The Transient Stress in an Elastic Half-space Excited by Impulsive
Loading over one Quarter of Its Surface*

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The propagation of transient waves in an elastic half space excited by a normal or tangential impulsive loading over one quarter of its surface is investigated. The solution is based on the stress function approach and the Laplace transform, the double Fourier transform method to non-axisymmetric dynamical problem. When the Laplace inverse transform is performed by means of Cagniard's method, it is important whether the formulation has, or has not, singularities on the real axis of Fourier transform plane. The stress transient in load region was compared with that on free surface. The numerical results are shown graphically for the stress variations versus time.

Key Words : Shock, Elastic Half-space, Impulsive Load, Quarter Region, Cagniard's Method, Stress Wave Propagation, Transient Stress

1. Introduction

The elasto-dynamic problems of a half space subjected to a concentrated impact load have been extensively studied. However, there seem to be few works which analysed the response to uniform impact load of the finite (1)-(3) region. Especially, the three dimensional non-axisymmetric problems have seldom been studied. The solution for the case when a uniform pressure is applied on a finite region of surface, will be given by the superposition of concentrated load problem.

The problem of one quadrant may be troublesome on account of calculation time, etc., in the evaluation of infinite double integration by performing the superposition.

It is important to obtain a direct solution in order to remove the difficulties arising in numerical method of superposition. The response of an elastic half space to the normal or tangential uniform pressure over the one quarter region, is analysed with aid of the three dimensional Cagniard's method published in [4]. An analytical process must be performed to note the region classified as under pressure or free surface and the point selected in the vertical plane containing the straight boundary of pressure, when the Fourier inverse integral is converted into Cagniard's integral.

2. Basic Equations

The boundary condition for a normal uniform impact pressure on the one quarter region of elastic half space as shown in

Fig.1 Coordinate system and impulsive load over one quarter region.

Fig.1, is given by

$$(\sigma_z)_{z=0} = -p_0 \delta(t) \lim_{\gamma \rightarrow 0} H(x) e^{-\gamma x} \lim_{\epsilon \rightarrow 0} H(y) e^{-\epsilon y} \quad \dots \dots \dots (1)$$

where $\delta(\)$ is Dirac's delta function and $H(\)$ is the Heaviside unit step function.

The stress components τ_{yz} and τ_{zx} on the surface of half space are given by

$$(\tau_{yz})_{z=0} = (\tau_{zx})_{z=0} = 0 \quad \dots \dots \dots (2)$$

Applying the Laplace transform and the double Fourier transforms to Eq.(1) and Eqs.(2), we have

$$\begin{aligned} (\bar{\sigma}_z)_{z=0}^{**} &= -p_0 \lim_{\gamma \rightarrow 0} \frac{1}{(\gamma + i p a)} \lim_{\epsilon \rightarrow 0} \frac{1}{(\epsilon + i p b)} \\ &= \frac{p_0}{b^2 a \beta} \quad \dots \dots \dots (3) \end{aligned}$$

$$(\bar{\tau}_{yz})_{z=0}^{**} = (\bar{\tau}_{zx})_{z=0}^{**} = 0 \quad \dots \dots \dots (4)$$

where

$$\sigma^{**} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma e^{-i p (a x + b y)} dx dy$$

$$\bar{\sigma} = \int_0^{\infty} \sigma e^{-p t} dt$$

The Fourier inverse transformed stress components are obtained from the boundary condition of Eq.(3) and Eq.(4).

$$\bar{\sigma}_x = \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{n_3 \left(a^2 - \frac{\nu}{2(1-\nu)} s_3^2 \right)}{a \beta F(\alpha, \beta)} e^{-p N_1} \right.$$

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$$\begin{aligned}
& + \frac{\alpha n_1 n_2}{\beta F(\alpha, \beta)} e^{-\beta N_1} \} dad\beta \\
\bar{\sigma}_y = & \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{n_3 \left(\beta^2 - \frac{\nu}{2(1-\nu)} s_1^2 \right)}{\alpha \beta F(\alpha, \beta)} e^{-\beta N_1} \right. \\
& \left. + \frac{\beta n_1 n_2}{\alpha F(\alpha, \beta)} e^{-\beta N_1} \right\} dad\beta \\
\bar{\sigma}_z = & \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{n_3^2}{\alpha \beta F(\alpha, \beta)} e^{-\beta N_1} \right. \\
& \left. - \frac{(\alpha^2 + \beta^2) n_1 n_2}{\alpha \beta F(\alpha, \beta)} e^{-\beta N_1} \right\} dad\beta \\
\bar{\tau}_{xy} = & \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -\frac{n_3}{F(\alpha, \beta)} e^{-\beta N_1} \right. \\
& \left. + \frac{n_1 n_2}{F(\alpha, \beta)} e^{-\beta N_1} \right\} dad\beta \\
\bar{\tau}_{yz} = & \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{n_1 n_3}{\alpha F(\alpha, \beta)} e^{-\beta N_1} \right. \\
& \left. + i \frac{n_1 n_3}{\alpha F(\alpha, \beta)} e^{-\beta N_1} \right\} dad\beta \\
\bar{\tau}_{zx} = & \frac{p_0}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -i \frac{n_1 n_3}{\beta F(\alpha, \beta)} e^{-\beta N_1} \right. \\
& \left. + i \frac{n_1 n_3}{\beta F(\alpha, \beta)} e^{-\beta N_1} \right\} dad\beta
\end{aligned}
\tag{5}$$

where $N_1 = n_1 z - i\alpha x - i\beta y$, $N_2 = n_2 z - i\alpha x - i\beta y$,
 $n_1 = \sqrt{\alpha^2 + \beta^2 + s_1^2}$, $n_2 = \sqrt{\alpha^2 + \beta^2 + s_2^2}$, $n_3 = (\alpha^2 + \beta^2 + s_1^2/2)$,
 $F(\alpha, \beta) = n_3^2 - (\alpha^2 + \beta^2) n_1 n_2$, $s_1 = 1/c_1$, $s_2 = 1/c_2$

c_1 is the velocity of a dilatational wave and c_2 that of a shearing wave. The brief formulation for Eq.(5) representing the stress components is given as

$$\bar{\sigma} = \frac{p_0}{(2\pi)^2} \{ (\bar{\sigma})_p + (\bar{\sigma})_s \} \tag{6}$$

The first term of right hand side of Eq.(6) is for a dilatational wave

$$(\bar{\sigma})_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_p(\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-\beta N_1} dad\beta \tag{7}$$

and the second term is for a shearing wave

$$(\bar{\sigma})_s = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_s(\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-\beta N_1} dad\beta \tag{8}$$

$$(\bar{\sigma})_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_p(\xi, \eta) e^{-\beta(n_1 z - i\xi r)} }{(\xi \cos \phi - \eta \sin \phi)(\xi \sin \phi + \eta \cos \phi) F(\xi, \eta)} d\xi d\eta \tag{10}$$

where $n_1 = \sqrt{\xi^2 + \eta^2 + s_1^2}$, $n_2 = \sqrt{\xi^2 + \eta^2 + s_2^2}$, $n_3 = (\xi^2 + \eta^2 + s_1^2/2)$, $F(\xi, \eta) = n_3^2 - (\xi^2 + \eta^2) n_1 n_2$

and j is taken as 1 and 2. The asymptotic of integrand of Eq.(10) for $\xi \rightarrow \infty$ is given by

$$\lim_{\xi \rightarrow \infty} \frac{f_p(\xi, \eta)}{(\xi \cos \phi - \eta \sin \phi)(\xi \sin \phi + \eta \cos \phi) F(\xi, \eta)} \rightarrow 0$$

To change the variable ξ to the time t , we put

$$n_1 z - i\xi r = t \tag{11}$$

The variable ξ is represented by the time t as follows,

$$\xi = \frac{1}{R} \{ it \cos \varphi \pm \sin \varphi \sqrt{t^2 - h_1^2(\eta)} \} \tag{12}$$

where $\cos \varphi = r/R$, $\sin \varphi = z/R$, $R^2 = x^2 + y^2 + z^2$

$$h_1^2(\eta) = R^2(\eta^2 + s_1^2) \tag{13}$$

Observing the locus of ξ to the time t of Eq.(12), we have Cagniard's paths, as shown in

Fig.2 Classified region in analytical process

Fig.3 Cagniard's inverse integral path corresponding to Eq.10.

Equation (7) and Eq.(8) represent the stress components to the first quadrant [I] in x - y coordinate system where the region [I] is classified as shown in Fig.2. In the case of an analytical processing of the classified region and the point in the vertical plane containing the coordinate axis, we must take note the exponential function of Eq.(7) and Eq.(8).

3. The Conversion of Fourier Inverse Transform Integral Path

3.1 Analysis of classified region [I] ($x > 0, y > 0, z > 0$)

In the case of the classified region [I], we may use Eq.(7) and Eq.(8) exactly as they are. We change the variables α and β to ξ and η .

$$\alpha = \xi \cos \phi - \eta \sin \phi, \beta = \xi \sin \phi + \eta \cos \phi \tag{9}$$

where $\cos \phi = x/r$, $\sin \phi = y/r$, $r = \sqrt{x^2 + y^2}$

Substituting Eq.(9) into Eq.(7) and Eq.(8), we have

Fig.3, which consist of the line integral path Γ and the contour of $\Gamma_1 \rightarrow [\Gamma_2, \Gamma_2', \Gamma_2''] \rightarrow \Gamma_3$. In this case, since the simple poles of $\xi = -\eta \cot \phi$ and $\xi = \eta \tan \phi$ are involved inside of the closed curve of Cagniard's path, the inverse transform may be superseded by the evaluation of residue. $i = \sqrt{\eta^2 + s^2}$ and $i\sqrt{\eta^2 + s^2}$ in Fig.3 are branch points. Since a pole $i\sqrt{\eta^2 + s^2}$ is outside of the closed curve of Cagniard's path, the residue theorem is not applicable. The integral contour passing through Γ_2 gives a response due to the dilatational wave. The contour of Γ_2' relates to the shearing wave and the path of Γ_2'' depends on the Von-Schmidt wave. Notation s_R is a reciprocal of Rayleigh speed c_R . Evaluating the residue and then converting the integral path, Eq. (10) may be rewritten as follows

$$(\bar{\sigma})_{[I]} = 2\pi i \left[\int_{-\infty}^{\infty} \frac{f_{[I]}(\eta \tan \phi, \eta) e^{-p(n_j z - i r \eta \tan \phi)}}{\eta / \cos \phi F(\eta \tan \phi, \eta)} d\eta + \int_{-\infty}^{\infty} \frac{f_{[I]}(-\eta \cot \phi, \eta) e^{-(n_j z + i r \eta \cot \phi)}}{-\eta / \sin \phi F(-\eta \cot \phi, \eta)} d\eta \right] + \int_{-\infty}^{\infty} d\eta \int_E^0 G(\xi, \eta) d\xi + \int_{-\infty}^{\infty} d\eta \int_0^F G(\xi, \eta) d\xi \quad (14)$$

where $G(\xi, \eta)$ is an integrand. It is necessary to obtain a formal solution by applying the Laplace inverse transform to Eq. (14). In this case, the Laplace inverse transform will be performed by the three-dimensional Cagniard's method.

3.2 Analysis for the classified region [II] ($x < 0, y > 0, z > 0$)

Considering that the classified region [II] is in the fourth quadrant, a brief formulation of stress components is given by

$$(\bar{\sigma})_{[II]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[II]}(\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-p(n_j z + i \alpha |x| - i \beta y)} d\alpha d\beta \quad (15)$$

Substituting $\alpha = \xi \cos \phi - \eta \sin \phi, \beta = -(\xi \sin \phi + \eta \cos \phi)$ into Eq. (15), we obtain

$$(\bar{\sigma})_{[II]} = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[II]}(\xi, \eta) e^{-p(n_j z + i \eta r)}}{(\xi \cos \phi - \eta \sin \phi)(\xi \sin \phi + \eta \cos \phi) F(\xi, \eta)} d\xi d\eta \quad (16)$$

where $\cos \phi = |x|/r, \sin \phi = y/r$

Changing the variable ξ to the time t , we put $t = n_j z + i \xi r$ of the augment of exponential function of Eq. (16). The variable ξ is represented by the time t as follows

$$\xi = -\frac{i t \cos \phi}{R} \pm \frac{\sin \phi \sqrt{t^2 - h^2(\eta)}}{R} \quad (17)$$

Considering the asymptotic integrand of Eq. (16) for $\xi \rightarrow \infty$, the Fourier inverse integral path may be converted into Cagniard's path as shown in Fig.4. In this case, since the simple poles of $\xi = \eta \tan \phi$ and $\xi = -\eta \cot \phi$ exist outside of the closed curve of Cagniard's path, the residue theorem can not be applied. With conversion of Eq. (16), we have

$$(\bar{\sigma})_{[II]} = - \left\{ \int_{-\infty}^{\infty} d\eta \int_E^0 G(\xi, \eta) d\xi + \int_{-\infty}^{\infty} d\eta \int_0^F G(\xi, \eta) d\xi \right\} \quad (18)$$

3.3 Analysis for the classified region [III] ($x < 0, y < 0, z > 0$)

Considering that classified region [III] is in the third quadrant, a brief formulation of stress components is given by

$$(\bar{\sigma})_{[III]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[III]}(\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-p(n_j z + i \alpha |x| + i \beta |y|)} d\alpha d\beta \quad (19)$$

Substituting $\alpha = \xi \cos \phi - \eta \sin \phi, \beta = \xi \sin \phi + \eta \cos \phi$ into Eq. (19), we have

$$(\bar{\sigma})_{[III]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[III]}(\xi, \eta) e^{-p(n_j z + i \eta r)}}{(\xi \cos \phi - \eta \sin \phi)(\xi \sin \phi + \eta \cos \phi) F(\xi, \eta)} d\xi d\eta \quad (20)$$

where $\cos \phi = |x|/r, \sin \phi = |y|/r$

In this analytical region, we also can use Eq. (17) in the conversion into Cagniard's path.

The Fourier inverse integral path of Eq. (20) may be changed as shown in Fig.4. With conversion of Eq. (20), the Laplace transformed stress is obtained as follows

$$(\bar{\sigma})_{[III]} = \int_{-\infty}^{\infty} d\eta \int_E^0 G(\xi, \eta) d\xi + \int_{-\infty}^{\infty} d\eta \int_0^F G(\xi, \eta) d\xi \quad (21)$$

3.4 Analysis for the classified region [IV] ($x > 0, y < 0, z > 0$)

Considering that the Classified region [IV] is in the second quadrant, a brief formulation of stress components is given by

$$(\bar{\sigma})_{[IV]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[IV]}(\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-p(n_j z - i \alpha x + i \beta |y|)} d\alpha d\beta \quad (22)$$

Substituting $\alpha = \xi \cos \phi - \eta \sin \phi, \beta = -(\xi \sin \phi + \eta \cos \phi)$ into Eq. (22), we obtain

$$(\bar{\sigma})_{[IV]} = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_{[IV]}(\xi, \eta) e^{-p(n_j z - i \eta r)}}{(\xi \cos \phi - \eta \sin \phi)(\xi \sin \phi + \eta \cos \phi) F(\xi, \eta)} d\xi d\eta \quad (23)$$

where $\cos \phi = x/r, \sin \phi = |y|/r$

Since Eq. (23) is similar to Eq. (10), we can use Cagniard's path as shown in Fig. 3. With conversion of Eq. (23), the Laplace transformed stress is obtained as follows

$$(\bar{\sigma})_{[q]} = -2\pi i \left[\int_{-\infty}^{\infty} \frac{f[\eta](\eta \tan \phi, \eta) e^{-p(\eta/z - i\eta \tan \phi)}}{\eta / \cos \phi F(\eta \tan \phi, \eta)} d\eta + \int_{-\infty}^{\infty} \frac{f[\eta](-\eta \cot \phi, \eta) e^{-p(\eta/z + i\eta \cot \phi)}}{-\eta / \sin \phi F(-\eta \cot \phi, \eta)} d\eta \right] - \left\{ \int_{-\infty}^{\infty} d\eta \int_{\xi}^{\infty} G(\xi, \eta) d\xi + \int_{-\infty}^{\infty} d\eta \int_{\xi}^{\infty} G(\xi, \eta) d\xi \right\} \dots (24)$$

3.5 Analysis for the selected point in the vertical plane containing x axis

When we select the analytical point in the vertical plane containing x axis, the coordinate value is $y=0$. Substituting this value into Eq. (7) or Eq. (8), we obtain

$$(\bar{\sigma})_{[q]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-p(\eta/z - i\alpha x)} d\alpha d\beta \dots (25)$$

It is necessary to discuss the conversion of integral path of the Fourier inverse transform with respect to variable α of Eq. (25). To make a change of variable α into the time t , we put the augment of exponential function of Eq. (25) $t = \eta/z - i\alpha x$ and obtain Eq. (26). The variable α is represented by the time t .

$$\alpha = i \frac{x}{x^2 + z^2} t \pm \frac{z \sqrt{t^2 - (x^2 + z^2)(\beta^2 + s^2)}}{x^2 + z^2} \dots (26)$$

Observing the locus of α to time t of Eq. (26), we have Cagniard's path as shown in Fig. 5 which consists of the line integral path Γ and the contour of $\Gamma_1 \rightarrow [\Gamma_2, \Gamma_2', \Gamma_2''] \rightarrow \Gamma_3$.

The simple pole at $\alpha=0$ exists inside of the closed curve of Cagniard's path. Evaluating the residue to the simple pole $\alpha=0$ and then integrating along the Cagniard's path, we have

$$(\bar{\sigma})_{[q]} = 4\pi i \int_0^{\infty} \frac{f[\eta](0, \beta)}{\beta F(0, \beta)} e^{-p\sqrt{\beta^2 + s^2}} d\beta + \int_{-\infty}^{\infty} d\beta \int_{\xi}^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} d\alpha + \int_{-\infty}^{\infty} d\beta \int_0^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} d\alpha \dots (27)$$

In the case of condition $x < 0$, the formulation representing the locus of α to the time t is given by

$$\alpha = -i \frac{x}{x^2 + z^2} t \pm \frac{z \sqrt{t^2 - (z^2 + x^2)(\beta^2 + s^2)}}{x^2 + z^2} \dots (28)$$

Then, we have Cagniard's path as shown in Fig. 6. Since the simple pole $\alpha=0$ may not be contained inside of closed curve of Cagniard's path, the residue theorem is not applicable. Therefore, in this case Eq. (25) may be rewritten as

$$(\bar{\sigma})_{[q]} = \int_{-\infty}^{\infty} d\beta \int_{\xi}^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} d\alpha + \int_{-\infty}^{\infty} d\beta \int_0^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} d\alpha \dots (29)$$

3.6 Analysis for the selected point in the vertical plane containing y axis

When we select the analytical point in the vertical plane containing y axis, it is necessary to replace the coordinate value x

Fig. 4 Cagniard's inverse integral path corresponding to Eq. 16.

Fig. 5 Cagniard's inverse integral path in the case of Eq. 25 for $x > 0$

Fig. 6 Cagniard's inverse integral path in the case of 25 for $x < 0$

of Eq. (26) or Eq. (28) with the value y , and rearrange the order of integrand with respect to variable α into the order of variable β in Eq. (27) or Eq. (29).

3.7 Analysis for the selected point on the z axis

In the case of the analytical point being selected on the z axis, it is necessary to substitute the coordinate value $x=0$ and $y=0$ into Eq. (7) or Eq. (8). We have then

$$(\bar{\sigma})_{[q]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f[\eta](\alpha, \beta)}{\alpha \beta F(\alpha, \beta)} e^{-p\eta/z} d\alpha d\beta \dots (30)$$

Since the integrand of Eq. (30) is an odd function with respect to variable α , the integration with respect to α will be zero.

However, the integrand has the simple pole at $\alpha=0$. Evaluating the residue of pole at $\alpha=0$, we have

$$(\bar{\sigma})_{[q]} = 2\pi i \int_{-\infty}^{\infty} \frac{f[\eta](0, \beta)}{\beta F(0, \beta)} e^{-p\sqrt{\beta^2 + s^2}} d\beta \dots (31)$$

Since the integrand of Eq. (31) is an

odd function with respect to variable β , the integration with respect to β will be zero.

However, the integrand of Eq. (31) has the simple pole at $\beta=0$. It is necessary to calculate the residue for the pole at $\beta=0$.

$$(\bar{\sigma})_{[0]} = -4\pi \frac{f_{[0]}(0,0)}{F(0,0)} e^{-\rho_0 z} \dots\dots\dots (32)$$

Applying the Laplace inverse transform to Eq. (32), we have the following formal solution

$$(\bar{\sigma})_{[0]} = \frac{-4\pi}{2\pi i} \int_{\Gamma} \frac{f_{[0]}(0,0)}{F(0,0)} e^{-\rho(t-s)z} d\rho \dots\dots (33)$$

The three dimensional Cagniard's method can not be applied to this case. An inte-

gration along Bromwich's line path is required. The stress components over all regions can not be represented by the identical formula obtained from the analysis. It is important to classify the analytical process by considering whether the selected point is or is not in the pressure region and in the vertical plane containing the pressure boundary.

4. Analytical Results

The formal solution to Eq. (14) for the classified region [I] may be obtained by the three dimensional Cagniard's method. The stress components after a shearing wave has passed, are given by

$$\begin{aligned} \sigma_x = & \frac{2\rho_0}{(2\pi)^2} \left[-\pi \operatorname{Im} \left\{ \frac{n_3 \frac{\nu}{2(1-\nu)}}{\eta_{RP1}/\cos\phi F(\eta_{RP1}\tan\phi, \eta_{RP1})} \left(\frac{d\eta_{RP1}}{dt} \right) \right\} \right. \\ & - \pi \operatorname{Im} \left\{ \frac{n_3 \left(\eta_{RP2}^2/\sin^2\phi - \frac{\nu}{2(1-\nu)} \right)}{\eta_{RP2}/\sin\phi F(-\eta_{RP2}\cot\phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right\} \\ & - \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{n_3 \sin\phi \cos\phi \{ (\xi_p^2 \cos^2\phi + \eta^2 \sin^2\phi)(\xi_p^2 - \eta^2) + 2\xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi) - \frac{\nu}{2(1-\nu)} (\xi_p^2 - \eta^2) \}}{((\xi_p^2 - \eta^2)^2 \sin^2\phi \cos^2\phi - \xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi)^2) F(\xi_p, \eta)} \right. \\ & \times \left(\frac{d\xi_p}{dt} \right) \Bigg] d\eta - \pi \operatorname{Im} \left\{ \frac{\eta_{RS2}/\sin\phi n_1 n_2}{F(-\eta_{RS2}\cot\phi, \eta_{RS2})} \left(\frac{d\eta_{RS2}}{dt} \right) \right\} \\ & + \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{(\xi_p^2 + \eta^2) \sin\phi \cos\phi n_1 n_2}{(\xi_p^2 \sin^2\phi - \eta^2 \cos^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \Bigg] \\ \sigma_y = & \frac{2\rho_0}{(2\pi)^2} \left[\operatorname{Im} \left\{ \pi \frac{n_3 \left(\eta_{RP1}^2/\cos^2\phi - \frac{\nu}{2(1-\nu)} \right)}{\eta_{RP1}/\cos\phi F(\eta_{RP1}\tan\phi, \eta_{RP1})} \left(\frac{d\eta_{RP1}}{dt} \right) \right\} \right. \\ & + \operatorname{Im} \left\{ \pi \frac{n_3 \frac{\nu}{2(1-\nu)}}{\eta_{RP2}/\sin\phi F(-\eta_{RP2}\cot\phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right\} \\ & - \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{n_3 \sin\phi \cos\phi \{ (\xi_p^2 \cos^2\phi + \eta^2 \sin^2\phi)(\xi_p^2 - \eta^2) - 2\xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi) - \frac{\nu}{2(1-\nu)} (\xi_p^2 - \eta^2) \}}{((\xi_p^2 - \eta^2)^2 \sin^2\phi \cos^2\phi - \xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi)^2) F(\xi_p, \eta)} \right. \\ & \times \left(\frac{d\xi_p}{dt} \right) \Bigg] d\eta - \pi \operatorname{Im} \left\{ \frac{\eta_{RS1}/\cos\phi n_1 n_2}{F(\eta_{RS1}\tan\phi, \eta_{RS1})} \left(\frac{d\eta_{RS1}}{dt} \right) \right\} \\ & + \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{(\xi_p^2 + \eta^2) \sin\phi \cos\phi n_1 n_2}{(\xi_p^2 \cos^2\phi - \eta^2 \sin^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \Bigg] \\ \sigma_z = & \frac{2\rho_0}{(2\pi)^2} \left[-\pi \operatorname{Im} \left\{ \frac{n_3^2}{\eta_{RP1}/\cos\phi F(\eta_{RP1}\tan\phi, \eta_{RP1})} \left(\frac{d\eta_{RP1}}{dt} \right) \right\} + \pi \operatorname{Im} \left\{ \frac{n_3^2}{\eta_{RP2}/\sin\phi F(-\eta_{RP2}\cot\phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right\} \right. \\ & \times \left(\frac{d\eta_{RP2}}{dt} \right) \Bigg] + \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{(\xi_p^2 - \eta^2) \sin\phi \cos\phi n_3^2}{((\xi_p^2 - \eta^2)^2 \sin^2\phi \cos^2\phi - \xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi)^2) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \\ & + \pi \operatorname{Im} \left\{ \frac{\eta_{RS1}^2 (1 + \tan^2\phi) n_1 n_2}{\eta_{RS1}/\cos\phi F(\eta_{RS1}\tan\phi, \eta_{RS1})} \left(\frac{d\eta_{RS1}}{dt} \right) \right\} - \pi \operatorname{Im} \left\{ \frac{\eta_{RS2}^2 (1 + \cot^2\phi) n_1 n_2}{\eta_{RS2}/\sin\phi F(-\eta_{RS2}\cot\phi, \eta_{RS2})} \left(\frac{d\eta_{RS2}}{dt} \right) \right\} \\ & - \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{(\xi_p^2 - \eta^4) \sin\phi \cos\phi n_1 n_2}{((\xi_p^2 - \eta^2)^2 \sin^2\phi \cos^2\phi - \xi_p^2 \eta^2 (\sin^2\phi - \cos^2\phi)^2) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \Bigg] \\ \tau_{xy} = & \frac{2\rho_0}{(2\pi)^2} \left[-\int_0^{\eta_p} \operatorname{Re} \left\{ \frac{n_3}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta + \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{n_1 n_2}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \right] \\ \tau_{yz} = & \frac{2\rho_0}{(2\pi)^2} \left[\pi \operatorname{Re} \left\{ \frac{n_1 n_3}{F(\eta_{RP1}\tan\phi, \eta_{RP1})} \left(\frac{d\eta_{RP1}}{dt} \right) \right\} + \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \cos\phi n_1 n_3}{(\xi_p^2 \cos^2\phi - \eta^2 \sin^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \right. \\ & - \pi \operatorname{Re} \left\{ \frac{n_1 n_3}{F(\eta_{RS1}\tan\phi, \eta_{RS1})} \left(\frac{d\eta_{RS1}}{dt} \right) \right\} - \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \cos\phi n_1 n_3}{(\xi_p^2 \sin^2\phi - \eta^2 \cos^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \Bigg] \\ \tau_{xz} = & \frac{2\rho_0}{(2\pi)^2} \left[\pi \operatorname{Re} \left\{ \frac{n_1 n_3}{F(-\eta_{RP2}\cot\phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right\} + \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \sin\phi n_1 n_2}{(\xi_p^2 \sin^2\phi - \eta^2 \cos^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \right. \\ & + \pi \operatorname{Re} \left\{ \frac{n_1 n_3}{F(-\eta_{RS2}\cot\phi, \eta_{RS2})} \left(\frac{d\eta_{RS2}}{dt} \right) \right\} - \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \sin\phi n_1 n_2}{(\xi_p^2 \sin^2\phi - \eta^2 \cos^2\phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \Bigg] \end{aligned} \dots\dots\dots (34)$$

where

$$\eta_{R1} = i \frac{r \tan\phi}{z^2 + R^2 \tan^2\phi} t + \frac{z \sqrt{t^2 (1 + \tan^2\phi) - l^2 (z^2 + R^2 \tan^2\phi)}}{z^2 + R^2 \tan^2\phi}$$

$$\frac{d\eta_{R1}}{dt} = i \frac{r \tan \phi}{z^2 + R^2 \tan^2 \phi} + \frac{z(1 + \tan^2 \phi)t}{(z^2 + R^2 \tan^2 \phi)\sqrt{l^2(1 + \tan^2 \phi) - l^2(z^2 + R^2 \tan^2 \phi)}} \quad (35)$$

$$\eta_{R2} = -i \frac{r \cot \phi}{z^2 + R^2 \cot^2 \phi} t + \frac{z\sqrt{l^2(1 + \cot^2 \phi) - l^2(z^2 + R^2 \cot^2 \phi)}}{z^2 + R^2 \cot^2 \phi}$$

$$\frac{d\eta_{R2}}{dt} = -i \frac{r \cot \phi}{z^2 + R^2 \cot^2 \phi} + \frac{z(1 + \cot^2 \phi)t}{(z^2 + R^2 \cot^2 \phi)\sqrt{l^2(1 + \cot^2 \phi) - l^2(z^2 + R^2 \cot^2 \phi)}} \quad (36)$$

$$\xi_{[1]} = \frac{1}{R} \{it \cos \varphi + \sin \varphi \sqrt{l^2 - h^2(\eta)}\}, \quad \frac{d\xi_{[1]}}{dt} = \frac{1}{R} \left\{ i \cos \varphi + \frac{t \sin \varphi}{\sqrt{l^2 - h^2(\eta)}} \right\} \quad (37)$$

Re is a real part of complex number and Im is an imaginary part. We have the relation of $l^2 = \kappa^2$ for the dilatational wave, with $\kappa^2 = (1 - 2\nu)/2(1 - \nu)$ and $l^2 = 1$ for the shearing wave. The boundary conditions for a uniformly distributed tangential load on one quadrant are given by

$$(\tau_{zx})_{z=0} = -q_1 \delta(t) \lim_{y \rightarrow 0} H(x) e^{-\gamma y} \lim_{x \rightarrow 0} H(y) e^{-\xi x} \quad (38)$$

$$(\sigma_x)_{z=0} = (\tau_{yz})_{z=0} = 0 \quad (39)$$

Analysing in the same way as with the normal load, the variation of stresses in the classified region [I] after the shearing wave has passed, is obtained as follows;

$$\sigma_x = \frac{2q_1}{(2\pi)^2} \left[-\pi \operatorname{Re} \frac{n_2 \left\{ \eta_{RP2}^2 / \sin^2 \phi - \frac{\nu}{2(1-\nu)} \right\}}{F(-\eta_{RP2} \cot \phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right.$$

$$- \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \sin \phi n_2 \left\{ \xi_p^2 \cos^2 \phi + \eta^2(1 + \cos^2 \phi) - \frac{\nu}{2(1-\nu)} \right\}}{(\xi_p^2 \sin^2 \phi - \eta^2 \cos^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta$$

$$+ \pi \operatorname{Re} \frac{n_2 n_3}{F(-\eta_{RS1} \cot \phi, \eta_{RS1})} \left(\frac{d\eta_{RS1}}{dt} \right)$$

$$+ \int_0^{\eta_s} \operatorname{Im} \left\{ \frac{\xi_s \sin \phi (n_2 + \xi_s^2 \sin^2 \phi - \eta^2 \cos^2 \phi) n_3 - 2\xi_s \sin \phi (\xi_s^2 \sin^2 \phi - \eta^2 \cos^2 \phi) n_1 n_2}{(\xi_s^2 \sin^2 \phi - \eta^2 \cos^2 \phi) n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \Big]$$

$$\sigma_y = \frac{2q_1}{(2\pi)^2} \left[\pi \operatorname{Re} \frac{n_2 \frac{\nu}{2(1-\nu)}}{F(-\eta_{RP2} \cot \phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) \right.$$

$$- \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \sin \phi n_2 \left\{ \xi_p^2 \sin^2 \phi - \eta^2 \cos^2 \phi - \frac{\nu}{2(1-\nu)} \right\}}{(\xi_p^2 \sin^2 \phi - \eta^2 \cos^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta$$

$$- \int_0^{\eta_s} \operatorname{Im} \left\{ \frac{\xi_s \sin \phi (n_3 - 2n_1 n_2)}{n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \Big]$$

$$\sigma_z = \frac{2q_1}{(2\pi)^2} \left[\pi \operatorname{Re} \frac{n_2 n_3}{F(-\eta_{RP2} \cot \phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) + \int_0^{\eta_p} \operatorname{Im} \left\{ \frac{\xi_p \sin \phi n_2 n_3}{(\xi_p^2 \sin^2 \phi - \eta^2 \cos^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \right.$$

$$- \pi \operatorname{Re} \frac{n_2 n_3}{F(-\eta_{RS2} \cot \phi, \eta_{RS2})} \left(\frac{d\eta_{RS2}}{dt} \right) - \int_0^{\eta_s} \operatorname{Im} \left\{ \frac{\xi_s \sin \phi n_2 n_3}{(\xi_s^2 \sin^2 \phi - \eta^2 \cos^2 \phi) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \Big]$$

$$\tau_{xy} = \frac{2q_1}{(2\pi)^2} \left[\int_0^{\eta_p} -\operatorname{Im} \left\{ \frac{\xi_p \cos \phi n_2}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta + \pi \operatorname{Re} \frac{\{n_2^2 n_3 + \eta_{RS1}^2 / \cos^2 \phi (n_3 - 2n_1 n_2)\}}{2F(\eta_{RS1} \tan \phi, \eta_{RS1}) n_2} \left(\frac{d\eta_{RS1}}{dt} \right) \right.$$

$$+ \int_0^{\eta_s} \operatorname{Im} \left\{ \frac{n_2^2 n_3 \xi_s \cos \phi + \xi_s \cos \phi \{ \xi_s^2 (\sin^2 \phi - \cos^2 \phi) + \eta^2 (2 \sin^2 \phi + 1) \} (n_3 - 2n_1 n_2)}{2(\xi_s^2 \cos^2 \phi - \eta^2 \sin^2 \phi) n_2 F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \Big]$$

$$\tau_{yz} = \frac{2q_1}{(2\pi)^2} \left[\int_0^{\eta_p} -\operatorname{Re} \left\{ \frac{n_1 n_2}{F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta + \int_0^{\eta_s} \operatorname{Re} \left\{ \frac{n_1 n_2}{F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \right]$$

$$\tau_{zx} = \frac{2q_1}{(2\pi)^2} \left[\pi \operatorname{Im} \frac{\eta_{RP2} / \sin \phi n_1 n_2}{F(-\eta_{RP2} \cot \phi, \eta_{RP2})} \left(\frac{d\eta_{RP2}}{dt} \right) - \int_0^{\eta_p} \operatorname{Re} \left\{ \frac{(\xi_p^2 + \eta^2) \sin \phi \cos \phi n_1 n_2}{(\xi_p^2 \sin^2 \phi - \eta^2 \cos^2 \phi) F(\xi_p, \eta)} \left(\frac{d\xi_p}{dt} \right) \right\} d\eta \right.$$

$$- \pi \operatorname{Im} \left\{ \frac{n_2^2 - n_1 n_2 (\eta_{RS1} / \cos \phi)^2}{\eta_{RS1} / \cos \phi F(\eta_{RS1} \tan \phi, \eta_{RS1})} \right\} \left(\frac{d\eta_{RS1}}{dt} \right) + \pi \operatorname{Im} \left\{ \frac{n_2^2}{\eta_{RP2} / \sin \phi F(-\eta_{RS2} \cot \phi, \eta_{RS2})} \right\} \left(\frac{d\eta_{RS2}}{dt} \right) \Big]$$

$$+ \int_0^{\eta_s} \operatorname{Re} \left\{ \frac{n_2^2 (\xi_s^2 - \eta^2) \sin \phi \cos \phi - n_1 n_2 \sin \phi \cos \phi \{ (\xi_s^2 - \eta^2) \times (\xi_s^2 \sin^2 \phi + \eta^2 \cos^2 \phi) + 2\xi_s^2 \eta^2 (\sin^2 \phi - \cos^2 \phi) \}}{((\xi_s^2 - \eta^2)^2 \sin^2 \phi \cos^2 \phi - \xi_s^2 \eta^2 (\sin^2 \phi - \cos^2 \phi)^2) F(\xi_s, \eta)} \left(\frac{d\xi_s}{dt} \right) \right\} d\eta \Big]$$

.....(40)

5. Numerical Evaluation

The variation of stresses at the selected points B_1 ($x=2.0$, $y=1.0$, $z=1.0$) involved in the classified region [I] under

a normal impulsive pressure and at the point B_2 ($x=-2.0$, $y=1.0$, $z=1.0$) contained in the classified region [II] under a free surface, is shown in Fig.8. The stresses at the point B_1 have a sudden rise upon the arrival of a dilatational wave P_m emanating from the point m_1 as shown in Fig.7 and then

Fig.7 Stress waves coming to observation point B_1 or B_2

Fig.8 The stress variation caused by normal impulsive load.

P_{nx} from the point nx on the straight boundary of pressure of $y=0$, P_{ny} from the point ny on the straight boundary of pressure of $x=0$, P_0 from the point 0 i.e., crossing of the x axis and y axis, subsequently in order of the shearing waves S_m , S_{nx} , S_{ny} , S_0 . The dilatational wave propagating along the surface emanates from the stress wave at the arrival point which is called the Von-Schmidt wave.

Fig.9 The stress variation caused by shearing impulsive load.

VS_{ny} caused by a dilatational wave propagating from the point ny which comes faster than the shearing wave S_{ny} affects the variation of stresses until the shearing wave S_{ny} reaches. VS_0 caused by the dilatational wave from the point 0 arrives faster than the shearing wave S_0 . Therefore, the variation of stresses is influenced by VS_0 until the shearing wave S_0 comes. Since the

shearing wave S_{nx} comes faster than the Von-Schmidt wave VS_{nx} , the effect of VS_{nx} does not appear. The shearing wave S_o finally arrives at the point B_1 . The stresses approach zero after the shearing wave S_o has gone. The thin lines as shown in Fig.8 represent the variation of stresses at the selected point B_2 . The variation of stresses at the point B_2 has a sudden rise upon the arrival of the dilatational wave P_{ny} emanating from the point ny on the straight boundary of pressure and then the shearing wave S_{ny} , P_o from the point O , subsequently the shearing wave S_o , the Von-Schmidt wave VS_o .

The magnitude of variation at the point B_2 is smaller than at the point B_1 . Figure 9 shows the variation of stress with the tangential impulsive uniformly distributed load. The selected point is similar to the case of normal load. The stress variation of point B_1 or B_2 have a sudden rise upon the arrival of different stress waves. The magnitude of variation of point B_1 appears to be larger than that of point B_2 .

6. Conclusions

The transient response of an elastic half space subjected to impact load in one quadrant region of its surface was analysed on the basis of the three dimensional elastodynamic theory. The analytical process must be performed to know the region classified as under pressure or free surface and the point selected in the vertical plane containing the straight boundary of pressure

,when the Fourier inverse transform integral path is converted into Cagniard's path.

Considering the stress variation from point of view of stress wave propagating, we gain some engineering knowledge.

(1) The stress have a sudden rise upon the arrival of stress waves emanating from the point in the pressure region above the selected point, the point on the straight boundary of pressure which is at the shortest distance from the selected point, the crossing point on the straight boundary of x axis and y axis.

(2) The magnitude of variation of the selected point under free surface, is small and rapidly decreases.

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