The Onset of Oscillation of an Elastic Plate Lying Spread out on a Liquid Layer under a Vertical Periodic Motion*

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A thin elastic plate lies spread out on a horizontally viscous liquid layer with a finite depth. The liquid layer is excited from its bottom by a vertical periodic force. In this paper, whether or not the elastic plate is excited parametrically is investigated theoretically and experimentally. The boundaries of the region of instability in the space of the amplitude and frequency of the imposed oscillation are found for the subharmonic response by a linear theory. They are dependent on the wave number of the disturbance. The critical amplitudes of the imposed oscillation, below which the plate is stable, have a minimum value with respect to the wave number of the disturbance. The critical condition for the onset of instability is determined by this minimum critical amplitude. These theoretical results are found to be in fairly good agreement with the experimental ones obtained by using a thin rubber plate.

Key Words : Fluid Vibration, Elastic Plate, Viscous Fluid Layer,
Vertical Oscillation, Subharmonic Resonance, Stability Curve,
Critical Amplitude.

1. Introduction

One of the simple methods which prevent a wave motion arising on a free surface of water is to cover its surface with a thin elastic plate such as a sheet. Such a method has been frequently used in various fields. A wave motion of the water surface in nature is generated by a wind. Meanwhile a wave motion of the water surface in a container is, in engineering practice, generated by an external oscillation, such as sloshing of a vessel. There are two ways of external oscillation, that is, the vertical and horizontal ones. In this paper, the system considered is assumed to be excited by an external vertical oscillation.

The study on the stability of the free surface of a liquid contained in a vessel under a vertical periodic motion undertaken by Benjamin and Ursell is famous as one of the fundamental investigations of the wave generation under an external vertical excitation⁽¹⁾. The studies from a point of view of practical technology on this kind of problem are referred to in a report on the flow-induced vibration edited by a group of specialists of the Japan Society of Mechanical Engineers⁽²⁾. An extension of Benjamin-Ursell's problem to the case when the bottom of the vessel containing a

liquid is made of an elastic plate has been studied⁽³⁾. In these investigations, the fluid is always assumed to be ideal. The force acting on the vessel by a fluid oscillation can be effectively found by an ideal fluid theory. However, the limitation of the amplitude of a wave motion which arises under an external oscillation of the liquid in a vessel can not be solved by such a theory.

In this paper, under the assumption that the liquid is a viscous fluid and is covered with a thin elastic plate, the critical condition for a wave motion of the free surface of a liquid layer lying on a horizontal plane to occur is found. If the wave length of a wave motion arising on the free surface is small compared with the horizontal extent of the free surface, the end effect of the boundary of the surface may be neglected. Then both the extent of the liquid layer and that of the elastic plate are assumed to be infinite. In the following, whether or not the system consisting of a liquid layer and a thin elastic plate can be excited by an external vertical oscillation, and if it is so, what is the critical condition on for a wave motion to arise are investigated theoretically and the result is compared with the experimental one.

2. Governing Equations

A viscous liquid layer with a depth H is at rest under the action of gravity on a horizontal plane with infinite extent. The layer of an incompressible fluid with the density ρ and the kinematic viscosity ν is covered with a thin elastic plate with

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the surface density m and the bending stiffness D . The system consisting of the liquid layer and the elastic plate is assumed to be vertically excited from the bottom plane by a given periodic force with the acceleration $f \cos \omega t$. Here, f and ω are the amplitude and the circular frequency of the acceleration, and t is the time.

Under the state that any disturbance does not exist at all in the system considered, the Cartesian coordinates (x, y, z) at rest relative to the oscillating wall are taken with the xy -plane along the underside of the elastic plate and with the z -axis vertically upward. Then, in this state, the pressure P in the liquid layer is governed by the equation

$$-\frac{1}{\rho} \frac{\partial P}{\partial z} - (g + f \cos \omega t) = 0 \quad (1)$$

where, g is the gravitational acceleration. If the space of the upperside of the elastic plate is assumed to be empty, the boundary condition for this equation is given as

$$z=0: P = m(g + f \cos \omega t) \quad (2)$$

Thus, in the state that there is no disturbance in the system, the pressure in the liquid layer varies periodically in time as

$$P = (m - \rho z)(g + f \cos \omega t) \quad (3)$$

Of course, both the plate and the liquid layer are at rest relative to the moving plane wall.

Now, a small disturbance is assumed to be caused in the system by the periodic excitation. In this disturbed state, the displacement of the elastic plate is denoted by $\eta(t, x, y)$, the velocity components of the liquid by $u(t, x, y, z)$, $v(t, x, y, z)$ and $w(t, x, y, z)$, and the pressure by $P(t, z) + p(t, x, y, z)$. Assuming the variable quantities η , u , v , w and p to be small, and neglecting the terms of the second order of these quantities, the linear equations governing the thin elastic plate and the viscous liquid are written as follows:

$$D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \eta - \left(N_1 \frac{\partial^2}{\partial x^2} + N_2 \frac{\partial^2}{\partial y^2} + 2N_3 \frac{\partial^2}{\partial x \partial y} \right) \eta + m \frac{\partial^2 \eta}{\partial t^2} = F \quad (4)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u \quad (5)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v \quad (6)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) w \quad (7)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (8)$$

where, F is the liquid force per unit area acting on the plate and is given remaining only the terms of the first order of the small quantities as follows:

$$F = p|_{z=0} - \rho(g + f \cos \omega t)\eta - 2\rho\nu \frac{\partial w}{\partial z} \Big|_{z=0} \quad (9)$$

N_1 , N_2 and N_3 are the given in-plane forces of the tension or compression in the x and y directions and of the shear respectively.

These equations are solved under the following boundary conditions: the no-slip condition on the horizontal plane,

$$z = -H: u = v = w = 0 \quad (10)$$

the kinematic condition meaning that the underside of the plate is moving together with the liquid,

$$z = 0: \frac{\partial \eta}{\partial t} = w \quad (11)$$

and the no-slip condition at the underside of the plate because the plate moves only in the z direction,

$$z = 0: u = v = 0 \quad (12)$$

Eqs.(4)-(12) obtained above are the governing equations of the disturbance arising in the externally excited system.

Using the equations of continuity and of motion, u , v and p are expressed in terms of the velocity component w in the z direction as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \quad (13)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) p = \rho \left(\frac{\partial}{\partial t} - \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right) \frac{\partial w}{\partial z} \quad (14)$$

Thus, the equations governing the considered system reduce to those written in the terms of only w and η as follows:

$$D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \eta - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(N_1 \frac{\partial^2}{\partial x^2} + N_2 \frac{\partial^2}{\partial y^2} + 2N_3 \frac{\partial^2}{\partial x \partial y} \right) \eta + m \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial^2 \eta}{\partial t^2} + \rho \nu \frac{\partial^3 w}{\partial z^3} \Big|_{z=0} + \rho(g + f \cos \omega t) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta = \rho \left\{ \frac{\partial}{\partial t} - 3\nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right\} \frac{\partial w}{\partial z} \Big|_{z=0} \quad (15)$$

$$\left\{ \frac{\partial}{\partial t} - \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right\} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) w = 0 \quad (16)$$

$$z = 0: w = \frac{\partial \eta}{\partial t} \quad (17)$$

$$\begin{aligned}\frac{\partial w}{\partial z} &= 0, \dots\dots\dots (18) \\ z = -H: w &= 0, \dots\dots\dots (19) \\ \frac{\partial w}{\partial z} &= 0, \dots\dots\dots (20)\end{aligned}$$

Fig. 1 Schematic diagram and coordinates.

Expanding the small quantities based on the disturbance into the Fourier components with the x and y components of the wave number, k_x and k_y , they are written as

$$\begin{aligned}\eta(t, x, y) &= \bar{\eta}(t) e^{i(k_x x + k_y y)} \dots\dots\dots (21) \\ w(t, x, y, z) &= \bar{w}(t, z) e^{i(k_x x + k_y y)} \dots\dots\dots (22)\end{aligned}$$

Introducing the following dimensionless quantities with the symbol $*$:

$$\left. \begin{aligned}z^* &= \frac{z}{H}, \quad k_x^* = H k_x, \quad k_y^* = H k_y, \\ t^* &= \frac{\sqrt{gH}}{H} t, \quad \bar{\eta}^* = \frac{\bar{\eta}}{H}, \quad \bar{w}^* = \frac{\bar{w}}{\sqrt{gH}}, \\ \nu^* &= \frac{\nu}{\sqrt{gH} \cdot H}, \quad D^* = \frac{D}{\rho g H^4}, \\ N_i^* &= \frac{N_i}{\rho g H^2} \quad (i=1, 2, 3), \\ m^* &= \frac{m}{\rho H}, \quad f^* = \frac{f}{g}, \quad \omega^* = \frac{H}{\sqrt{gH}} \omega.\end{aligned} \right\} \dots\dots\dots (23)$$

the governing equations reduce to those of the amplitude functions, $\bar{w}^*$ and $\bar{\eta}^*$, as follows:

$$\begin{aligned}D^* k^{*2} \bar{\eta}^* + k^{*2} (N_1^* k_x^{*2} + N_2^* k_y^{*2} + 2N_3^* k_x^* k_y^*) \bar{\eta}^* + (1 + f^* \cos \omega^* t^*) k^{*2} \bar{\eta}^* \\ + m^* k^{*2} \frac{d^2 \bar{\eta}^*}{dt^{*2}} - \nu^* \frac{\partial^3 \bar{w}^*}{\partial z^{*3}} \Big|_{z^*=0} \\ = - \left(\frac{\partial}{\partial t^*} + 3\nu^* k^{*2} \right) \frac{\partial \bar{w}^*}{\partial z^*} \Big|_{z^*=0},\end{aligned} \dots\dots\dots (24)$$

$$\left\{ \frac{\partial}{\partial t^*} - \nu^* \left(\frac{\partial^2}{\partial z^{*2}} - k^{*2} \right) \right\} \left(\frac{\partial^2}{\partial z^{*2}} - k^{*2} \right) \bar{w}^* = 0, \dots\dots\dots (25)$$

$$z^*=0: \bar{w}^* = \frac{d\bar{\eta}^*}{dt^*}, \dots\dots\dots (26)$$

$$\frac{\partial \bar{w}^*}{\partial z^*} = 0, \dots\dots\dots (27)$$

$$z^*=-1: \bar{w}^* = 0, \dots\dots\dots (28)$$

$$\frac{\partial \bar{w}^*}{\partial z^*} = 0. \dots\dots\dots (29)$$

Here, k^* is the magnitude $(k_x^{*2} + k_y^{*2})^{1/2}$ of the vector of the wave number. Then, the problem to be considered is found to be governed by seven nondimensional parameters, D^* , N^* , m^* regarding the plate, ν^* regarding the fluid, f^* , ω^* regarding the external force and k^* regarding the disturbance. It should be noted that, for simplicity, the symbol $*$ denoting nondimensionality is omitted in the following analysis. It should also be noted that, although the right-hand side of Eq.(24) to be equal to zero is found from the boundary condition Eq.(27), it is retained deliberately in Eq.(24) so that the case of an ideal fluid can be analyzed in the section 3 by using the equation governing the case of a viscous fluid.

3. Ideal Fluid Case

Consider the case that a flow of the liquid layer is regarded as one of an ideal fluid. The governing equations for such a case are obtained setting the parameter ν meaning the magnitude of the viscosity equal to zero and neglecting Eqs.(27) and (29) derived from the no-slip condition as follows:

$$\begin{aligned}D k^2 \bar{\eta} + k^2 (N_1 k_x^2 + N_2 k_y^2 + 2N_3 k_x k_y) \bar{\eta} \\ + (1 + f \cos \omega t) k^2 \bar{\eta} + m k^2 \frac{d^2 \bar{\eta}}{dt^2} = - \frac{\partial^2 \bar{w}}{\partial t \partial z} \Big|_{z=0},\end{aligned} \dots\dots\dots (30)$$

$$\frac{\partial^2 \bar{w}}{\partial z^2} - k^2 \bar{w} = 0, \dots\dots\dots (31)$$

$$z=0: \bar{w} = \frac{d\bar{\eta}}{dt}, \dots\dots\dots (32)$$

$$z=-1: \bar{w} = 0. \dots\dots\dots (33)$$

The formal solution of the equation of motion (31) with the boundary conditions (32) and (33) is found as

$$\bar{w} = \frac{d\bar{\eta}}{dt} \frac{\sinh k(z+1)}{\sinh k}. \dots\dots\dots (34)$$

Substituting this equation into the equation of motion for the plate (30), the governing equation for the case of the ideal fluid is found as

$$\begin{aligned}\left(\frac{1}{k \tanh k} + m \right) \frac{d^2 \bar{\eta}}{dt^2} \\ + (D k^4 + N k^2 + 1 + f \cos \omega t) \bar{\eta} \\ = 0.\end{aligned} \dots\dots\dots (35)$$

Here, for simplicity, a new parameter is introduced, namely:

$$N k^2 \equiv N_1 k_x^2 + N_2 k_y^2 + 2N_3 k_x k_y.$$

In this equation, if both the surface density m and the bending rigidity D of the plate are set equal to zero, Eq.(35) corresponds to the equation governing the stability of the wave motion of a liquid in a vessel under a vertical periodic excitation derived by Benjamin & Ursell⁽¹⁾. It is noticed that N denoting an in-plane force can take both a positive value and a negative value, while

the surface tension coefficient takes only a positive value.

Since Eq.(35) is one of the typical Mathieu equations, the subharmonic response is well known to be observed as the strongest resonance⁽⁴⁾. Therefore, only the boundary of the instability due to the subharmonic response is discussed in the following. The solution of Eq.(35) is assumed to be in the form as

$$\bar{\eta}(t) = A^+ \cos \frac{\omega}{2} t + A^- \sin \frac{\omega}{2} t \quad (36)$$

Substituting this solution into Eq.(35), the following equations are derived :

$$\left\{ -\left(\frac{1}{k \tanh k} + m \right) \left(\frac{\omega}{2} \right)^2 + Dk^4 + Nk^2 + 1 + \frac{f}{2} \right\} A^+ = 0 \quad (37)$$

$$\left\{ -\left(\frac{1}{k \tanh k} + m \right) \left(\frac{\omega}{2} \right)^2 + Dk^4 + Nk^2 + 1 - \frac{f}{2} \right\} A^- = 0 \quad (38)$$

The instability boundary for the subharmonic response is found from the solvable condition for these equations governing the constants A^+ and A^- . It is given by two curves as

$$-\left(\frac{1}{k \tanh k} + m \right) \left(\frac{\omega}{2} \right)^2 + Dk^4 + Nk^2 + 1 \pm \frac{f}{2} = 0 \quad (39)$$

Under the condition that the wave number is prescribed, the frequency ω_i of the externally imposed oscillation at which the disturbances become most excitable, that is, they can be excited by the smallest amplitude of the external oscillation is given as

$$\left(\frac{\omega_i}{2} \right)^2 = \frac{Dk^4 + Nk^2 + 1}{\frac{1}{k \tanh k} + m} \quad (40)$$

It is found from Eqs.(39) and (40) that the exciting frequency increases as the tensile in-plane force and the bending stiffness increase. However, they have no effects on the extent of the instability region. On the other hand, the exciting frequency decreases as the surface density and the compressive in-plane force increase. The extent of the instability region decreases as the surface density increases. If the boundaries of the instability region for various wave numbers are described in the ω - f plane, they are expressed by a pair of two curves bent slightly as shown in Fig.2. Because f is a quadratic equation with respect to ω . In Fig.2, the region bounded with two curves issuing from a point on the ω -axis is an unstable region. The boundary of the instability for a given wave number in Fig.2 corresponds to that of the Benjamin-Ursell problem for the liquid layer covered with

a thin elastic plate. Since the liquid layer considered is horizontally infinite, it is likely that disturbances with any wave number exist. This fact is crucially different from the case of a wave motion in a finite vessel. Then, if the system is excited by a periodic force with a given frequency, this vibrating system always corresponds to a point in the unstable region for a disturbance with a given wave number, that is, on the right side of the boundary of instability. It is, therefore, found in the case of the ideal fluid that if the system is shaken externally, a wave motion always arises. This is very different from the case of the liquid motion in a vessel.

Fig. 2 The boundaries of the instability region in the case of the ideal fluid.

4. Viscous Fluid Case

The partial differential system, Eqs.(24) to (29), which fully take the viscous effect in the fluid flow into account, is difficult to solve analytically. However, if only the subharmonic response, of which the frequency is equal to half that of the external periodic excitation, is treated in the same way as the ideal fluid case, the partial differential equations considered here can be reduced to ordinary differential equations. If the amplitude function $\bar{w}(t, z)$ of the velocity component in the z direction is expressed in the following form :

$$\bar{w}(t, z) = \frac{\omega}{2} \left\{ W^+(z) \cos \frac{\omega}{2} t + W^-(z) \sin \frac{\omega}{2} t \right\} \quad (41)$$

the amplitude function $\bar{\eta}(t)$ of the displacement of the plate is written, taking the kinematic condition, Eq.(26), at the underside of the plate into account, as

$$\bar{\eta}(t) = -W^-(0) \cos \frac{\omega}{2} t + W^+(0) \sin \frac{\omega}{2} t \quad (42)$$

Substituting these formal solutions (41) and (42) into Eqs.(24), (25), (27), (28) and (29), the governing equations are rewritten in the form of the ordinary differential system as

$$\nu \left(\frac{d^2}{dz^2} - k^2 \right) W^+ = \frac{\omega}{2} \left(\frac{d^2}{dz^2} - k^2 \right) W^-, \dots (43)$$

$$\nu \left(\frac{d^2}{dz^2} - k^2 \right) W^- = -\frac{\omega}{2} \left(\frac{d^2}{dz^2} - k^2 \right) W^+, \dots (44)$$

$z=0$:

$$\nu \frac{\omega}{2} \frac{d^3 W^+}{dz^3} + k^2 \left\{ Dk^4 + Nk^2 + 1 - m \left(\frac{\omega}{2} \right)^2 + \frac{f}{2} \right\} W^- = 0, \dots (45)$$

$$\nu \frac{\omega}{2} \frac{d^3 W^-}{dz^3} - k^2 \left\{ Dk^4 + Nk^2 + 1 - m \left(\frac{\omega}{2} \right)^2 - \frac{f}{2} \right\} W^+ = 0, \dots (46)$$

$$\frac{dW^+}{dz} = \frac{dW^-}{dz} = 0, \dots (47)$$

$z=-1$:

$$W^+ = W^- = \frac{dW^+}{dz} = \frac{dW^-}{dz} = 0, \dots (48)$$

In the following, these homogeneous differential equations are numerically integrated by using the Runge-Kutta method, and the eigen relation

$$f(\omega, f, k; \nu, D, N, m) = 0 \dots (49)$$

is obtained. A question of under what condition the excitation arises is mainly discussed.

5. Minimum Amplitude of Acceleration

The eigen relation (49) obtained by solving the equation system in the section 4 gives a critical condition for the subharmonic excitation to occur in this system, that is, a boundary of the instability region. The boundaries of the instability regions are shown in Fig.3 taking the various wave numbers of the wave motion arising in the system as the parameter. In this figure, the properties of the fluid and the plate are assumed to be constant. In the region on the right side of the curve corresponding to each wave number, a disturbance with its wave number can arise. It is found from this figure that the boundary of the instability region does not intersect the ω -axis. That is, an excitation in the considered system is likely to happen only when the amplitude of the acceleration externally exerted is larger than a finite critical value. This fact is found crucially different from the case of the ideal fluid. The critical amplitude f_c of the acceleration above which a disturbance arises is a function of the wave number. It is found from Fig.3 that the critical amplitude takes a minimum value for the disturbance of a special wave number, that is, the amplitude of the given acceleration increases as the

wave number is large or small compared with the special wave number. In this system, a disturbance with any wave number is likely to arise. Therefore, if a curve which envelops the boundaries of the instability regions for various wave numbers is drawn in the ω - f space as shown in Fig.3, this envelope is the neutral stability curve of the system considered. Whereas, in the case of the ideal fluid all the regions in the ω - f plane are unstable regions, in this case only the region on the right side of the neutral stability curve is an unstable region. A minimum amplitude $f_{\min}$ of the acceleration above which a disturbance always arises is found on the neutral stability curve. This result is derived from the fact that the viscous effect is taken into account. Because the difference between the ideal fluid theory in the section 3 and the viscous fluid one in the section 4 is in only one point, that is the fact that the viscous effect is taken into account in the latter theory. It is found, therefore, that the viscosity of the fluid directly prevents the wave motion from arising in the liquid layer covered with a thin elastic plate. Then, the stiffness, the own weight and the in-plane force of the plate are found to be ineffective for the prevention of the wave motion.

Fig. 3 The boundaries of the instability region and the neutral stability curve in the case of the viscous fluid.

6. Viscous Effects

It is found in the previous section that only the viscosity is directly effective for the prevention of the wave motion. Then it becomes important to discuss the viscous effect. The effect of the viscosity on the boundary of the instability region for the disturbance with a given wave number is shown in Fig.4. As the viscosity increases, the critical amplitude of the acceleration externally exerted increases. In the vicinity of the upper branch of the boundary of the instability region, the viscosity is found from Fig.4 to highly stabilize the system. In the vicinity of the lower branch of the boundary of the

instability region, it slightly destabilizes the system. The frequency corresponding to the critical amplitude of the acceleration is decreased by the viscous effect.

Fig. 4 The effect of the viscosity on the boundary of the instability region.

Fig. 5 The effect of the viscosity on the neutral stability curve.

Fig. 6 The relation between the frequency of the vertical oscillation and the wave number of the disturbance on the neutral stability curve.

The effect of the viscosity on the neutral stability curve which envelops the boundary of the instability region is shown in the ω - f plane in Fig.5. As the viscosity increases, the neutral stability curves in the ω - f plane are shifted to the right side. That is, the increase of the viscosity stabilizes the system over all the intervals of the frequency. The broken line in Fig.5 shows the state that makes the amplitude of the acceleration minimum, that is, makes it f_{min} . The frequency corresponding to f_{min} increases as the viscosity increases. Since, in the vicinity of the broken line, the neutral stability curve is nearly parallel to the ω -axis, if the amplitude of the acceleration increases beyond a critical value, the excitation arises irrespective of the frequency of the periodic excitation. Thus, whether or not the instability arises is found to mainly depend on the magnitude of the amplitude of the acceleration and hardly depend on its frequency. The relations between the wave number and the frequency on the neutral stability curves shown in Fig.5 are described in Fig.6. The wavelength of the disturbance arising in the system decreases as the frequency increases. As the viscosity increases under the condition of a constant frequency, the wavelength decreases slightly. The effect of the viscosity on the eigen functions W^+ and W^- is shown in Figs.7 and 8. It is found from these figures that in the viscous case the fluid near the rigid wall is hard to move unlike the ideal case.

Fig. 7 The viscous effect on the eigen function W^+ .

Fig. 8 The viscous effect on the eigen function W^- .

7. Effect of Plate

The effects of the bending stiffness, the surface density and the in-plane force of the plate on the instability characteristics are shown in Figs.9 to 11. The critical frequency of the acceleration making the system most unstable increases as the bending stiffness and the in-plane force increase. Meanwhile it decreases as the surface density increases. These results are the same as in the case of the ideal fluid. The critical amplitude of the acceleration making the system most unstable increases as the bending stiffness and the in-plane force increase, and it decreases as the surface density increases. Thus, the increase in the bending stiffness and the tensile in-plane force is found to stabilize the system shifting the neutral stability curve to the right side. Inversely, the increase of surface density is found to slightly destabilize the system.

Fig. 9 The effect of the bending stiffness.

Fig.10 The effect of the surface density.

Fig.11 The effect of the in-plane force.

8. Comparison between the Theoretical Results and the Experimental Ones

An aqueous solution of glycerine put in a square vessel with the side length 33 cm and the depth 4 cm is prepared for the experiment. The surface of the solution of glycerine is covered by a rubber plate with the thickness 0.5 mm of which the edges are free. This liquid layer is vertically excited by using a mechanical vibrator of the Matsudaira type. The experiment is performed varying the frequency and the amplitude. It was difficult to take the experimental data over a wide range of the frequencies on account of the capacity of the vibrator. The bending stiffness of the rubber plate is directly measured in every experiment because the material property of the rubber is very changeable in the course of time. The bending stiffness of the plate is found in the following preliminary experiment: the circular plate of the rubber with the surface density m is horizontally clamped on the upper edge of a vertical hollow cylinder with the inner radius R . The displacement ξ_0 at the center of the plate is measured under the action of gravity by using a powerful microscope. The bending rigidity D in the dimensional form is determined using the formula $mgR^4/64\xi_0$ (5). As a result, the dimensionless bending rigidity is determined as $D^*=1.008$. The wave motion in the condition a little beyond the critical condition is observed by taking a photograph from the upper side of the rubber. The wavelength of the disturbance is measured from this photograph. One of the experimental data measured is shown in the ω - f plane and ω - k plane in the dimensionless form as Figs.12 and 13. The theoretical values corresponding to these experimental data are shown with a solid curve in these figures. The minimum amplitude f_{min} of the acceleration obtained experimentally is nearly equal to 0.5. As expected from the theory, the critical amplitudes of the acceleration are found experimentally to

Fig.12 Comparison of the neutral stability curves found experimentally and theoretically.

vary a little over a comparatively wide range of the frequencies. They are slightly large compared with that of theory. The neutral stability curve found experimentally fairly well agrees with that of the theory. It is found from Fig.13 that the relation between the frequency of the vibrator and the wave number of the disturbance completely agrees with the theoretical one. Therefore, it is concluded that the

Fig.13 Comparison of the relations between the frequency and the wave number found experimentally and theoretically.

frequency of the exciting disturbance is equal to half the frequency of the vibrator. The wavelength of the disturbance obtained from the experiment is found from Fig.13 to vary continuously with the frequency of the vibrator. Thus, it is inferred that the end effect of the rubber and of the vessel is negligible. The agreement between theory and experiment is fairly good as mentioned above.

Fig.14 A two-dimensional wave motion (kinematic viscosity $0.33 \text{ cm}^2/\text{s}$, depth 1 cm, frequency 399 rpm, amplitude 0.38 cm and wavelength 8.4 cm).

Lastly, the photographs of a wave motion of the rubber in the excited state slightly beyond the critical condition are shown in Figs.14 and 15. They are taken from upper side of the rubber. It should be noted that the wave motions in the excited state near the critical condition are experimentally found to be almost two-dimensional as shown in Fig.14 and to be scarcely three-dimensional as shown in Fig.15.

9. Conclusions

The instability of a liquid layer covered with a thin elastic plate under a vertical periodic motion is studied theoretically. As a result, it is found that (1) the boundaries of the instability region are obtained for various wave numbers, (2) the neutral stability curve is determined as a curve enveloping their boundary, (3) there exists a minimum value in the critical amplitudes of the acceleration making the system first unstable, (4) this fact is caused from the viscosity of the fluid, (5) the tensile in-plane force and the bending stiffness of the plate make the system stable, while the surface density of the plate makes it

Fig.15 A three-dimensional wave motion (kinematic viscosity $0.33 \text{ cm}^2/\text{s}$, depth 1 cm, frequency 813 rpm, amplitude 0.11 cm and wavelength 5.3 cm).

unstable, and (6) disturbances are excited by a $1/2$ subharmonic response. These results are fully confirmed by the experiment undertaken.

References

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