The Transient Stress in an Elastic Half-space subjected to a Semi-infinite Line Load varying as Unit Step Function on Its Surface*

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This paper presents an exact solution for the transient response in the interior of an elastic half space subjected to a normal or a tangential semi-infinite line load varying with time as a Heaviside unit step function on its surface. The solution is based on the stress function approach, the Laplace transform and the double Fourier transform method applied to a non-axisymmetric dynamical problem. The Laplace inverse transform which is performed by the three-dimensional Cagniard's technique gives an exact solution for the stress wave propagations. The results of numerical evaluation are shown graphically for the stress variation versus time in the interior of a half space.

Key Words : Shock, Elastic Half-space, Impact line Load, Laplace Transform, Cagniard's Method, Stress Wave Propagation, Transient Stress

1. Introduction

The analysis of an elasto-dynamic problem for a half space loaded on finite region appears to be few. This paper studied the transient stress in the interior of a half space subjected to a normal or shearing semi-infinite line load varying with a unit step function. By simple superposition, the solution will be obtained for a half space under a finite line load. The boundary element method, which is much employed in stress analysis, is based on the superposition of solution to a simple boundary condition.

Therefore, we feel that the basic elasto-dynamic problems should be studied.

The response⁽¹⁾⁻⁽⁶⁾ of an elastic half space to an impulsive load acting on its surface is mostly analysed with the aid of the Laplace transform method. However, if the evaluation of residue or branch cut integral is no longer possible, it often becomes troublesome to get a formal solution. Cagniard's method⁽⁷⁾⁽⁸⁾ is a rational one in the various steps of inversion. Cagniard's method has been applied to the two-dimensional elastodynamic and axisymmetric problems, but not to the three-dimensional nonaxisymmetric problem.⁽⁹⁾⁽¹⁰⁾ The present work shows how the three-dimensional Cagniard's method can be applied. The response of an elastic half space to a normal impact or a tangential semi-infinite line load is analysed with aid of the three-dimensional Cagniard's method which is developed here.

2. Basic Equations

Substituting the displacements u , v , w of Eqs.(1) expressed by the derivatives of stress functions φ_0 , λ_2 , λ_3 with respect to the rectangular coordinates (x, y, z) into the governing equation, we have the following wave equation of Eq.(2).

$$\begin{aligned} 2Gu &= \frac{\partial \varphi_0}{\partial x} - \frac{\partial \lambda_2}{\partial z} + \frac{\partial \lambda_3}{\partial y}, 2Gv = \frac{\partial \varphi_0}{\partial y} - \frac{\partial \lambda_3}{\partial x} \\ 2Gw &= \frac{\partial \varphi_0}{\partial z} + \frac{\partial \lambda_2}{\partial x} \end{aligned} \quad \dots\dots\dots (1)$$

$$\left(\nabla^2 - \frac{1}{c_j^2} \frac{\partial^2}{\partial t^2} \right) (\varphi_0, \lambda_2, \lambda_3) = 0 \quad \dots\dots\dots (2)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$c_j (j=1,2)$ is the velocity of stress wave.

The stress function φ_0 has the dilatational wave velocity c_1 and λ_2 , λ_3 have the shearing wave velocity c_2 in Eq.(2). Applying the Laplace transform and the double Fourier transforms to Eq.(2), we have

$$\begin{aligned} \frac{\partial^2}{\partial z^2} (\bar{\varphi}_0^{**}, \bar{\lambda}_2^{**}, \bar{\lambda}_3^{**}) - (\alpha^2 + \beta^2 + s_j^2) \bar{p}^2 &= 0 \\ s_j &= 1/c_j \end{aligned} \quad \dots\dots\dots (3)$$

where $\bar{f}^{**} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f e^{-i p(\alpha x + \beta y)} dx dy$, $\bar{f} = \int_0^{\infty} f e^{-pt} dt$

The general solutions of Eqs.(3) are given as follows

$$\bar{\varphi}_0^{**} = A e^{-p n_1 z}, \bar{\lambda}_2^{**} = B e^{-p n_2 z}, \bar{\lambda}_3^{**} = C e^{-p n_3 z} \quad \dots\dots\dots (4)$$

where $n_1 = \sqrt{\alpha^2 + \beta^2 + s_1^2}$, $n_2 = \sqrt{\alpha^2 + \beta^2 + s_2^2}$, $(S_1/S_2)^2 = \kappa^2$, $\kappa^2 = (1-2\gamma)/2(1-\gamma)$

A, B and C are unknown functions which may be determined by the boundary conditions. Notation γ is Poisson's ratio.

Using Eq.(4), the Fourier inverse transformed stress components are obtained as

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$$\begin{aligned}
\bar{\sigma}_x &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -A \left(\alpha^2 - \frac{\gamma}{2(1-\gamma)} \right) e^{-pN_1} + (Bian_2 - Ca\beta) e^{-pN_2} \right\} dad\beta \\
\bar{\sigma}_y &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -A \left(\beta^2 - \frac{\gamma}{2(1-\gamma)} \right) e^{-pN_1} + Ca\beta e^{-pN_2} \right\} dad\beta \\
\bar{\sigma}_z &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ An_3 e^{-pN_1} - Bian_2 e^{-pN_2} \right\} dad\beta \\
\bar{\tau}_{xy} &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -Aa\beta e^{-pN_1} + \frac{1}{2} [Bi\beta n_2 + C(\alpha^2 - \beta^2)] e^{-pN_2} \right\} dad\beta \\
\bar{\tau}_{yz} &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -Ai\beta n_1 e^{-pN_1} - \frac{1}{2} (Ba\beta - Cian_2) e^{-pN_2} \right\} dad\beta \\
\bar{\tau}_{zx} &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -Aian_1 e^{-pN_1} - \frac{1}{2} [B(\alpha^2 + n_1^2) + Ci\beta n_2] e^{-pN_2} \right\} dad\beta
\end{aligned} \quad (5)$$

where $N_1 = n_1 z - iax - i\beta y$, $N_2 = n_2 z - iax - i\beta y$ (6)

The formulations of boundary condition in which a semi-infinite line load acts on the half space in the form of a unit step function $H(t)$ as shown in Fig.1 are given as follows

$$(\sigma_x)_{z=0} = -p_0 H(t) \delta(y) \lim_{\epsilon \rightarrow 0} H(x) e^{-\epsilon x} \quad (7)$$

$$(\tau_{yz})_{z=0} = (\tau_{zx})_{z=0} = 0 \quad (8)$$

where $\delta(\cdot)$ is Dirac's delta function. Substituting the Laplace transforms and the Fourier transforms of Eq.(7) and Eq.(8) into the third, fifth and sixth ones of Eqs.(5), the unknown functions A, B and C are determined as

$$A = p_0 i \frac{n_3}{aF(\alpha, \beta)}, B = p_0 \frac{(\alpha^2 + \beta^2)n_1}{a^2 F(\alpha, \beta)}, C = p_0 i \frac{\beta n_1 n_2}{a^2 F(\alpha, \beta)} \quad (9)$$

The formulations of boundary conditions to the shearing impact load are given as

$$(\tau_{zx})_{z=0} = -q_1 H(t) \delta(y) \lim_{\epsilon \rightarrow 0} H(x) e^{-\epsilon x} \quad (10)$$

$$(\sigma_x)_{z=0} = (\tau_{yz})_{z=0} = 0 \quad (11)$$

The unknown functions A, B and C determined by Eq.(10) and Eq.(11), are expressed by

$$A = q_1 \frac{n_2}{F(\alpha, \beta)}, B = -q_1 i \frac{n_3}{aF(\alpha, \beta)}, C = -q_1 \frac{\beta(n_3 - 2n_1 n_2)}{an_2 F(\alpha, \beta)} \quad (12)$$

where

$$F(\alpha, \beta) = n_3^2 - (\alpha^2 + \beta^2)n_1 n_2 \quad (13)$$

Substituting Eqs.(9) and Eqs.(12) into Eq.(5), and applying the double Fourier inverse transforms, we have in the case of normal impact load

$$\begin{aligned}
\sigma_x &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{n_3 \left(\alpha^2 - \frac{\gamma}{2(1-\gamma)} s_2^2 \right)}{aF(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} + i \frac{an_1 n_2}{F(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta \\
\sigma_y &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{n_3 \left(\beta^2 - \frac{\gamma}{2(1-\gamma)} s_2^2 \right)}{aF(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} + i \frac{\beta^2 n_1 n_2}{aF(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta \\
\sigma_z &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ i \frac{n_3^2}{aF(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} - i \frac{(\alpha^2 + \beta^2)n_1 n_2}{aF(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta \\
\tau_{xy} &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{\beta n_3}{F(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} + i \frac{\beta n_1 n_2}{F(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta \\
\tau_{yz} &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\beta n_1 n_3}{aF(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} - \frac{\beta n_1 n_3}{aF(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta \\
\tau_{zx} &= \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{n_1 n_3}{F(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} - \frac{n_1 n_3}{F(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta
\end{aligned} \quad (14)$$

In the case of tangential impact load,

$$\begin{aligned}
\sigma_x &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{n_2 \left(\alpha^2 - \frac{\gamma}{2(1-\gamma)} s_2^2 \right)}{F(\alpha, \beta)} e^{-p(n_1 z - iax - i\beta y)} \right. \\
&\quad \left. + \frac{n_2^2 n_3 + \beta^2 (n_3 - 2n_1 n_2)}{n_2 F(\alpha, \beta)} e^{-p(n_2 z - iax - i\beta y)} \right\} dad\beta
\end{aligned}$$

Fig.1 Coordinate system and semi-infinite line load acting on an elastic half space.

$$\begin{aligned}
\bar{\sigma}_y &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{n_2 \left(\beta^2 - \frac{\gamma}{2(1-\gamma)} s_2^2 \right)}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} + \frac{\beta^2 (n_3 - 2n_1 n_2)}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} \right\} d\alpha d\beta \\
\bar{\sigma}_z &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{n_2 n_3}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} - \frac{n_2 n_3}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} \right\} d\alpha d\beta \\
\bar{\tau}_{xy} &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{\alpha \beta n_2}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} + \frac{\beta n_2^2 n_3 + \beta (\beta^2 - \alpha^2) (n_3 - 2n_1 n_2)}{2\alpha n_2 F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} \right\} d\alpha d\beta \\
\bar{\tau}_{yz} &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{\beta n_1 n_2}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} + i \frac{\beta n_1 n_2}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} \right\} d\alpha d\beta \\
\bar{\tau}_{zx} &= \frac{q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{\alpha n_1 n_2}{F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} + i \frac{n_2^2 - \beta^2 n_1 n_2}{\alpha F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} \right\} d\alpha d\beta
\end{aligned}
\tag{15}$$

3. The Three-dimensional Cagniard's method

A brief form of Eqs.(14) or Eqs.(15) is shown as

$$\bar{\sigma} = \frac{(p_0, q_1)}{(2\pi)^2} \{ (\bar{\sigma})_p + (\bar{\sigma})_s \} \tag{16}$$

The transformed stress components consist of the term $(\bar{\sigma})_p$ related with the dilatational wave

$$(\bar{\sigma})_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\alpha, \beta)}{\alpha F(\alpha, \beta)} e^{-p(n_1 x - i \alpha x - i \beta y)} d\alpha d\beta \tag{17}$$

and the term $(\bar{\sigma})_s$ of shearing wave

$$(\bar{\sigma})_s = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\alpha, \beta)}{\alpha F(\alpha, \beta)} e^{-p(n_2 x - i \alpha x - i \beta y)} d\alpha d\beta \tag{18}$$

We make a change of variables α and β as follows

$$\alpha = \xi \cos \phi - \eta \sin \phi, \beta = \xi \sin \phi + \eta \cos \phi$$

where $\cos \phi = x/r$, $\sin \phi = y/r$, $r = \sqrt{x^2 + y^2}$

Substituting the above equations into Eq.(17) and Eq.(18), we have

$$(\bar{\sigma})_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\xi, \eta) e^{-p(n_1 x - i \xi r)}}{(\xi \cos \phi - \eta \sin \phi) F(\xi, \eta)} d\xi d\eta \tag{19}$$

$$(\bar{\sigma})_s = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\xi, \eta) e^{-p(n_2 x - i \xi r)}}{(\xi \cos \phi - \eta \sin \phi)} d\xi d\eta \tag{20}$$

where $n_1 = \sqrt{\xi^2 + \eta^2 + s_1^2}$, $n_2 = \sqrt{\xi^2 + \eta^2 + s_2^2}$, $n_3 = \xi^2 + \eta^2 + s_3^2/2$

The asymptotic of integrand of Eq.(19) and Eq.(20) for $\xi \rightarrow \infty$ is given as follows

$$\lim_{\xi \rightarrow \infty} \frac{f(\xi, \eta)}{(\xi \cos \phi - \eta \sin \phi) F(\xi, \eta)} \rightarrow 0 \tag{21}$$

To make a change of variable ξ , we put $t = n_j z - i r \xi$, $j=1, 2$

The variable ξ is represented by the time t as follows

$$\xi_j = \frac{1}{R} \{ i t \cos \phi \pm \sin \phi \sqrt{t^2 - h_j^2(\eta)} \} \tag{22}$$

where $\sin \phi = z/R$, $\cos \phi = r/R$, $R^2 = x^2 + y^2 + z^2$, $h_j^2(\eta) = R^2(\eta^2 + s_j^2)$

Therefore, the infinite line integral of Eq.(19) may be converted to a closed curve as shown in Fig.2. Cagniard's path changed from Eq.(20) is shown in Fig.3. In Fig.2 and Fig.3, $i\sqrt{\eta^2 + s_1^2}$ and $i\sqrt{\eta^2 + s_2^2}$ are branch points. Since the pole of $i\sqrt{\eta^2 + s_1^2}$

Fig.2 Cagniard's inverse integral path corresponding to Eq.19.

Fig.3 Cagniard's inverse integral path corresponding to Eq.20.

is outside of the closed curve of Cagniard's path, the residue theorem is not applicable.

Notation s_R is the reciprocal of the Rayleigh speed c_R . Cagniard's path which is related to the dilatational wave passes the point $i \cos \phi \sqrt{\eta^2 + s_1^2}$ on the imaginary axis as shown in Fig.2. Since the closed curve is under branch point $i \cos \phi \sqrt{\eta^2 + s_1^2}$, we have the relation $i \cos \phi \sqrt{\eta^2 + s_1^2} < i \sqrt{\eta^2 + s_1^2}$. Consequently, the evaluation of branch cut is not necessary. Cagniard's path relating to the shearing wave passes the point of $i \cos \phi \sqrt{\eta^2 + s_2^2}$ on the imaginary axis under the branch point $i \sqrt{\eta^2 + s_2^2}$ in the domain $\cos \phi < \sqrt{\eta^2 + s_1^2} / \sqrt{\eta^2 + s_2^2}$.

In the range $\cos \phi > \sqrt{\eta^2 + s_1^2} / \sqrt{\eta^2 + s_2^2}$, since the point $i \cos \phi \sqrt{\eta^2 + s_2^2}$ on the imaginary axis comes between the branch points $i \sqrt{\eta^2 + s_1^2}$ and $i \sqrt{\eta^2 + s_2^2}$, we must perform the branch cut integral along the point A, B, C, and D. From evaluation for Eq.(19) and Eq.(20), we obtain

$$\begin{aligned}
(\bar{\sigma})_p &= 2\pi i \int_{-\infty}^{\infty} G_R(\eta \tan \phi, \eta) e^{-p(n_1 x - i r \eta \tan \phi)} d\eta \\
&+ \int_{-\infty}^{\infty} d\eta \int_{E_p}^{A_p} G(\xi, \eta) e^{-p n_1} d\xi \\
&+ \int_{-\infty}^{\infty} d\eta \int_{A_p}^{F_p} G(\xi, \eta) e^{-p n_2} d\xi
\end{aligned}
\tag{23}$$

$$(\bar{\sigma})_s = 2\pi i \int_{-\infty}^{\infty} G_R(\eta \tan \phi, \eta) e^{-\rho(\eta \tan \phi - i\eta \tan \phi)} d\eta + \int_{-\infty}^{\infty} d\eta \int_{E_s}^{A_s} G(\xi, \eta) e^{-\rho \xi} d\xi + \int_{-\infty}^{\infty} d\eta \int_{A_s}^{F_s} G(\xi, \eta) e^{-\rho \xi} d\xi \\ + \int_{-\infty}^{\infty} d\eta \int_{C_s}^{D_s} G(\xi, \eta) e^{-\rho \xi} d\xi + \int_{-\infty}^{\infty} d\eta \int_{D_s}^{F_s} G(\xi, \eta) e^{-\rho \xi} d\xi \dots (24)$$

where $G_R(\eta \tan \phi, \eta) = \frac{f(\eta \tan \phi, \eta)}{F(\eta \tan \phi, \eta)}$, $G(\xi, \eta) = \frac{f(\xi, \eta)}{(\xi \cos \phi - \eta \sin \phi)F(\xi, \eta)}$

3.1 The Laplace inverse transform by the three dimensional Cagniard's method applied to the transformed stress obtained by the residue

The first term of the right hand side of Eq.(23) or Eq.(24) is the Fourier inverse transform integral with respect to variable η which is obtained by the evaluation of residue around the pole in Fig.2 or Fig.3. In this case, the Laplace inverse transform could be performed by the two-dimensional Cagniard's method. Converting the variable η into the time t , the first term of the right hand side of Eq.(23) is changed as follows

$$\rho(\bar{\sigma})_p = -2\pi i \left[H\left(t - \frac{\kappa m_1}{m_2}\right) \int_0^{\infty} \left\{ \frac{f(\eta_{RP}^{\oplus} \tan \phi, \eta_{RP}^{\oplus})}{F(\eta_{RP}^{\oplus} \tan \phi, \eta_{RP}^{\oplus})} \left(\frac{d\eta_{RP}^{\oplus}}{dt} \right) - \frac{f(\eta_{RP}^{\ominus} \tan \phi, \eta_{RP}^{\ominus})}{F(\eta_{RP}^{\ominus} \tan \phi, \eta_{RP}^{\ominus})} \left(\frac{d\eta_{RP}^{\ominus}}{dt} \right) \right\} e^{-\rho t} dt \right] \dots (25)$$

where $m_1^2 = z^2 + R^2 \tan^2 \phi$, $m_2^2 = 1 + \tan^2 \phi$

$$\left(\frac{\eta_{RP}^{\oplus}}{\eta_{RP}^{\ominus}} \right) = i \frac{r \tan \phi}{m_1^2} t \pm \frac{z \sqrt{t^2 m_2^2 - \kappa^2 m_1^2}}{m_1^2}, \left(\frac{d\eta_{RP}^{\oplus}/dt}{d\eta_{RP}^{\ominus}/dt} \right) = i \frac{r \tan \phi}{m_1^2} \pm \frac{z t m_2^2}{m_1^2 \sqrt{t^2 m_2^2 - \kappa^2 m_1^2}} \dots (26)$$

Since the right hand side of Eq.(25) is a Laplace transform formula, the integrand gives the formal solution.

$$\rho(\bar{\sigma})_p = -2\pi H\left(t - \frac{\kappa m_1}{m_2}\right) \left(\frac{\text{Re}}{\text{Im}} \right) \frac{f(\eta_{RP} \tan \phi, \eta_{RP})}{F(\eta_{RP} \tan \phi, \eta_{RP})} \left(\frac{d\eta_{RP}}{dt} \right) \dots (27)$$

Equation (27) is a conjugate complex function. Re(real part) or Im(imaginary part) is selected from the characteristic of function of Eq.(27). The Laplace inverse transform for the first term of Eq.(24) also could be performed in the same way.

3.2 The Laplace inverse transform by the three-dimensional Cagniard's method applied to the transformed stress obtained by the line integral

The second term and the third term of the right hand side of Eq.(23) or the second term and fifth term of the right hand side of Eq.(24) are line integrals resulting from the conversion of the Fourier inverse transform. Substituting Eq.(22) into the second and third ones of Eq.(23), we have

$$\rho(\bar{\sigma})_p = \int_{-\infty}^{\infty} d\eta \int_{E_p}^{A_p} G(t, \eta) \left(\frac{d\xi_p^{\oplus}}{dt} \right) e^{-\rho t} dt + \int_{-\infty}^{\infty} d\eta \int_{A_p}^{F_p} G(t, \eta) \left(\frac{d\xi_p^{\ominus}}{dt} \right) e^{-\rho t} dt \dots (28)$$

where

$$\left(\frac{\xi_p^{\oplus}}{\xi_p^{\ominus}} \right) = \left\{ \frac{it \cos \phi}{R} \pm \frac{\sin \phi \sqrt{t^2 - h_1^2(\eta)}}{R} \right\}, \left(\frac{d\xi_p^{\oplus}/dt}{d\xi_p^{\ominus}/dt} \right) = \left\{ \frac{i \cos \phi}{R} \pm \frac{t \sin \phi}{R \sqrt{t^2 - h_1^2(\eta)}} \right\} \dots (29)$$

$t_1 = R s_1$ is the arrival time of a dilatational wave. The time t must satisfy the condition of $t > R \sqrt{\eta^2 + s_1^2}$. Considering this condition, the permitted region of η is given by

$$\sqrt{(t/R)^2 - s_1^2} > \eta_p > -\sqrt{(t/R)^2 - s_1^2} \dots (30)$$

Applying Eq.(30) to the limit of integral of Eq.(28) with respect to η , we have

$$\rho(\bar{\sigma})_p = -H(t - h_1(\eta)) \int_{-\eta_p}^{\eta_p} d\eta \int_0^{\infty} \left\{ G(t, \eta) \left(\frac{d\xi_p^{\oplus}}{dt} \right) - G(t, \eta) \left(\frac{d\xi_p^{\ominus}}{dt} \right) \right\} e^{-\rho t} dt \dots (31)$$

The integrand of Eq.(31) is a formal solution of the stress component $\rho(\sigma)_p$.

$$\rho(\sigma)_p = -H(t - h_1(\eta)) \int_{-\eta_p}^{\eta_p} \left\{ G(t, \eta) \left(\frac{d\xi_p^{\oplus}}{dt} \right) - G(t, \eta) \left(\frac{d\xi_p^{\ominus}}{dt} \right) \right\} d\eta$$

Considering that the integrand is a conjugate complex function and that only the even function is picked up, we have

$$\rho(\sigma)_p = -2H(t - h_1(\eta)) \int_0^{\eta_p} \left(\frac{\text{Re}}{\text{Im}} \right) G(t, \eta) \left(\frac{d\xi_p}{dt} \right) d\eta \dots (32)$$

The Laplace inverse transform in the case of Eq.(24) may be performed in the same way.

3.3 The Laplace inverse transform by the three-dimensional Cagniard's method applied to the transformed stress obtained by the branch cut

The third and fourth terms of right hand side of Eq.(24) show the branch cut with respect to the variable ξ . In this case, it is convenient to represent the variable ξ with branch points or an original point given by the polar coordinate as shown in Fig.4.

$$\begin{aligned}\xi &= i\sqrt{\eta^2 + s_1^2} + \rho_2 e^{i\theta_2} = -i\sqrt{\eta^2 + s_1^2} + \rho_3 e^{i\theta_3} \\ &= i\sqrt{\eta^2 + s_2^2} \rho_1 e^{i\theta_1} = -i\sqrt{\eta^2 + s_2^2} + \rho_4 e^{i\theta_4} \\ &= \rho e^{i\theta}\end{aligned}$$

Using the above relation, $n_1 = \sqrt{\xi^2 + \eta^2 + s_1^2}$ and $n_2 = \sqrt{\xi^2 + \eta^2 + s_2^2}$ become $n_1 = \sqrt{\rho_2 \rho_3} e^{i(\theta_2 + \theta_3)/2}$ and $n_2 = \sqrt{\rho_1 \rho_4} e^{i(\theta_1 + \theta_4)/2}$, respectively. In the case of the contour $A_s \rightarrow B_s$, we have $\theta_1 = -\pi/2$, $\theta_2 = -3\pi/2$, $\theta_3 = \pi/2$, $\theta_4 = \theta = \pi/2$. Substituting these values into the above formulation, we have $n_1 = -i\sqrt{\rho^2 - \eta^2 - s_1^2}$, $n_2 = \sqrt{\eta^2 + s_2^2 - \rho^2}$, $n_3 = \eta^2 + s_2^2/2 - \rho^2$, $\xi = i\rho$. Applying these relations to the third term of Eq.(24), we have

$$\begin{aligned}&\int_{-\infty}^{\infty} d\eta \int_{A_s}^{B_s} G(\xi, \eta) e^{-PN_s} d\eta \\ &= -i \int_{-\infty}^{\infty} d\eta \int_{\sqrt{\eta^2 + s_2^2} \cos \varphi}^{\sqrt{\eta^2 + s_1^2}} G_{AB}(i\rho, \eta) e^{-P(n_2 z + r\rho)} d\rho\end{aligned}\quad (33)$$

Putting $t = n_2 z + r\rho$ to make a change of variable ρ in the time t , we have

$$\left. \begin{aligned}\rho &= \left\{ \frac{t \cos \varphi}{R} + \frac{\sin \varphi \sqrt{h_2^2(\eta) - t^2}}{R} \right\} \\ \frac{d\rho}{dt} &= \left\{ \frac{\cos \varphi}{R} - \frac{t \sin \varphi}{R \sqrt{h_2^2(\eta) - t^2}} \right\}\end{aligned} \right\} \quad (34)$$

To convert the upper limit of integral of Eq.(33) to the time t , we put

$$\sqrt{\eta^2 + s_1^2} = \left\{ \frac{t \cos \varphi}{R} + \frac{\sin \varphi \sqrt{h_2^2(\eta) - t^2}}{R} \right\}$$

The upper limit of integral with respect to the time t is obtained

$$t = \sqrt{s_2^2 - s_1^2} z + \sqrt{\eta^2 + s_1^2} r \quad (35)$$

If we put $\eta = 0$ in Eq.(35), we have the time $t_1 = \sqrt{s_2^2 - s_1^2} z + s_1 r$ which denotes the arrival of the Von-Schmidt wave. The relation of the time t and the arrival time of Von-Schmidt wave t_1 keeps the condition of $t > t_1$.

Considering this condition, the permitted region of η is given by

$$\begin{aligned}&\sqrt{\left(\frac{t - z\sqrt{s_2^2 - s_1^2}}{r} \right)^2 - s_1^2} > \eta_v \\ &> -\sqrt{\left(\frac{t - z\sqrt{s_2^2 - s_1^2}}{r} \right)^2 - s_1^2} \quad (36)\end{aligned}$$

To change the lower limit of integral of Eq.(33) to the time t , we put

Fig.4 Representation of branch cut integral by polar coordinate

$$\sqrt{\eta^2 + s_2^2} \cos \varphi = \left\{ \frac{t \cos \varphi}{R} + \frac{\sin \varphi \sqrt{h_2^2(\eta) - t^2}}{R} \right\}$$

The lower limit of integral with respect to the time t is obtained from the above formula

$$t = R\sqrt{\eta^2 + s_2^2} \quad (37)$$

If we put $\eta = 0$ in Eq.(37), we have the time $t_2 = s_2 R$ at which the shearing wave comes. After the shearing wave has passed, the line integral path gives $E_s \rightarrow A_s \rightarrow D_s \rightarrow F_s$.

Consequently, Eq.(33) will be reduced to Eq.(38)

$$\begin{aligned}AB(\sigma)_v &= 2\{H(t - t_1) - H(t - t_2)\} \int_0^{\eta_v} d\eta \\ &\times \int_0^{\infty} i G_{AB}(i\rho, \eta) \left(\frac{d\rho}{dt} \right) e^{-P\rho} dt \quad (38)\end{aligned}$$

Since Eq.(38) is a Laplace transform integral, the integrand gives the following formal solution.

$$\begin{aligned}AB(\sigma)_v &= 2\{H(t - t_1) - H(t - t_2)\} \\ &\times \int_0^{\eta_v} i G(i\rho, \eta) \left(\frac{d\rho}{dt} \right) d\eta \quad (39)\end{aligned}$$

In the case of the contour passing through $C_s \rightarrow D_s$ as shown in Fig.3, the procedure of the Laplace inverse transform is similar to that employed in Eq.(39). Superimposing the analytical result on the Eq.(39), we have

$$\begin{aligned}(\sigma)_v &= 2\{H(t - t_1) - H(t - t_2)\} \\ &\times \int_0^{\eta_v} i \{G_{AB}(i\rho, \eta) - G_{CD}(i\rho, \eta)\} \left(\frac{d\rho}{dt} \right) d\eta\end{aligned}$$

Since the integrand of the above equation is a conjugate function, we may rewrite as follows

$$\begin{aligned}(\sigma)_v &= 2\{H(t - t_1) - H(t - t_2)\} \\ &\times \int_0^{\eta_v} \left(\frac{Re}{Im} \right) G(i\rho, \eta) \left(\frac{d\rho}{dt} \right) d\eta \quad (40)\end{aligned}$$

The variations of stress components with the normal impact load obtained by the analytical procedure as described above of those after the arrival of shearing wave ($t > s_2 R$) are shown.

$$\begin{aligned}
\sigma_x &= \frac{2p_0}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \cos \phi n_3 \left\{ \xi_p^2 \cos^2 \phi - \eta^2 \sin^2 \phi - \frac{\gamma}{2(1-\gamma)} \right\}}{(\xi_p^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. - \pi \operatorname{Re} \frac{\frac{\gamma}{2(1-\gamma)} n_3}{F(\eta_{RP} \tan \phi, \eta_{RP})} \left(\frac{d\eta_{RP}}{dt} \right) - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \cos \phi n_1 n_2}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\sigma_y &= \frac{2p_0}{(2\pi)^2} \left[\pi \operatorname{Re} \frac{n_3 \left(\eta_{RP}^2 / \cos^2 \phi - \frac{\gamma}{2(1-\gamma)} \right)}{F(\eta_{RP} \tan \phi, \eta_{RP})} \left(\frac{d\eta_{RP}}{dt} \right) \right. \\
&\quad \left. + \int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \cos \phi n_3 \left\{ \xi_p^2 \sin^2 \phi + \eta^2 (1 + \sin^2 \phi) - \frac{\gamma}{2(1-\gamma)} \right\}}{(\xi_p^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. - \pi \operatorname{Re} \frac{\eta_{RS}^2 / \cos^2 \phi n_1 n_2}{F(\eta_{RS} \tan \phi, \eta_{RS})} \left(\frac{d\eta_{RS}}{dt} \right) - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \cos \phi \left\{ \xi_s^2 \sin^2 \phi + \eta^2 (1 + \sin^2 \phi) \right\} n_1 n_2}{(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\sigma_z &= \frac{2p_0}{(2\pi)^2} \left[-\pi \operatorname{Re} \frac{n_3^2}{F(\eta_{RP} \tan \phi, \eta_{RP})} \left(\frac{d\eta_{RP}}{dt} \right) - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \cos \phi n_3^2}{(\xi_p^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. + \pi \operatorname{Re} \frac{\eta_{RS}^2 / \cos^2 \phi n_1 n_2}{F(\eta_{RS} \tan \phi, \eta_{RS})} \left(\frac{d\eta_{RS}}{dt} \right) + \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \cos \phi (\xi_s^2 + \eta^2) n_1 n_2}{(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{xy} &= \frac{2p_0}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \sin \phi n_3}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \sin \phi n_1 n_2}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{yz} &= \frac{2p_0}{(2\pi)^2} \left[-\pi \operatorname{Im} \frac{\eta_{RP} / \cos \phi n_1 n_2}{F(\eta_{RP} \tan \phi, \eta_{RP})} \left(\frac{d\eta_{RP}}{dt} \right) + \int_0^{\eta_p} \operatorname{Re} \frac{(\xi_p^2 + \eta^2) \sin \phi \cos \phi n_1 n_3}{(\xi_p^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. + \pi \operatorname{Im} \frac{\eta_{RS} / \cos \phi n_1 n_2}{F(\eta_{RS} \tan \phi, \eta_{RS})} \left(\frac{d\eta_{RS}}{dt} \right) - \int_0^{\eta_p} \operatorname{Re} \frac{(\xi_s^2 + \eta^2) \sin \phi \cos \phi n_1 n_3}{(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{zx} &= \frac{2p_0}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Re} \frac{n_1 n_3}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta - \int_0^{\eta_p} \operatorname{Re} \frac{n_1 n_3}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \dots\dots\dots (41)
\end{aligned}$$

The variation of stresses after the arrival of shearing wave ($t > S_2 R$) with the tangential impact load are as follows

$$\begin{aligned}
\sigma_x &= \frac{2q_1}{(2\pi)^2} \left[-\int_0^{\eta_p} \operatorname{Re} \frac{n_2 \left\{ \xi_p^2 \cos^2 \phi + \eta^2 \sin^2 \phi - \frac{\gamma}{2(1-\gamma)} \right\}}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. + \int_0^{\eta_p} \operatorname{Re} \frac{n_2^2 n_3 + (\xi_s^2 \sin^2 \phi + \eta^2 \cos^2 \phi) (n_3 - 2n_1 n_2)}{n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\sigma_y &= \frac{2q_1}{(2\pi)^2} \left[-\int_0^{\eta_p} \operatorname{Re} \frac{n_2 \left\{ \xi_p^2 \sin^2 \phi + \eta^2 \cos^2 \phi - \frac{\gamma}{2(1-\gamma)} \right\}}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. + \int_0^{\eta_p} \operatorname{Re} \frac{(\xi_s^2 \sin^2 \phi + \eta^2 \cos^2 \phi) (n_3 - 2n_1 n_2)}{n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\sigma_z &= \frac{2q_1}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Re} \frac{n_2 n_3}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta - \int_0^{\eta_p} \operatorname{Re} \frac{n_2 n_3}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{xy} &= \frac{2q_1}{(2\pi)^2} \left[-\int_0^{\eta_p} \operatorname{Re} \frac{n_2 (\xi_p^2 - \eta^2) \sin \phi \cos \phi}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta \right. \\
&\quad \left. - \pi \operatorname{Im} \frac{\eta_{RS} / \cos \phi \{ n_2^2 n_3 + \eta_{RS}^2 (\cos^2 \phi - \sin^2 \phi)^2 (n_3 - 2n_1 n_2) / \cos^2 \phi \}}{2n_2 F(\eta_{RS} \tan \phi, \eta_{RS})} \left(\frac{d\eta_{RS}}{dt} \right) \right. \\
&\quad \left. + \int_0^{\eta_p} \operatorname{Re} \frac{\sin \phi \cos \phi \{ (\xi_s^2 + \eta^2) n_2^2 n_3 + ((\xi_s^2 - \eta^2) (\sin^2 \phi - \cos^2 \phi) + 4\xi_s^2 \eta^2) (n_3 - 2n_1 n_2) \}}{2(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{yz} &= \frac{2q_1}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \sin \phi n_1 n_2}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \sin \phi n_1 n_2}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \\
\tau_{zx} &= \frac{2q_1}{(2\pi)^2} \left[\int_0^{\eta_p} \operatorname{Im} \frac{\xi_p \cos \phi n_1 n_2}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) d\eta - \pi \operatorname{Re} \frac{n_3^2 - (\eta_{RS}^2 / \cos^2 \phi) n_1 n_2}{F(\eta_{RS} \tan \phi, \eta_{RS})} \left(\frac{d\eta_{RS}}{dt} \right) \right. \\
&\quad \left. - \int_0^{\eta_p} \operatorname{Im} \frac{\xi_s \cos \phi \{ n_3^2 - \xi_s^2 \sin^2 \phi - \eta^2 (1 + \sin^2 \phi) \}}{(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) d\eta \right] \dots\dots\dots (42)
\end{aligned}$$

4. Numerical Evaluation

Equation (32) represents the stress components relating to the dilatational wave which are shown by the definite integral with respect to variable η . The integrands of Eq. (32) calculated for $R=1, \nu=0, \theta=60^\circ$, at $\tau=3$ with $\gamma=0.3$ changing the value of every $\Delta n = \eta_p / n$ (n , the number of divisions) step from $\eta=0$ to $\eta=\eta_p$ are shown in Fig. 5. Since the integrands tend to diverge as $\eta \rightarrow \eta_p$, the

numerical integration of Eq. (32) would deviate far from the correct value. Introducing an auxiliary function $G(t, \eta_p) / \sqrt{t^2 - h_1^2(\eta)}$ $\times (d\xi_p / dt)$ into Eq. (32), we have

$$\begin{aligned}
\rho(\sigma)_p &= -2H(t - h_1(\eta)) \\
&\times \left\{ \int_0^{\eta_p} \left(\operatorname{Re} \frac{1}{R} \left[\frac{G(t, \eta)}{\sqrt{t^2 - h_1^2(\eta)}} \right] \left(\frac{d\xi_p}{dt} \right) \right. \right. \\
&\quad \left. \left. - \frac{G(t, \eta_p)}{\sqrt{t^2 - h_1^2(\eta)}} \left(\frac{d\xi_p}{dt} \right) \right] d\eta \right\}
\end{aligned}$$

$$+ \frac{\pi(\text{Re})}{2(\text{Im})} \frac{G(t, \eta_p)}{R} \left(\frac{d\tilde{\xi}_p}{dt} \right)_{(t=\eta_p)} \} \dots\dots\dots (43)$$

where

$$\frac{d\tilde{\xi}_p}{dt} = i \cos \varphi \sqrt{t^2 - h^2(\eta)} + t \sin \varphi$$

Evaluating the integrand of Eq.(43) with respect to η , we have a closed curve as shown in Fig.6. Consequently, we can perform the numerical integral of Eq.(43). Besides, the second term of Eq.(43) keeps a definite value. Fig.8 shows the variation of stresses with $\gamma=0.3$ for $R=1.0$, at $\theta=0^\circ, 30^\circ, 45^\circ, 60^\circ$ in the vertical x - z plane ($y=0$) containing the position of the normal line load. The stresses at $\theta=0^\circ$ shown in Fig.8-a make a sudden variation at the arrival of a dilatational wave emanating from the point O of load distribu-

tion end as shown in Fig.7. The stress σ_x has a sudden rise upon the arrival of the first front. After this front has passed, σ_x gradually decreases until the shearing wave arrival. After the second front has passed, σ_x approaches the steady state solution σ_{xs} . The steady state solution to the normal load or the tangential load is obtained by the integration of result of Ref. (14) or Ref.(15). The stress after the second front has passed, must be similar to the steady state solution. The stress σ_y changes to a tension from a compression after the dilatational wave P_0 has gone. Upon the arrival of a shearing wave, σ_y has the maximum value, and then approaches the steady state solution with gradual decrease. The stress σ_z has a major compression jump when the first front P_0 comes. Although the stress makes a temporary decrease after the dilata-

Fig.5 Test for convergence of integrand of Eq.32.

Fig.6 Convergence of modified integrand of Eq.43.

Fig.7 Stress waves emanating from the point under line load.

(a) $R=1, \theta=0^\circ; (y=0)$

(c) $R=1, \theta=45^\circ; (y=0)$

(b) $R=1, \theta=30^\circ; (y=0)$

(d) $R=1, \theta=60^\circ; (y=0)$

Fig.8 The stress variation caused by normal impact line load.

tional wave P_0 has passed, σ_x has a summit to increase again before the arrival of the second front S_0 . After the shearing wave has passed, σ_x approaches the steady state σ_{xs} with gradual decrease. The stress τ_{xz} has a sudden positive rise at the arrival of dilatational wave. After the first front has passed, τ_{xz} monotonically increases. τ_{xz} has a maximum value upon the arrival of a shearing wave.

After the shearing wave has passed, τ_{xz} makes a sudden fall and then approaches the steady state τ_{xzs} with gradual decrease. The stresses at $\theta=30^\circ$ shown in Fig.8-b make a sudden variation upon the arrival of the dilatational wave P_m emanating from the point m_0 as shown in Fig.7 in the vertical plane containing the normal line load, and then P_0 from the point O of the distribution end of the line load, subsequently shearing wave S_m, S_0 .

After the shearing wave S_0 has passed, the stresses gradually approach the steady state. At this point, also the stress σ_x has the largest magnitude of variation. The stresses at $\theta=45^\circ$ shown in Fig.8-c have a larger variation than at the other observation point ($\theta=0^\circ, \theta=30^\circ$). However, the shape of variation is similar to that at $\theta=30^\circ$. The stress at $\theta=60^\circ$ shown in Fig.8-d makes a complex variation, since the Von-Schmidt wave $V S_0$ arrives before the arrival of the shearing wave S_0 . In the case as described above, the stresses τ_{xy}, τ_{yz} are zero at $y=0$. Fig.9 shows the variation of stresses with $\gamma=0.3$ for $R=1.0$ at $\theta=0^\circ, 30^\circ, 45^\circ, 60^\circ$ in the vertical $x-z$ plane ($y=0$) containing the position of tangential line load. The stresses at $\theta=0^\circ$ shown in Fig.9-a have a sudden variation upon the arrival of a dilatational wave prop-

agating from the point O of the load distribution end as shown in Fig.7. Although the stresses σ_x, σ_y have a sudden rise in a tension, σ_x, σ_y change to a compression from a tension after the first front P_0 has passed.

Until the shearing wave arrives, the σ_x and σ_y have the same value. The stress σ_x suddenly changes to a tension from a compression upon the arrival of the second front S_0 . The magnitude of σ_x at this time is larger than that at the arrival of the dilatational wave. σ_x gradually approaches the steady state, after the shearing wave has gone. The stress σ_y changes a little at the arrival of the second front S_0 , and then approaches the steady state σ_{ys} . The stress σ_z has a sudden rise in tension upon the arrival of a dilatational wave P_0 . σ_z continues to increase until the second front arrives.

σ_x has the maximum value upon the arrival of the second front S_0 . After the shearing wave has passed, σ_x has a sudden fall for a short time, and then approaches the steady state σ_{xs} with gradual decrease. The stress τ_{xz} is zero at the arrival of a dilatational wave. τ_{xz} has a major negative jump at the arrival of shearing wave S_0 . τ_{xz} rapidly decreases after the second front has passed, and then gradually approaches the steady state τ_{xzs} . The stresses at $\theta=30^\circ$ shown in Fig.9-b have a sudden variation at the arrival of dilatational wave P_m emanating from the point m_0 as shown in Fig.7 existing in the vertical plane containing the location of a tangential line load and then P_0 from the point O of the distribution end of a line load, subsequently shearing waves S_m, S_0 . The stress τ_{xz} has a minor positive

Fig.9 The stress variation caused by shearing impact line load.

jump at the arrival of dilatational wave P_0 .

After the stress waves P_m, P_0 have passed, τ_{xz} gradually increases. Upon the arrival of a shearing wave S_m , τ_{xz} has a major negative jump. After the stress wave S_0 has passed, τ_{xz} has a sudden negative fall, and then gradually approaches the steady state $\tau_{xz\infty}$. The other stress components have a larger variation at the arrival of a shearing wave than that at the arrival of a dilatational wave.

The stresses at $\theta=45^\circ$ shown in Fig.9-c have a larger variation than that at the other observation point ($\theta=0^\circ, \theta=30^\circ$).

However, the shape of variation is similar to that at $\theta=30^\circ$. The stresses at $\theta=60^\circ$ shown in Fig.9-d have a complex variation, since the Von-Schmidt wave VS_0 reaches before the shearing wave S_0 . In the case of $y=0$, the stresses τ_{xy} and τ_{yz} are zero.

5. Conclusions

So far, it has been accepted that Cagniard's method is applicable to the two dimensional elastodynamic and axisymmetric problems. This paper shows how Cagniard's method can be extended to the three dimensional non-axisymmetric problem. The formal solution by the three dimensional Cagniard's method is given by the definite integral. However, the integrand is a divergent at the integral limit. Since the formal solution to the two dimensional elastodynamic and axisymmetric problem is given by an algebraic formula, there is no longer any problem of the divergence. The divergence can be removed by rewriting the integrand in terms of the auxiliary function. The three dimensional Cagniard's method developed here is applicable to the problem of an impact semi-infinite line load. The result is valid, since the stress value after the stress wave has passed, agrees with the steady state solution.

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