Unsteady Flow through a Cascade of Airfoils with Steady Loading*

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Incompressible potential flow through a cascade of staggered thin airfoils with steady loading in general, unsteady, in-phase motion is considered. With the assumptions of the Kutta condition and linearized wakes, exact analytical expressions are derived for pressure distribution, lift and moment, using conformal mapping applied to the velocity field. Mean attack angle and parabolic camber are considered to give steady loading, and then the results are expressed in closed-form as quadratures, and reduce to the well-known relations for thin airfoil theory, as the solidity decreases to zero. Typical results are given in the figures. They should serve as a measure of the accuracy of numerical or approximate solutions, as well as representing in a simple way the effects of steady loading, stagger and solidity on unsteady cascade aerodynamics.

Key Words : Unsteady Flow, Linear Theory, Cascade, Lift, Moment

1. Introduction

Unsteady flow in airfoil cascades has been studied for many years, since it has many important applications to problems of turbomachinery. With the rapid development of computer technology, a number of numerical methods, including the singularity method, the finite difference method and the finite element method, have made it possible to calculate a fairly complicated unsteady cascade flow. To the contrary, examples of the analytical solutions are not so numerous, though there are well-known results by Karman and Sears⁽¹⁾⁽²⁾ in the case of unsteady flow around an isolated airfoil. Especially in the case of staggered cascade, quite a few reports⁽³⁾⁽⁴⁾ gave the analytical solutions.

Recently two of the present authors have attacked the aerodynamics of unsteady in-phase motion of a staggered cascade of flat plates (with zero mean attack angle). Familiar method of thin airfoil theory with shed vorticity was used, and analytical expressions for pressure, lift and moment were obtained. In the present paper, we will extend the theory⁽⁵⁾ to the cases in which airfoils of cascade have steady loading, without losing the essential feature that all the results are expressed in terms with a close affinity to isolated unsteady airfoil theory⁽⁶⁾⁽⁷⁾.

Following simplifications and assumptions are introduced ;

- (1) The fluid is incompressible and inviscid, and the flow is two-dimensional and irrotational.
- (2) The steady loading is not so large as to disturb the linearity of the wake.
- (3) The ratio of the unsteady perturbation velocity to the mean flow velocity is so small that its square can be neglected.
- (4) Free vortices are shed from the trailing edges and are carried downstream linearly with the mean velocity U .
- (5) The Kutta condition is satisfied at the trailing edge.
- (6) The unsteady condition is limited to the in-phase oscillation.

Analytical approach needs many assumptions and sometimes involves some complicated calculations. However, once the solutions can be expressed in a simple form, there is a great virtue that the physical meaning of it is easy to understand. The results of the present theory are cast into the same form as isolated airfoil theory, and divided into four terms, which are quasi-steady, wake, apparent mass terms and the term caused by the steady loading. First, the pressure, lift and moment for general in-phase motion are derived and specialized to harmonic motion. Mean attack angle and parabolic camber are considered to give steady loading, and the specific formulas for translatory motion and sinusoidal gusts are derived.

2. Nomenclature

d = spacing
 b = semichord length
 β = stagger angle
 ω = angular velocity of oscillation
 α = mean attack angle
 h = camber
 ρ = fluid density
 U = mean free stream velocity

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L = lift
 M = moment
 p = pressure
 Γ = circulation
 i = unit of imaginary number with respect to space
 j = unit of imaginary number with respect to time
 $z = x+iy$ = complex coordinate in physical plane
 $\zeta = \xi+i\eta$ = complex coordinate in mapped plane
 R, K, K^*, J = parameters of mapping function
 $\omega = vb/U$ = reduced frequency
 ϵ = shed vorticity distribution in the wake

Suffixes

o = quasi-steady term
 w = wake term
 s = steady term

3. Conformal mapping function and general equations of the steady flow

We start with the staggered cascade of flat plates representing the airfoil chord lines, as shown by Fig.1, in the physical $z=x+iy$ plane. They are transformed conformally into a circle of radius unity in the $\zeta=\xi+i\eta$ plane, shown in Fig.2, with leading edge at B' and the trailing edge at A' . The line joining A' and B' makes an angle θ_T with respect to the ξ axis. The transformation is the well-known logarithmic one:

$$z = \frac{d}{2\pi} \left[e^{-i\theta} \ln \frac{R+\zeta}{R-\zeta} + e^{i\theta} \ln \frac{\zeta+R^{-1}}{\zeta-R^{-1}} \right] \dots (1)$$

The points $-R$ and R are singularities of the transformation which represent in the plane the regions at $x=\pm\infty$ in the z plane.

The angle θ in the circle plane, measured counter-clockwise from the axis, denotes points on the unit circle which correspond to points x on the central airfoil (for which $y=0$) in the cascade.

$$z = x(\theta) + i y(\theta) \quad (n=0, \pm 1, \pm 2 \dots) \dots (2)$$

The relation between x and θ is found from the transformation Eq.(1) to be

$$x(\theta) = \frac{d}{\pi} \left[\cos \beta \operatorname{arctanh} \left(\frac{\cos \theta}{K} \right) + \sin \beta \operatorname{arctan} \left(\frac{\sin \theta}{K^*} \right) \right] \dots (3)$$

where K and K^* are related to the transformation parameter R by

$$K = (R+R^{-1})/2 \quad K^* = (R-R^{-1})/2 \dots (4)$$

The leading and trailing edges correspond to the angles θ_L and θ_T in the circle plane. The physical geometrical parameters chord $2b$, spacing along the cascade axis d , and stagger angle β (see Fig.1) determine the parameter R, K, K^*, θ_L and θ_T . These relations follow from Eq.(1) and its derivative, since θ_T is a singular point of transformation on the circle. It is shown in the reference (8) that

$$\left. \begin{aligned} \theta_L &= \theta_T + \pi, \quad \tan \theta_T = \frac{K}{K^*} \tan \beta \\ \cos \theta_T &= K \cos \beta / J, \quad \sin \theta_T = K^* \sin \beta / J \\ J &= \sqrt{K^2 - \sin^2 \beta}, \quad K^* = \sqrt{K^2 - 1} \end{aligned} \right\} (5)$$

The non-dimensional functions $X(\theta)$ and $X^*(\theta)$ are defined for later convenience as

$$\begin{aligned} X(\theta) &= x(\theta)/b \\ X^*(\theta) &= \frac{d}{\pi b} \left[\sin \beta \operatorname{arctanh} \left(\frac{\cos \theta}{K} \right) - \cos \beta \operatorname{arctan} \left(\frac{\sin \theta}{K^*} \right) \right] \dots (6) \end{aligned}$$

According to the reference (8) in which the steady flow through a staggered cascade of thin airfoils was analyzed, the steady circulation and moment can be expressed by using the distribution of perturbation velocities in the y direction v' as

$$\frac{\Gamma}{b} = -\frac{KK^*}{J} \frac{d}{\pi b} \int_0^{2\pi} \frac{1 + \cos(\phi - \theta_T)}{K^2 - \cos^2 \phi} v'(\phi) d\phi \dots (7)$$

$$\frac{M}{\rho U b^2} = -\frac{Jd}{\pi b} \int_0^{2\pi} \frac{\sin(\phi - \theta_T)}{K^2 - \cos^2 \phi} X^*(\phi) v'(\phi) d\phi \dots (8)$$

Above relations give the perturbation velocity in the x direction u_{FP} , the circulation Γ_{FP} and the moment M_{FP} of a staggered cascade of flat plates which is located in the uniform flow with mean attack angle α as

$$u_{FP} = \frac{UaKK^*}{J^2} \left[\tan \frac{\theta - \theta_T}{2} - \frac{\sin \beta \cos \beta}{KK^*} \right] \equiv Ua u_{1FP} \dots (9.a)$$

$$\frac{\Gamma_{FP}}{b} = \frac{2Uad}{Jb} \dots (9.b)$$

$$\frac{M_{FP}}{\rho U b^2} = 2\pi Ua \left(\frac{d}{\pi b} \right)^2 \ln \frac{K^*}{K} \dots (9.c)$$

Equation (9.b) shows that the coefficient of interference of the cascade is equal to $d/\pi bJ$.

4. Unsteady pressure distribution, lift and moment

The components of unsteady velocity on the airfoil in the x and y directions, q_x and q_y , of the cascade with steady loading are expressed as

$$q_x = U + u_s + u_d, \quad q_y = v_s + v_d \dots (10)$$

where U is the vector mean of the far up- and downstream of the cascade, u_s and v_s are the components of the steady perturbation velocity, and u_d and v_d are the components of the unsteady perturbation velocity respectively. Here, both steady and unsteady perturbation velocities are assumed to be small, thus

$$u_s, v_s, u_d, v_d \ll U \dots (11)$$

Furthermore, u_d is divided into two terms,

$$u_d = u_l + (u_o + u_w) \dots (12)$$

where u_I is an unsteady imposed velocity, and $(u_0 + u_w)$ is a velocity due to unsteady vorticity distribution on the airfoils and wakes.

From an unsteady Bernoulli's equation, the local pressure p on the airfoil is

$$\begin{aligned} \frac{p}{\rho} &= -\frac{1}{2}(q_s^2 + q_w^2) - \frac{\partial \phi}{\partial t} \\ &= -\frac{1}{2}[(U + u_s + u_w)^2 + (v_s + v_w)^2] \\ &\quad - \int_{-b}^x \frac{\partial q_s}{\partial t} dx + P_1(t) \end{aligned} \quad (13)$$

where $P_1(t)$ is an unknown function of t . Since only the pressure difference between upper and lower surfaces is significant practically, hereafter terms which are function of t and constant all over the flowfield will be expressed like $P_1(t)$, and will not be calculated.

The pressure expressed by Eq.(13) contains the steady component p_s , which is written as

$$\frac{p_s}{\rho} = -\frac{1}{2}[(U + u_s)^2 + v_s^2] \quad (14)$$

Subtracting Eq.(14) from Eq.(13) and neglecting higher order quantities, the unsteady pressure Δp becomes

$$\begin{aligned} \frac{\Delta p}{\rho} &= \frac{p - p_s}{\rho} = -(U + u_s)u_w - \int_{-b}^x \frac{\partial u_w}{\partial t} dx + P_1(t) \\ &= -u_s(u_I + u_0 + u_w) - \left[Uu_I + \int_{-b}^x \frac{\partial u_I}{\partial t} dx \right] \\ &\quad - \left[U(u_0 + u_w) + \int_{-b}^x \frac{\partial (u_0 + u_w)}{\partial t} dx \right] + P_1(t) \end{aligned}$$

The first term in the above can be partly neglected because $u_s(u_0 + u_w)$ becomes higher order quantities. The second term is the same at the upper and lower surface and consequently it makes no contribution to the lift and moment. As the combination of the second term and an unknown function $P_1(t)$, a new unknown function $P_2(t, x)$ is introduced. Thus, unsteady pressure is expressed as

$$\frac{\Delta p}{\rho} = -u_s u_I - U(u_0 + u_w) - \int_{-b}^x \frac{\partial (u_0 + u_w)}{\partial t} dx + P_2(t, x) \quad (15)$$

The second and third terms of Eq.(15) correspond to the result without steady loading⁽⁵⁾. Thus, the general expression of the fluctuating pressure becomes

$$\frac{\Delta p}{\rho U} = -\frac{u_s u_I}{U} - u_0 - \frac{KK^*}{2dJ} u_{I,FP} \int_b^s \varepsilon(x) R_s \left(\frac{1}{\sqrt{s^2 - 1}} \right) dx - \frac{1}{U} \frac{\partial}{\partial t} \int_{\theta_r + \pi}^{\theta} v_{\theta ONC}^* d\theta' + P_3(t, x) \quad (16)$$

where the shed wake vortex sheet $\varepsilon(x)$ lies along x axis from $x=b$ to $x=B$ and at corresponding points for all other airfoils. Let ζ_w be an arbitrary point on the wake, then s is defined as

$$s = \frac{\zeta_w e^{-i\theta_r} + \zeta_w^{-1} e^{i\theta_r}}{2} \quad (17)$$

And $v_{\theta ONC}^*$ in Eq.(16) is related to the perturbation velocity v' as

$$v_{\theta ONC}^* = \frac{dJ}{2\pi^2} \int_0^{2\pi} \frac{\sin(\phi - \theta_r)}{K^2 - \cos^2 \phi} v'(\phi) \cot \frac{\theta - \phi}{2} d\phi \quad (18)$$

The unsteady lift ΔL is obtained by integrating the pressure Eq.(16) over x as

$$\frac{\Delta L}{\rho U b} = \int_0^{2\pi} -\frac{u_s u_I}{U} \frac{dX}{d\theta} d\theta + \frac{\Gamma_0}{b} + \frac{KK^*}{bJ} \int_b^s \varepsilon(x) R_s \left(\frac{1}{\sqrt{s^2 - 1}} \right) dx + \frac{b}{U} \frac{d}{dt} \left(\frac{M_0}{\rho U b^2} \right) \quad (19)$$

The unsteady moment (positive nose-up) about the airfoil center ΔM is obtained by integrating $-X \Delta p$ over x as

$$\begin{aligned} \frac{\Delta M}{\rho U b} &= \int_0^{2\pi} \frac{X u_s u_I}{U} \frac{dX}{d\theta} d\theta + \frac{M_0}{\rho U b^2} + \frac{KK^*}{bJ} \frac{d}{dt} \int_b^s \varepsilon(x) R_s \left(\frac{1}{\sqrt{s^2 - 1}} \right) dx \\ &\quad + \frac{1}{U} \frac{d}{dt} \int_0^{2\pi} \frac{x}{b^2} \frac{dX}{d\theta} d\theta \int_{\theta_r + \pi}^{\theta} v_{\theta ONC}^* d\theta' \end{aligned} \quad (20)$$

5. Steady state harmonic motion

The general expressions of the unsteady pressure, lift and moment were given in the preceding section as Eqs.(16), (19) and (20). They contain the unknown function of

Fig.1 Physical plane

Fig.2 Mapping plane

shed vorticity distribution $\epsilon(x, t)$, which must be determined by solving the integral equation. However, if we limit the unsteady condition to the steady state harmonic one, $\epsilon(x, t)$ can be obtained easily, because the form of the wake dependence on x is known. The wake convects with the stream speed U so $\epsilon(x, t)$ is proportional to $\exp(j\omega t - j\omega x/U)$. Thus, $\epsilon(x, t)$ can be written as

$$\epsilon(x, t) = g b^{-1} \exp(j\omega t - j\omega x/U) \quad (21)$$

where g is a complex constant of proportionality. The complex constant g is found by using the fact that the rate of change of total circulation is equal but of opposite sign, to the vorticity shed from the trailing edge per unit time. Thus, the wake strength is found as

$$\epsilon(x, t) = \frac{-e^{-j\omega x} \Gamma_0/b}{KC_0 + KC_1} \quad (22)$$

where ω is the reduced frequency $\omega b/U$, and KC_0, KC_1 are defined as

$$KC_0 = \int_1^\infty e^{-j\omega x} \left[\frac{KK^*}{J^2} R_s \left(\frac{1}{\sqrt{s^2-1}} \right) - \frac{\cos \beta}{J} \right] dX + \frac{\cos \beta e^{-j\omega}}{j\omega J} \quad (23a)$$

$$KC_1 = \int_1^\infty e^{-j\omega x} \left[\frac{KK^*}{J^2} R_s \left(\frac{s}{\sqrt{s^2-1}} \right) - 1 \right] dX + \frac{e^{-j\omega}}{j\omega} \quad (23b)$$

They are the staggered cascade equivalents of the modified Bessel functions of the second kind $K_0(j\omega)$ and $K_1(j\omega)$ which appear in isolated unsteady airfoil theory, to which they reduce in the limit $d \rightarrow \infty$.

From these relations, the expressions of the unsteady pressure, lift and moment become

$$\frac{\Delta p}{\rho U} = -\frac{u_s u_l}{U} - u_0 - u_{1FF} \frac{J}{2d} \Gamma_0(CC-1) - j\omega \int_{\theta_r+\pi}^{\theta} \frac{v_{s0nc}^*}{b} d\theta' + P_s(t, x) \quad (24)$$

$$\frac{\Delta L}{\rho U b} = \frac{Jd}{\pi b U} \int_0^{2\pi} u_s u_l \frac{\sin(\theta-\theta_r)}{K^2 - \cos^2 \theta} d\theta + \frac{\Gamma_0}{b} + \frac{\Gamma_0}{b} (CC-1) + j\omega \left(\frac{M_0}{\rho U b^2} \right) \quad (25)$$

$$\begin{aligned} \frac{\Delta M}{\rho U b} = & -\frac{Jd}{\pi b U} \int_0^{2\pi} u_s u_l \frac{X \sin(\theta-\theta_r)}{K^2 - \cos^2 \theta} d\theta + \frac{M_0}{\rho U b^2} + \frac{Jd}{\pi b} \ln \frac{K}{K^*} \frac{\Gamma_0}{b} (CC-1) \\ & + \frac{j\omega}{b} \int_0^{2\pi} X \frac{dx}{d\theta} d\theta \int_{\theta_r+\pi}^{\theta} v_{s0nc}^* d\theta' \quad (26) \end{aligned}$$

where CC is a cascade analogy to the Theodorsen function, defined as

$$CC = \frac{KC_1}{KC_0 + KC_1} \quad (27)$$

6. Applications of the harmonic motion

6.1 The flat plate cascades with angle of attack

The effect of the mean angle of attack, α , will be considered. The steady velocity perturbation in the x direction on the airfoil, u_s , is expressed as

$$u_s = U\alpha u_{1FF} \quad (28)$$

Two types of imposed velocity perturbation in the x direction will be considered.

6.1.1 Translatory motion in the x direction

The imposed velocity perturbation in the x direction, u_l , is given as

$$u_l = W e^{j\omega t} \quad (29)$$

Then the unsteady lift and moment become

$$\frac{\Delta L_s^{(0)}}{\rho U b} = W a e^{j\omega t} 2\pi \frac{d}{\pi b J} \left[CC + 1 + j\omega \frac{Jd}{\pi b} \ln \frac{K}{K^*} \right] \quad (30)$$

$$\frac{\Delta M_s^{(0)}}{\rho U b^2} = W a e^{j\omega t} 2\pi \left(\frac{d}{\pi b} \right)^2 \ln \frac{K}{K^*} (CC + 1) \quad (31)$$

6.1.2 Sinusoidal gust in the x direction

The imposed velocity u_l is given as

$$u_l = W e^{j\omega t} e^{-j\omega x} \quad (32)$$

This case can represent inlet disturbances convecting through the cascade with mean velocity U .

Then the unsteady lift can be expressed as

$$\begin{aligned} \frac{\Delta L_s^{(0)}}{\rho U b} = & W \cdot a \cdot e^{j\omega t} 2\pi \frac{d}{\pi b J} \left[JC_0 + j JC_1 \right. \\ & \left. + \frac{\pi b J}{d} SC \right] \quad (33) \end{aligned}$$

where $JC_0(\omega)$, $JC_1(\omega)$ are the cascade equivalents of Bessel functions $J_0(\omega)$, $J_1(\omega)$ to which they reduce as $d \rightarrow \infty$. They are defined as

$$JC_0(\omega) = \frac{KK^*}{\pi} \int_0^\pi \frac{\cos \omega X}{K^2 - \cos^2 \phi} d\phi \quad (34)$$

$$JC_1(\omega) = \frac{KK^*}{\pi} \int_0^\pi \frac{\sin \omega X \cos(\phi - \theta_r)}{K^2 - \cos^2 \phi} d\phi \quad (35)$$

and SC is a cascade analogy to the Sears function defined as

$$\begin{aligned} SC = & \frac{d}{\pi b J} [(JC_0 - j JC_1) CC + j JC_1] \\ = & \frac{d}{\pi b J} \frac{1}{KC_0 + KC_1} \quad (36) \end{aligned}$$

Use of Eq. (26) gives the moment as

$$\frac{\Delta M_z^{(s)}}{\rho U b^2} = W a e^{j\omega t} 2\pi \frac{d}{\pi b} \left[J \ln \frac{K}{K^*} SC + \frac{1}{J} \left(\frac{JC_0^* - JC_2}{2} + j JC_1^* \right) \right] \quad (37)$$

where JC_0 and JC_1 are the cascade versions of Bessel functions defined as

$$JC_0(\omega) = \frac{KK^*}{\pi} \int_0^\pi \frac{\cos \omega X}{K^2 - \cos^2 \theta} \left[X \cos(\theta - \theta_T) - \frac{J^2 X^*}{KK^*} \sin(\theta - \theta_T) \right] d\theta \quad (38)$$

$$JC_2(\omega) = -\frac{KK^*}{\pi} \int_0^\pi \frac{\cos \omega X}{K^2 - \cos^2 \theta} \times \left[X \cos(\theta - \theta_T) + \frac{J^2 X^*}{KK^*} \sin(\theta - \theta_T) \right] d\theta \quad (39)$$

6.2 The cascade of cambered airfoils

The effects of camber will be examined hereafter. As shown in Fig. 7, camber y of the airfoil is assumed to be parabolic with maximum camber h at the center of the chord, thus

$$y = h(1 - x^2/b^2)$$

By using the relation $dy/dx = -2hx/b$,

the steady velocity perturbation of this parabolic arc airfoils in cascade is given as

$$u_s = -\frac{2h}{b} U X^*(\theta) \quad (40)$$

6.2.1 Translatory motion in the x direction

The imposed velocity is

$$u_t = W e^{j\omega t} \quad (41)$$

Equations (25) and (26) give the lift and moment as

$$\frac{\Delta L_z^{(s)}}{\rho U b} = W \frac{2h}{b} e^{j\omega t} 2\pi \left(\frac{d}{\pi b} \right)^2 \ln \frac{K}{K^*} (CC+1) \quad (42)$$

$$\frac{\Delta M_z^{(s)}}{\rho U b^2} = W \frac{2h}{b} e^{j\omega t} \left[2\pi \left(\frac{d}{\pi b} \right)^3 J \times \ln^2 \frac{K}{K^*} (CC-1) + j\omega l \right] \quad (43 \cdot a)$$

$$I = \frac{KK^*}{J} \frac{d}{\pi b} \left[\frac{1}{3} \int_0^\pi X^3 \frac{\cos(\theta - \theta_T)}{K^2 - \cos^2 \theta} d\theta - J \frac{d}{\pi b} \ln \frac{K}{K^*} \int_0^\pi X^2 \frac{d\theta}{K^2 - \cos^2 \theta} \right] \quad (43 \cdot b)$$

Fig. 3 The lift due to translatory motion with mean attack angle $\Delta L_z^{(s)} / \Delta L_{z0}^{(s)}$ ($\alpha \neq 0, h = 0$)

Fig. 4 The moment due to translatory motion with mean angle of attack and the lift due to translatory motion with camber $\Delta M_z^{(s)} / \Delta M_{z0}^{(s)}$ ($\alpha \neq 0, h = 0$), $\Delta L_z^{(s)} / \Delta L_{z0}^{(s)}$ ($\alpha = 0, h \neq 0$)

Fig. 5 The lift due to sinusoidal gust with mean angle of attack $\Delta L_z^{(s)} / \Delta L_{z0}^{(s)}$ ($\alpha \neq 0, h = 0$)

Fig. 6 The moment due to sinusoidal gust with mean angle of attack $\Delta M_z^{(s)} / \Delta M_{z0}^{(s)}$ ($\alpha \neq 0, h = 0$)

6.2.2 Sinusoidal gust in the x direction

The imposed velocity is

$$u_1 = W e^{j\omega t} e^{-j\omega x} \quad (44)$$

The unsteady lift is obtained from Eq.(25) as

$$\frac{\Delta L_h^{(s)}}{\rho U b} = W \frac{h}{b} e^{j\omega t} 2\pi \frac{d}{\pi b J} \left[CC(JC_1^* - JC_2 - 2jJC_1^*) - JC_0^* + JC_2 + \frac{4}{\omega} JC_1 \right] \quad (45)$$

where $JC_1^*(\omega)$ is also equivalent to $J_1(\omega)$ and defined as

$$JC_1^*(\omega) = \frac{KK^*}{\pi} \int_0^\pi \frac{\sin \omega X}{K^2 - \cos^2 \theta} X d\theta \quad (46)$$

There is a well known relation of Bessel functions

$$J_0(\omega) + J_2(\omega) = 2/\omega J_1(\omega)$$

The equivalent relation also holds for the cascade version of Bessel functions

$$JC_0^*(\omega) + JC_2(\omega) = \frac{2}{\omega} JC_1(\omega) \quad (47)$$

The expression of moment in this case includes necessarily double integral terms, so that they can be written only with impractical complexity. So we refrain from referring to it here.

7. Results of calculation

To obtain numerical results for the lift and moment derived in this paper, only numerical quadratures are needed. There are two geometric parameters: the stagger angle β and the solidity $\sigma = 2b/d$. In addition, the reduced frequency ω enters. The only numerical complication is the calculation of the integrals KC_0 and KC_1 , because of the complicated relation between $s(\zeta_w)$ of Eq.(17) and x .

To present the numerical results, we normalize the lift and moment by the proper values, which will be denoted by a double zero subscript.

$$\Delta L_{h00}^{(s)} = \Delta L_{h00}^{(s)} = 2\pi \rho U b a W e^{j\omega t} \quad (48)$$

$$\Delta M_{h00}^{(s)} = \Delta M_{h00}^{(s)} = \pi \rho U b^2 a W e^{j\omega t} \quad (49)$$

Fig.7 Cambered airfoil cascade

$$\Delta L_{h00}^{(s)} = \Delta L_{h00}^{(s)} = 2\pi \rho U b \frac{h}{b} W e^{j\omega t} \quad (50)$$

$$\Delta M_{h00}^{(s)} = \frac{\pi}{2} \rho U b^2 \frac{h}{b} W e^{j\omega t} \quad (51)$$

Calculations have been made over a range of β , σ and ω . As examples we show the results for $\beta = 30^\circ$ in the figures as plots with the real part on the abscissa and the imaginary part on the ordinate.

Figure 3 shows the lift due to translatory motion with mean angle of attack. The higher solidity cascades have a real part which is nearly independent of reduced frequency, and an imaginary part which increases approximately proportional to ω .

Figure 4 shows the moment due to translatory motion with mean angle of attack and the lift due to translatory motion with camber. Normalized values of them are equal each other. Since apparent mass term of them becomes zero, all

Fig.8 The moment due to translatory motion with camber
 $\Delta M_h^{(s)} / \Delta M_{h00}^{(s)}$ ($\alpha = 0, h \neq 0$)

Fig.9 The lift due to sinusoidal gust with camber
 $\Delta L_h^{(s)} / \Delta L_{h00}^{(s)}$ ($\alpha = 0, h \neq 0$)

the imaginary parts are negative, and as the solidity increases, they become very small, while the real part becomes nearly constant, independent of ω . Figures 5 and 6 show the lift and moment due to sinusoidal gust with angle of attack. Both of them show the familiar spiral into the origin as ω increases. Figure 8 shows the moment due to translatory motion with camber, which is equal to the negative of the moment due to pitching without steady loading⁽⁵⁾. Its real part also becomes independent of ω at the higher solidities. At the lower solidities, it is nearly independent of ω and α for $\omega > 1$. Finally, the lift due to sinusoidal gust with camber is displayed in Fig. 9. It shows the same spiral character as the case with angle of attack Fig. 5.

8. Conclusions

The incompressible flow through a staggered cascade of thin airfoils with steady loading in unsteady in-phase motion has been solved analytically by conformal mapping. This is an extension of the work of Shoji, Ohashi and Kemp to the cases with steady loading.

First we considered a general unsteady motion, and presented expressions for pressure distribution, lift and moment. They each appear in the form from isolated unsteady airfoil theory, having a term caused by steady loading, a quasi-steady term, an apparent mass term and a wake term. Mean attack angle and parabolic camber are considered to give steady loading. Application is made to two important harmonic cases of translatory motion and a sinusoidal gust. Typical results for a stagger angle of 30 deg. are given in the figures. The only

numerical complication lies in the calculation of the integrals KC_0 and KC_1 , but once they are obtained, we can apply the solution to any unsteady conditions (harmonic) with only numerical quadratures.

Although in-phase oscillation is not usually of much physical interest, the results obtained should prove useful as check cases for numerical and approximate solutions of more complex unsteady cascade problems. They also show the interesting relations, for example, the moment with mean attack angle and the lift with camber due to translatory motion are equal each other, the relation of Eq. (36), and the pressure of sinusoidal gusts has the same form as flat plate cascades with constant upwash. These relations are considered also to help understand the physical meaning of the solutions of the unsteady cascade flow.

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