance points corresponding to the k_v series, i.e., f_v , occur. Due to the constraint by spring k_v , these frequencies are correspondingly higher than those of the unconstrained wheel (cf. Table 1).

(2) When the point of spring support is different from the excited point [Fig.4(c)], the resonance points f_v and f_s coexist as mentioned in section 4.1. But in the present case, the influence of k_s is so small that f_s is actually coincident with f_{ms} . Hence, the numbers of resonance points k_s in Fig.4(c) are the same as those of resonance points f_{ms} in Fig.4(a). We see no resonance point f_s separated from f_{ms} of the axially symmetric mode in Fig.4(c) (cf. Table 1). This is understood from the fact that ω_s separated from ω_{os} does not exist in Eq.(18).

(3) In the high frequency region, f_v approaches f_{ms} . Namely the spring effect is increasingly negligible and the characteristics of Figs.4(b) and (c) approach the characteristic of the individual wheel in Fig.4(a) as the frequency is increased.

(4) The characteristic modes corresponding to the resonance of the k_v series always have a loop on the diameter that passes through the elastically supported point independently of the position of the excited point. The characteristic modes of the k_s series are approximately coincident with those of the individual wheel, that is, of the unconstrained wheel, but they always have a node on the diameter that passes through the elastically supported point.

From a comparison of the theoretical and experimental results about the mechanical impedance, the results are summarized as follows:

Fig.5 Examples of characteristic mode at resonance point ($\circ$: excited point, $\odot$: point of spring support. The solid lines represent the nodal lines. The numbers in the figure correspond to those of Fig.4)

(5) The experimental and theoretical results agree very well up to the limited frequency of the vibrator (8 kHz), though the anti-resonance peaks are in part crushed. The experimental results prove very well the theoretical relationship between the position supported elastically and the aspect of the occurrence of resonance points. This shows the validity of the present method of analysis. The natural frequencies higher than those in the figures were also checked by impact response and they agreed well with the theoretical ones. In the modes of lower order, however, the experimental resonance frequencies are lower than the respective theoretical ones due to the imperfection of the boundary condition "fixed" at the inner circumference.

The resonance in the vicinity of 2.5kHz in the figures was always caused by the vibrator used.

5. Conclusions

Corresponding to the steady-state response of the rail already formulated, the steady-state response of a rotating wheel in the axial direction has been formulated, and the method of analysis of the coupled vibration between a rail and a rotating wheel in the horizontal direction was demonstrated. The method can be applied readily if the contact forces acting at a point of contact on the outer circumference of a rotating wheel are concretely given. The authors obtained the mechanical impedance of a stationary wheel and established the validity of the analysis by a comparison with the experimental results.

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