Tension of an Elastic Strip with Two Pairs of Semicircular Notches*
(Interference Effects of Stress Concentrations)

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This paper deals with the stress concentrations of an elastic strip with two pairs of semicircular notches under tension. The method of solution is based upon an application of Green's functions for body force problems of a strip. That is, a principle of the method of solution is to distribute body forces in the interior of a strip with no notches and to determine the intensities of the body forces distributed so as to satisfy the boundary conditions of the notches. Numerical results are obtained systematically for various kinds of notch sizes. An investigation is made for interference effects of stress concentrations due to two pairs of notches and the influences of radii of notches and distance between two pairs of notches on the stress concentration factors.

Key Words : Elasticity, Stress Concentration, Form Factor, Notch, Strip, Body Force Method, Green's Function, Tension, Stress Analysis

1. Introduction

This paper deals with the stress concentration problems of a strip with two pairs of semicircular notches under tension (compression) in the two dimensional theory of elasticity. The problem has been treated by Atumia, but there are few numerical results in his paper.

It is well known that form factors (stress concentration factors) are relaxed by multiple notches due to interference effects of stress concentrations. In previous papers, we have investigated the stress concentration problems of a solid cylinder with two circumferential grooves under tension or torsion and pointed out that there are cases of $\alpha_1\alpha_2$ in tension, whereas the relation $\alpha_1\alpha_2$ always holds in torsion, where $\alpha_i$ represents the form factor for $i$ grooves. In this paper, it is shown that there exist cases of $\alpha_1\alpha_2$ in tension of a strip with two pairs of notches, similar to a solid cylinder with two grooves under tension.

The fundamental principle of the method of solution is an application of Green's functions for body force problems of a strip, and is the so-called body force method as investigated by Nisitani. That is, the method of solution used here is to distribute body forces so as to satisfy boundary conditions of the surfaces of notches. Green's functions are defined as a solution to the problem of a strip subjected to concentrated forces acting on four points in the interior of the strip and satisfying the boundary conditions of stress free from applied surface forces.

It may be also noted that the body forces are distributed on a finite number of points, whereas the body forces are distributed continuously in Nisitani's method.

Performing numerical calculations systematically for various sizes of two pairs of notches, we investigate the effects of the distance between two pairs of notches and the radii of semicircular notches on the form factors.

2. Boundary Conditions

We consider tension of a strip of a width $2C$ with two pairs of semicircular notches of equal radii r symmetrically located on the straight edges as shown in Fig.1, where we used rectangular coordinates $(x,y)$. In Fig.1, $2C$ represents the distance between the two pairs of notches. If we assume that the strip is subjected to a uniform stress $\sigma_y=0$ at a sufficiently large distance from the notches and all the surfaces of the strip are free from applied surface forces, then the boundary conditions of the problem are expressed as follows:

- \[ x^2+c_1 |y| \leq -r, r+rsy |y| \leq c_2, \quad c_2=0 \]
- \[ x^2+c_1 |y| \leq 0 \]
- \[ x^2+c_1 |y|=c_2, c_2=0 \]

where $c_1$ and $c_2$ are normal and shear stresses developed on the notched surfaces $x_1, y_1$ and are expressed by

- \[ N_x = a_1 \sin^2 \theta + a_2 \cos^2 \theta + 2r_0 \sin \theta \cos \theta \]
- \[ + r_0 \sin \theta \cos \theta \]

where $\theta$ is an angle between a normal to a notched surface and the straight edge as shown in Fig.2.
3. Basic Equations

It is well known that the displacement \( U(x, y) \) of a two dimensional elastic solid satisfies the following equation

\[
\mu \left( \frac{1}{1} \nabla \phi \right) + k \nabla \cdot \nabla U + F = 0
\]

where \( \mu \) is the shear modulus of elasticity, \( F(x, y) \) is a body force, \( \nabla^2 \) is Laplacian in two dimensions and \( k \) is expressed by

\[
k = \begin{cases} 
\frac{1}{1} & \text{(in plane stress)} \\
\frac{1}{1 - 2\nu} & \text{(in plane strain)} 
\end{cases} 
\]

where \( \nu \) is Poisson's ratio.

The displacement \( U \) satisfying Eq.(3) can be expressed by

\[
2\mu \nabla^2 \phi = 2(1 - \nu) \nabla \cdot \nabla \phi - \frac{2(1 - \nu)k}{1 + k} \nabla \cdot \nabla \phi
\]

provided that the stress function \( \phi(x, y) \) satisfies the following equation

\[
\nabla^2 \phi = \frac{1}{1 - \nu} F
\]

4. Green's Function

In order to obtain a solution to the problem mentioned above, we shall apply Green's functions for body force problems of a strip. Green's functions used here are defined as a solution to the body force problem of an infinite strip satisfying the boundary conditions

\[
x = \pm c, |y| < \infty; \sigma_x = \sigma_y = 0
\]

and subjected to the concentrated forces

\[
F_1 = \delta(x - a) - \delta(x + a)) \delta(y - b) + \delta(y + b) \\
F_2 = \delta(x - a) + \delta(x + a)) \delta(y - b) - \delta(y + b)
\]

acting at four points \( (x = \pm a, y = \pm b) \) in the interior of the strip. In Eq.(8), \( \delta(\cdot) \) is a Dirac delta function and \( F_j \) is the \( j \)-th component of the body force \( F \) in \( x \) \((j = 1)\) and \( y \) \((j = 2)\) directions.

From Eqs.(5) and (6), we can find Green's functions defined above as follows:

(i) Green's functions for the body force \( F_1 \):

\[
\begin{align*}
\sigma_x^1 & = \sigma_x \text{ of Eq.(20)} \\
\sigma_y^1 & = \sigma_y \text{ of Eq.(24)} \\
r_{11} & = r_{11} \text{ with G of Eq.(26)}
\end{align*}
\]

(ii) Green's functions for the body force \( F_2 \):

\[
\begin{align*}
\sigma_x^2 & = \sigma_x \text{ of Eq.(21)} \\
\sigma_y^2 & = \sigma_y \text{ of Eq.(24)} \\
r_{11} & = r_{11} \text{ with G of Eq.(27)}
\end{align*}
\]

where Eqs.(20) and so on are shown in Appendix.

5. Method of Solution

From the symmetry of the problem with respect to the \( x \)- and \( y \)-axes, we may consider only the quarter part \((0,0,0,0)(r,0,0,0)\) of
the strip as shown in Fig. 2. The curved line of a radius \( r \) in Fig. 2 is the plane to become the surface of a semicircular notch and the dotted line of a radius \( e \) is the plane of distribution of the body forces of Eq. (8). It is easily seen that the stresses at a point \((x, y)\) of the strip due to the body forces distributed on the dotted line and a uniform tension \( \sigma_{\psi} = \sigma_{\psi} \) are expressed by

\[
\begin{align*}
\sigma_1 &= \int_{r_{\alpha}}^{r_{\alpha'}} f_1(\theta) \sigma_1' \, d\theta \\
\tau_{\alpha \theta} &= \int_{r_{\alpha}}^{r_{\alpha'}} f_2(\theta) \sigma_1' \, d\theta \\
+ f_3(\theta) \left[ \sigma_1' \right]_{r_{\alpha}}^{r_{\alpha'}} \, d\theta + \frac{\psi}{2} \left[ \sigma_1' \right]_{\theta=\pi}^{\theta=0} \\
\end{align*}
\]

\[
\text{(11)}
\]

where \( f_1(\phi) \), \( f_3(\phi) \), are arbitrary functions corresponding to intensities of the body forces distributed, \( \psi \) is the angle from the straight edge \((x=C)\) to a normal (radius \( e \)) at a point \((x, y)\) on the plane of distribution of the body forces as shown in Fig. 2 and \( \sigma(\phi) \) and \( b(\phi) \) represent

\[
\begin{align*}
\sigma(\phi) &= c - e \sin \phi \\
b(\phi) &= s - e \cos \phi
\end{align*}
\]

\[
\text{(12)}
\]

From the definition of Green's functions used here, Eq. (11) satisfies completely both the boundary conditions (i) and (ii) of Eq. (1) as well as the fundamental equation (3) of elasticity, regardless of the values of the functions \( f_1(\phi) \). Therefore, if we choose the values of the functions \( f_1(\phi) \) such that Eq. (11) satisfies the boundary condition (ii) of Eq. (1), then Eq. (11) becomes a solution to the present problem.

Application of the boundary condition (ii) of Eq. (1) to Eq. (11) yields

\[
\begin{align*}
-\rho \cos \theta &= \int_{r_{\alpha}}^{r_{\alpha'}} f_1(\theta) \sigma_1' \, d\theta \\
-\rho \sin \theta \cos \theta &= \int_{r_{\alpha}}^{r_{\alpha'}} f_3(\theta) \sigma_1' \, d\theta \\
+ f_3(\theta) \left[ \sigma_1' \right]_{r_{\alpha}}^{r_{\alpha'}} \, d\theta + \frac{\psi}{2} \left[ \sigma_1' \right]_{\theta=\pi}^{\theta=0}
\end{align*}
\]

\[
\text{(13)}
\]

where the coordinates \((x_{i}, y_{i})\) represent a point on the plane of the radius \( r \) which is to become a notched surface, and \( \sigma_{\psi} \) and \( \tau_{\alpha \theta} \), \((j=1, 2)\), represent

\[
\begin{align*}
\sigma_{\psi}' &= \sigma_{\psi} \sin \theta + \sigma_{\psi} \cos \theta \\
+ 2 \tau_{\alpha \theta} \sin \theta \cos \theta \\
\tau_{\alpha \theta}' &= -\sigma_{\psi} \sin \theta \cos \theta \\
+ \tau_{\alpha \theta} \left( \sin \theta + \cos \theta \right)
\end{align*}
\]

\[
\text{(14)}
\]

Expressions (13) are dual integral equations with unknown functions \( f_1(\phi) \). To solve the integral equations by reducing them to a set of linear algebraic equations, we perform an appropriate discretization of the body forces distributed. For this purpose, we shall assume

\[
\begin{align*}
f_1(\theta) &= \sum_{n=1}^{N} f_1^n(\phi - \psi_n) \\
(0 < \psi_n < \pi; \ n = 1 \sim N)
\end{align*}
\]

\[
\text{(15)}
\]

where \( f_1^n \) are arbitrary constants corresponding to the intensities of the body forces distributed, and \( \psi_n \) is an angle \( \psi \) corresponding to a point \((x, y)\) where body forces act. Assumption of Eq. (15) means that the body forces are distributed on \( N \) isolated points

\[
\begin{align*}
\sigma_{\psi} &= c - e \sin \psi \psi_n \\
\tau_{\alpha \theta} &= s - e \cos \psi \psi_n
\end{align*}
\]

\[
\text{(16)}
\]

Substitution of Eq. (15) into Eq. (13) yields

\[
\begin{align*}
-\rho \cos \theta &= \sum_{n=1}^{N} f_1^n(\sigma_{\psi}^{n}) \\
-\rho \sin \theta \cos \theta &= \sum_{n=1}^{N} f_3^n(\sigma_{\psi}^{n}) \\
+ f_3^n(\sigma_{\psi}^{n}) \left[ \sigma_{\psi}^{n} \right]_{r_{\alpha}}^{r_{\alpha'}} \, d\theta + \frac{\psi}{2} \left[ \sigma_{\psi}^{n} \right]_{\theta=\pi}^{\theta=0} \\
\end{align*}
\]

\[
\text{(17)}
\]

Since there are \( 2N \) arbitrary constants \( f_1^n \) in Eq. (17), we shall consider \( N \) points

\[
\begin{align*}
x_i &= c - r \sin \psi_i \\
y_i &= s - r \cos \psi_i \\
(0 < \psi_i < \pi; \ i = 1 \sim N)
\end{align*}
\]

\[
\text{(18)}
\]

on the plane of a radius \( r \) which is to become the boundary of a notch and choose the values of \( f_1^n \) which satisfy Eq. (17) at these selected points. It follows from the above that we can construct a set of simultaneous linear equations. Solving the linear algebraic equations and substituting their solutions \( f_1^n \) into the expressions

\[
\begin{align*}
\sigma_1 &= \sum_{n=1}^{N} f_1^n \sigma_1^{n} \\
\tau_{\alpha \theta} &= \sum_{n=1}^{N} f_1^n \tau_{\alpha \theta}^{n} \\
+ f_3^n(\sigma_{\psi}^{n}) \left[ \sigma_{\psi}^{n} \right]_{r_{\alpha}}^{r_{\alpha'}} \, d\theta + \frac{\psi}{2} \left[ \sigma_{\psi}^{n} \right]_{\theta=\pi}^{\theta=0} \\
\end{align*}
\]

\[
\text{(19)}
\]

then we can obtain stresses at any point \((x, y)\) in the strip with two pairs of semicircular notches.

6. Validity of the Method of Solution (Solutions for a pair of semicircular notches)

For the confirmation of the validity of the method of solution described above, it was applied to the problem of a strip with a pair of semicircular notches. This problem has been investigated by Ishida\(^1\), Nishitani\(^2\), Lin\(^3\), Appli\(^4\), and so on, and their results (form factors) for the problem are shown in Table 1 with the present results. The value 3.065 for \( r/c = 0.0 \) in Table 1 was referred to the references (11) and (12). From Table 1, we see that the present results are in good agreement with the known one in the accuracy within 0.15.

7. Numerical Results

Based on the method of solution described in the previous chapters, numerical calculations were performed. The values \( e/c \) and \( N \) used in the numerical calculations are shown in Table 2. As for the \( N \) points (\( \psi = \psi_n \))
where body forces are distributed and as for the N points \((\theta = \theta_j)\) where the boundary conditions are to be satisfied, we take the N points corresponding to the angles determined by dividing the central angle \(\pi\) of a semicircular notch into \((N+1)\) equal angles.

As mentioned in the previous papers by Atsumi, the maximum stress \(\sigma_0\) and \(\sigma_0\) found by Atsumi, the maximum stress \(\sigma_0\) for the cases of two notches or two pairs of notches might appear not at the bottom \((\theta = 90^\circ)\) of a notch but at a point \((\theta = 90^\circ)\) shifted in the y-direction on the notched surface. Figure 3 shows the relation between the positions \((\theta = \theta_j)\) where the maximum stress \(\sigma_0\) is found, s/c and r/c. It is seen from Fig.3 that the relation is substantially of the same character as that of a solid cylinder with two notches investigated in the previous paper, and the positions \((\theta = \theta_j)\) more away from the bottom of a notch in the y-direction with a decrease of s/c under the condition of s/c=constant.

Table 1 gives the form factors \(\frac{\sigma_0 \max}{\sigma_0 \min}\) obtained in the paper, where \(\sigma_0 \max\) and \(\sigma_0 \min\) are the maximum and mean values of the stress \(\sigma_0\) at the cross section where the maximum stress appears. In Table 1, the numbers \((1),(2)\) and \((3)\) represent \(\sigma_0\) calculated by \((1)\) \(\sigma_0 \max\) and \(\sigma_0 \min\) at the values of the minimum cross section \((\theta = 90^\circ)\), \(\sigma_0 \max\) and \(\sigma_0 \min\) at the cross section \((\theta = 90^\circ)\), and \(\sigma_0 \max\) and \(\sigma_0 \min\) at the minimum cross section \((\theta = 90^\circ)\), respectively.

By the way, two values for the form factor of the problem have been given by Atsumi in the case of s/c=0.5 as follows:

\[
\begin{align*}
\frac{r}{c} & \quad 0.1 \quad 0.25 \quad 0.5 \\
\frac{a_0}{a_1} & \quad 2.68 \quad 2.76 \quad 2.56 \\
\end{align*}
\]

The present results agree with Atsumi's results in the accuracy within 0.5%

Figure 4 shows the relation between \(\frac{a_0}{a_1}\) and s/c with the parameter r/c, where \(a_0\) is a form factor shown in Table 1. Figure 5 shows the relation between \(\frac{a_0}{a_1}\) and r/c with the parameter s/c. From these figures we see that (i) the interference effects (a phenomenon of \(a_0 > a_1 < 1.0\)) due to two pairs of notches occur in the condition of r/c<0.55; \(a_0\) in the condition of r/c<0.55 there are cases of \(a_0 > a_1 > 1.0\); (ii) the interference effects are larger when r/c is about 0.3 under the condition of s/c=constant; (iv) the interference effects are larger for smaller values of s/c under the condition of r/c=constant.

Figure 6 shows the distribution of the stresses \(\sigma_0\) in the cases of r/c=0.3, where the solid lines, the dotted lines and the dot chain lines are the results for s/c=0.9, 0.6 and 0.3, respectively. In Fig. 6, y/s=1.0 represents the distribution on the minimum cross section and y/s=0.0 does that on the symmetrical cross section. From Fig. 6 we see that the stress distributions on the minimum cross section (y/s=1.0) do not make much difference according to change of s/c, whereas the distribution on the symmetrical cross section (y/s=0.0) approaches a uniform one with an increase of s/c.

S. Concluding Remarks

The principal results of this paper are summarized as follows:

(1) A method of solution is proposed for the stress concentration problems of a strip with two pairs of semicircular notches under tension, by applying Green's functions for body force problems of a strip.

(2) The form factors for the problem are shown systematically for various sizes of notches, and the interference effects (a phenomenon of \(a_0 > a_1 < 1.0\)) where \(a_0\) is a form factor for 1 pairs of notches) of stress concentrations are investigated. It is also shown that there are cases of \(a_0 > a_1 > 1.0\).

(3) For verification, the method of solution proposed is applied to the problem of a strip with a pair of semicircular notches. The present results are in good agreement with the known ones.

Table 2 The values of \(a/c\) and N used in numerical calculations.

<table>
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<tr>
<th>r/c</th>
<th>0.1</th>
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<th>0.5</th>
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Table 1 Form factors \(a_0\) for a strip with a pair of notches.

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<th>Ishida</th>
<th>Misinito</th>
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Fig. 3. The positions (angle \(\theta_j\)) where the maximum stress \(\sigma_0\) appears.
Table 3 Form factors $a_2$ for a strip with two pairs of notches.

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Acknowledgments

The authors are indebted to Messrs. Ichihara, T., Kawashima, T. and Sakaida, Y. (graduates of Meiji University) for their help in numerical calculations.

Appendix

Expressions of Eqs. (20)–(27) mentioned in Eqs. (9) and (10) are as follows:

\[ \sigma_2 = -\frac{1}{4\pi c \alpha} \left[ 3 + \xi (x Q_3 - Q_0) - 2(1 + \xi)Q_3 \right] - \frac{1}{\pi \alpha} \left[ (x - 3x Q_3 - Q_0) + 2(1 + \xi)Q_3 \right] \]

\[ \sigma_3 = -\frac{1}{4\pi c \alpha} \left[ (x - 3x Q_3 - Q_0) + 2(1 + \xi)Q_3 \right] \]

\[ \tau_{23} = -\frac{1}{4\pi c \alpha} \left[ \frac{3}{2} (3 + \xi)Q_3 - 2(1 + \xi)Q_3 \right] - 2(1 + \xi) \frac{\pi \alpha}{2} Q_3 \]

\[ \sigma_5 = -\frac{1}{4\pi c \alpha} \left[ (1 + 3\xi) \frac{x Q_3}{2} - 2(1 + \xi)Q_3 \right] - 2(1 + \xi) \frac{\pi \alpha}{2} Q_3 \]

\[ \sigma_7 = -\frac{1}{4\pi c \alpha} \left[ (1 - 3\xi) \frac{x Q_3}{2} + 2(1 + \xi)Q_3 \right] + 2(1 + \xi) \frac{\pi \alpha}{2} Q_3 \]

\[ \tau_{57} = -\frac{1}{4\pi c \alpha} \left[ (1 - \xi)(y Q_3 - y')Q_3 + 2(1 + \xi)Q_3 \right] + \frac{\pi \alpha}{2} Q_3 \]

In these expressions, \( Q_1 \) and \( Q_4 \) represent

\[ Q_1 = \frac{1}{\pi \alpha} (1 + \xi) \cot \frac{\pi x}{c} \]

\[ Q_2 = \frac{1}{\pi \alpha} \frac{1}{x^2} \sinh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_3 = \frac{1}{\pi \alpha} \frac{1}{x^2} \sinh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_4 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_5 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_6 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_7 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_8 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_9 = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{10} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{11} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{12} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{13} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{14} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{15} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{16} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{17} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{18} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{19} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

\[ Q_{20} = \frac{1}{\pi \alpha} \frac{1}{x^2} \cosh \frac{y y'}{x(x^2 + y')^2} \]

Fig. 4 Interference effects of stress concentrations.

Fig. 5 Interference effects of stress concentrations.

Fig. 6 Distributions of the stresses \( \sigma_x \).
\[Q_{st} = \frac{1}{a_1 a_2} \left( \frac{cy + y'}{x_1 + y'} \right) \]
\[Q_{s4} = -\frac{1}{a_1 a_2} \left( -1 \right) \left( \frac{cy + y'}{x_1 + y'} \right) \]

where
\[x_1 = x - a, \ y_1 = y + b\]
\[x_1 = x - (2nc + a), \ x_1 = x + (2nc + a)\]
\[x_2 = x - (2nc - a), \ x_2 = x + (2nc - a)\]

\[\begin{align*}
\sigma_x &= \int_{x_1}^{x_2} \frac{G}{D} \left( \tanh \beta \epsilon + \beta \epsilon \cosh \beta \epsilon \right) \\
\sigma_y &= \int_{y_1}^{y_2} \frac{G}{D} \left( \tanh \beta \epsilon - \beta \epsilon \cosh \beta \epsilon \right) \\
\tau_{xy} &= \int_{x_1}^{x_2} \frac{G}{D} \left( \beta \epsilon \sinh \beta \epsilon \right) \\
\end{align*}
\]

where
\[D = \sinh \beta \epsilon + \beta \epsilon \cosh \beta \epsilon - \beta \epsilon \tanh \beta \epsilon \sinh \beta \epsilon\]

Expression \(G\) is, as \(\gamma = e^{-\beta \epsilon}\), as follows:

(i) for the body force \(F_1\):
\[G = -\frac{2 \cos \beta \epsilon}{\pi} e^{-\beta \epsilon} \left[ 1 + \frac{1 + \frac{\xi}{2}}{1 - \gamma} (c - a) \right] \]
\[+ \gamma \left[ e^{-\beta \epsilon} \left[ 1 + \frac{1 + \frac{\xi}{2}}{1 - \gamma} (c - a) \right] \right] \]
\[e^{-\beta \epsilon} \left[ 1 + \frac{1 + \frac{\xi}{2}}{1 - \gamma} (c - a) \right] \]

where \(\xi = \nu\) for plane stress and \(\xi = \nu/(1-\nu)\) for plane strain. The notation \(\nu\) represents Poisson's ratio.

References