Transient Thermal Stresses
in a Half Plane with a Crack Parallel to the Surface

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Transient thermal stresses in a half plane with a crack parallel to the surface are investigated. It is assumed that the half plane is initially at uniform temperature, and subjected to stepwise temperature rise on the surface. The problem is formulated as integral equations. From their numerical solutions, variations of the stress intensity factors with time are obtained.

Key Words: Elasticity, Thermoelasticity, Thermal Stress, Crack, Half Plane, Stress Intensity Factor

1. Introduction

When elastic bodies contain cracks, there arise severe stresses near the crack tips. Since they may cause fractures of materials, it is important to analyze thermal stresses induced in such elastic bodies. Till now, investigations on the steady state thermal stresses in an infinite plane with a crack in the and the transient thermal stresses in a half plane with an edge crack perpendicular to the surface have been reported. But analyses become considerably difficult in the cases that cracks disturb transient heat flows, and studies on such problems are few.

In this paper, we analyze transient thermal stresses in a half plane with a crack parallel to its surface. The half plane is initially at uniform temperature, and then it is subjected to stepwise temperature rise on the surface. In the analysis, temperature distributions are expressed in the integral forms, and an integral equation is derived for the Laplace transform of the unknown temperature difference between the upper and the lower crack surfaces. From this temperature solution, a particular solution of thermal stresses is obtained by the use of thermoelastic potentials. Since they do not satisfy the boundary conditions, certain isothermal displacement fields must be superimposed to them in order to realize traction-free surfaces. This problem is formulated in the form of integral equations for the unknown displacement differences between the crack surfaces. The integral equations are solved by the use of the collocation methods and the variations of stress intensity factors with time are obtained after numerical Laplace inversions.

2. Temperature Distributions

We consider a half plane (plane strain)

\[ T(x,t) = \frac{1}{2} \left( T_0 + T_0' \right) \quad \text{for} \quad -\infty < x_1 < \infty, x_2 = 0 \quad \text{(1)} \]

Further, the initial condition is given by

\[ T(x,t) = T_0 \quad \text{if} \quad t = 0 \quad \text{(2)} \]

and the boundary conditions take the form

\[ \frac{\partial T}{\partial x_1}(x_1,a,x_2=0) = 0 \quad \text{(3)} \]

\[ T(x_1,a,x_2=h) = 0 \quad \text{(4)} \]

\[ T(x_1,\infty) = 0 \quad \text{(5)} \]

where \( \kappa \) denotes the thermal diffusivity.

Let \( T \) be the Laplace transform of \( T \), that is

\[ T = \int_0^\infty T e^{-\kappa t} dt \quad \text{(6)} \]

Then equations satisfied by \( T \) are derived from Eqs. (1)-(5) as follows:

\[ \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x_1^2} \quad \text{(7)} \]

![Fig. 1 Geometry]

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\[ T^e = \frac{1}{\kappa} T_0 \] \hspace{1cm} (8)

\[ \frac{\partial}{\partial x_1} T = 0 \hspace{1cm} (|x| < a, x_1 = h) \] \hspace{1cm} (9)

\[ T = 0 \hspace{1cm} (x_1 \to \infty) \] \hspace{1cm} (10)

When there is no crack in the half plane, its temperature \( T^e(x_1, x_2) \) is expressed in the form

\[ T^e(x_1, x_2) = \frac{T_0}{p} e^{-\sqrt{2\pi}x_2} \] \hspace{1cm} (11)

Further, the temperature field \( T^e \) induced by the insulated crack is obtained in the following manner.

The fundamental solution \( U(P,t;Q,0) \) of the heat conduction equation is the temperature of an infinite plane at point \( P \) and time \( t \), when there exists an instantaneous point source at point \( Q \) and time \( 0 \). The Laplace transform of the fundamental solution satisfies the equation

\[ \mathcal{L}U_{nm} = \hat{P} \hat{Q} - \frac{1}{\kappa} (P - Q) \] \hspace{1cm} (12)

\[ \mathcal{T}^e(P) = \int_{\Sigma} j^*(Q) U^*(Q,P) dx_1 d\omega_2 - \int_{\Sigma} \Delta T^e(Q)(km/Q) U^*(Q,P) dx_1 d\omega_2 \] \hspace{1cm} (15)

where \( \Sigma \) is the unit outward normal vector of the crack surface, \( \{ \} \) denotes the quantity concerning the upper [lower] crack surface \( x_1 = h + 0[h-0] \). Moreover, we put

\[ \sum j^*(Q) = (km/Q) T^e(P) + (km/Q) T^e(P)^* \] \hspace{1cm} (16)

Here it will be noticed for temperature field \( T^e \) that the sum of heat flux and the difference in temperature are equal to zero at the upper and the lower crack surfaces. On the other hand, the heat flux of \( T^e \) at each crack surface is in the opposite direction but has the same magnitude. Hence the temperature \( T^e(x_1, x_2) \) of the half space with a crack is given by superposing the temperature given by Eq. (12) over that by (11) as follows:

\[ T^e(P) = \frac{T_0}{p} e^{-\sqrt{2\pi}x_2} - k \int_{x} \Delta T^e(P)(x_1, x_2) \frac{\partial}{\partial x_2} U^*(Q, P) \bigg|_{x_2 = h} dx_1 \] \hspace{1cm} (17)

For an arbitrary temperature difference \( \Delta T \), the temperature field (17) satisfies the heat conduction equation, the initial condition (7) and the boundary conditions (8) and (10). From the relation \( f_0 = -k T_0 \), the heat flux \( f \) accompanying temperature (17) can be obtained

\[ f^e(P) = \frac{k}{p} \sqrt{2\pi} e^{-\sqrt{2\pi}x_2} + k \int_{x} \Delta T^e(P)(x_1, x_2) \frac{3\partial}{\partial x_2} U^*(Q, P) \bigg|_{x_2 = h} dx_1 \] \hspace{1cm} (18)

where the finite part integral is taken for divergent integral in the case of the point \( P \) situated on the crack surface.

Then substituting Eq. (18) into the remaining boundary condition (9) and putting \( q = \sqrt{2\pi} \), \( r = |x_1 - y|, r^* = (|x_1 - y|^2 + 4h^2)^{1/2} \), in the form

\[ \frac{q}{\pi} \int_{x} \Delta T^e(P)(x_1, x_2) \frac{3\partial}{\partial x_2} U^*(Q, P) \bigg|_{x_2 = h} dx_1 \] \hspace{1cm} (19)

3. Stress Distributions

Since the surface of the half plane and the crack surfaces are free from tractions, the boundary conditions of this problem are written in the form

\[ \sigma_0 = \sigma_1 = 0 \hspace{1cm} (|x| < a, x_1 = 0) \] \hspace{1cm} (20)

\[ \sigma_1 = \sigma_0 = 0 \hspace{1cm} (|x| < a, x_1 = h) \] \hspace{1cm} (21)

A particular displacement field induced under the temperature field \( T^e \) is given by the thermoelastic potential

\[ \hat{Q} = \alpha T + \text{Harmonic Function} \] \hspace{1cm} (22)

in which \( \alpha \) is the thermal expansion coefficient and \( \nu \) is Poisson's ratio. From Eqs. (17) and (22), it follows that

\[ U(P, Q) = \frac{1}{2\pi} K_0 \sqrt{\frac{R}{\pi}} \] \hspace{1cm} (13)

where \( \delta(x) \) denotes Dirac's delta function, \( K_0(x) \) a modified Bessel function of second kind and order \( \nu \), \( R \) the distance between points \( P \) and \( Q \). The fundamental solution \( U^*(P, Q) \) of a half plane whose surface is kept at zero temperature, is given by combining two fundamental solutions of an infinite plane:

\[ U^*(P, Q) = U(P, Q) - U(P, Q') \] \hspace{1cm} (14)

in which \( Q' = (x_1, -x_2) \) indicates the reflection of \( Q(x_1, x_2) \).

From Eqs. (7), (12) and (14) together with the divergent theorem of Gauss, we have the representation formula for \( T^e(P) \) in the form

\[ T^e(P) = \int_{\Sigma} j^*(Q) U^*(Q, P) dx_1 d\omega_2 - \int_{\Sigma} \Delta T^e(Q)(km/Q) U^*(Q, P) dx_1 d\omega_2 \] \hspace{1cm} (15)
\[ \sigma(P) = \frac{1 + \nu}{2(1 - \nu)} T \rho e^{-\sqrt{\nu} \rho x} - \frac{\alpha}{2(1 - \nu)} \int_{a}^{b} T(x') \frac{\partial}{\partial x} \left[ (K_a \sqrt{\nu} P + \ln(\sqrt{\nu} P)) \right] dx' \]
\[ - \left[ K_a \left( \sqrt{\nu} P \right) + \ln\left( \sqrt{\nu} P \right) \right] \bigg|_{x_m}^{x_0} \]  

where \( R^* \) is the distance between points \( P \) and \( Q^* \). Then the components of stress tensor \( \sigma_i \), \( \sigma_i \), \( \sigma_i \) are given by
\[ \sigma_i = -2 \mu \sigma_{i} \], \( \sigma_i = -2 \mu \sigma_{i} \), \( \sigma_i = 2 \mu \sigma_{i} \)  

This state of stress does not satisfy the boundary conditions (20) and (21), since there arise the shear stress \( \sigma_i(x, 0) \) on the surface of the half plane, and the normal stress \( \sigma_i(x, h) \) and the shear stress \( \sigma_i(x, h) \) on the crack surface.

Utilizing Somigliana's identities, the isothermal displacement field of the half plane is expressed in the form
\[ u_i(P) = \int_{S} n_i(Q) U_i^{(n)}(P, Q, P) ds(Q) - \int_{S} d u_i(Q) (n_i(Q) C_{n m}) U_i^{(n)}(P, Q, P) ds(Q) \]  

where \( S \) denotes the surface of the half plane, \( S \) the crack surface, \( 
\) the traction applied on \( S \), \( du \) the displacement difference between the upper and the lower crack surfaces, and \( C_{n m} \) is the elastic constants tensor
\[ C_{n m} = \frac{2}{1 - 2v} \delta_{nm} + \mu \delta_{nm} \]  

Further, \( U_i^{(n)} \) denotes the fundamental displacement solution of the half plane, that is the \( \sigma \) direction component of the displacement at point \( P(x, y, z) \) of a half plane when a unit concentrated force is applied at point \( Q(x, y, z) \) in \( \gamma \) direction. It is represented by the complex stress functions \( \varphi \) and \( \varphi^{(n)} \) in the form
\[ U_i^{(n)}(x, y, z) = \frac{1}{2\pi i} \int \left[ (3 - 4v) \varphi^{(n)}(z) - 4v \varphi^{(n)}(z) \right] dx \]
\[ \varphi^{(n)}(z) = \frac{1}{8\pi(1-\nu)} \left[ \ln(-z) + (3 - 4v) \ln(z) + (1 - \nu) \frac{1}{z - 1} \right] \]
\[ \varphi^{(n)}(z) = \frac{1}{8\pi(1-\nu)} \left[ \ln(-z) + (3 - 4v) \ln(z) + (1 - \nu) \frac{1}{z - 1} \right] \]
\[ \left( \gamma = 1, 2 \right) \]  

where \( i \) is \( \sqrt{-1} \), \( z = x + iy, x = x + iy \) and \( - \) indicates a complex conjugate.

If we put \( \sigma_i = \sigma_i(x, 0), \sigma_f = 0 \) and superimpose the displacement (25) on the particular solutions derived from Eq. (23), then the boundary conditions (20) on the surface of the half plane are satisfied. On the other hand, the stress causing the displacement (25) is
\[ \sigma_i(P) = \int_{S} n_i(Q) C_{n m} U_i^{(n)}(P, Q, P) ds(Q) - \int_{S} d u_i(Q) C_{n m} (T_i^{(n)}(P, Q, P)) ds(Q) \]  

Hence from the remaining boundary conditions (21), we have the integral equation for the unknown \( du \) in the form
\[ - \frac{\mu}{2\pi(1-\nu)} \int_{S} \left[ du_i(x, y) - du_i(y, x) \right] \left[ \frac{1}{(x - y)^2} \right] dy \]
\[ + \frac{4u}{\pi(1-\nu)} \int_{S} \left[ du_i(x, y) + du_i(y, x) \right] \left[ \frac{h}{(x - y)^2 + 4h^2} \right] dy \]
\[ = \int_{S} \sigma_i(x, y) \left[ (x - y) - \frac{h}{(x - y)^2 + 4h^2} \right] dy - \sigma_i(x, h) - \sigma_i(x, h) \]  

The solution \( du_i(x, y) \) approaches zero as \( x \to \pm a \) with order \( 1/2 \). Accordingly, it can be expressed in the form
\[ du_i(x, y) = \sqrt{1 - (x/a)^2} d_i(x, y) \]  

The stress intensity factors \( K_a \) and \( K_0 \) at the right \( (x = a) \) and left \( (x = -a) \) crack tips are given by
\[ K_{a}= \frac{\sqrt{\pi \mu}}{2(1-\nu)} \]  
\[ K_{a}= \frac{\sqrt{\pi \mu}}{2(1-\nu)} \]  

(30)
4. Numerical Results

Since it is difficult to obtain analytical solutions of the integral equation (19), we adopt numerical techniques. Taking into consideration the singular behaviors near $x_i = \pm a$, we assume an unknown function $\Delta T(x_i)$ in the form

$$\Delta T(x_i) = \sqrt{1 - (x_i/a)^2} \sum e_i U_n(x_i/a)$$

(31)

where $e_i$ are unknown constants to be determined and $U_n(x)$ is Chebyshev polynomial of second kind. When Eq. (31) is substituted into Eq. (19), integrals in the left hand side which contain singular kernels can be evaluated by integral formulas

$$\int_{-1}^{1} (1 - Y^2) U_n(Y) (X - Y)^{m-1} dY$$

$$= \pi (m+1) U_n(X) \quad (|X| < 1)$$

$$\int_{-1}^{1} (1 - Y^2) U_n(Y) \sin \alpha Y dY$$

$$= \pi \left\{ \begin{array}{l}
\frac{1}{2} \left[ X^m \cos \frac{m}{2} \right] \\
\frac{1}{2} \frac{1}{n+2} T_n(X) - \frac{1}{n} T_n(X) \quad (n \geq 1)
\end{array} \right. \quad (m \neq 0)$$

and integrals which contain regular kernels can be evaluated numerically by the use of the quadrature formula

$$\int_{-1}^{1} f(Y) dY$$

$$= \sum_{m=0}^{2M} \sin^2 \frac{m-1}{M} \pi \frac{1}{2M}$$

Hence we assume that both sides of the equation coincide with each other at points $x_i = a \cos (\pi (2N+1)) (i=1, 2, \ldots, N)$, which are the zeros of $U_n(x/a)$, and we have the simultaneous equations for $e_i (i=0, 1, \ldots, N-1)$. From their solutions, $\Delta T$ and the temperature distribution $\tilde{T}$ for arbitrary Laplace transform parameter $\rho$ can be determined.

In order to obtain variations of temperature with time, we utilize the numerical integration techniques attributed to Schapery. In this method, the original function $f(t)$ is developed into the series

$$f(t) = A + Bt + \frac{1}{\rho^2} C_0$$

(32)

where $A$ is a suitably chosen constant. The Laplace transform of Eq. (32) is written in the form

$$\rho f(\rho) = A + B \rho + \frac{1}{\rho^2} C_0$$

(33)

Accordingly, putting $\rho = 0$ in Eq. (32) we have

$$A = \lim_{\rho \to 0} \frac{1}{\rho^2} C_0$$

Similarly, Eq. (33) leads to the relation

$$B = \lim_{\rho \to 0} \rho f(\rho)$$

Furthermore, $C_0$ is determined from the values of $f(\lambda_i)$ by solving the simultaneous equations, which are derived by setting $\rho = \lambda_i (i=1, 2, \ldots, L)$ in Eq. (33).

We carried out several numerical calculations, with the number of terms in the Chebyshev expansion $N = 8$, and the

![Fig.2 Temperature Distributions on $x_i$ axis (k/a=5)]

![Fig.3 Variations of the Stress Intensity Factors with Time]

![Fig.4 Relations between the Location of the Crack and the Maximum Values of the Stress Intensity Factors]
number of points for the numerical quadrature $M=16$. Number of matching points for the numerical Laplace inversion is $L=15$ and $\lambda_1$ is selected as a geometrical sequence $\lambda_1 r^{-1}$, in which $\lambda_1=10^{-4}$ and $r=2.5$. The temperature distributions on $x$ axis in the case of $h/a=5$ and the Fourier number $Nt/a^2$ = 1, 10, 100 are shown in Fig.2. The solid lines represent the results about a half plane with a crack, and the broken lines those about a half plane without a crack. It is observed that temperature distributions are discontinuous at the crack surface, and the effects of the crack diminish at a distance.

Similarly, stress distributions are obtained from the numerical solutions of Eq.(28). Fig.3 shows the variations of the stress intensity factors $K$ and $K_s$ with time. As the crack becomes closer to the surface of the half plane, the time when the stress intensity factor takes its maximum value comes earlier. The relation be-

tween these maxima and the crack locations $h/a$ is shown in Fig.4.

References