Analysis of Plane-strain Work Hardening in Sheet Alloys and Its Effect on Forming Limit*

By Yasuyoshi FUKUI**, Kenji NAKANISHI***, and Shun-ichi OKAMURA****

Using commercial 70/30 brass, OFHC copper, and 1100-O aluminum sheet alloys, the work hardening behavior and limit strain in plane-strain condition have been studied by means of the uniaxial and plane-strain tensile tests in relation to the effect of empirical work hardening law equation and yield function. The parameter m of Hill's non-polynomial yield function is confirmed to be the function of strain by applying Wagoner's rigorous method. The best-fit empirical work hardening law equation describing the relation between logarithmic work hardening rate and effective strain is Voce law equation (saturation law) for brass, Swift law equation (modified power law) for copper, and Hollomon law equation (power law) for aluminum. The results indicate that theoretical ε₁₀ of forming limit in plane-strain condition are almost independent of m-value and theoretical ε₁₀ based on best-fit work hardening law equation agree well with experimental ε₁₀. The conditions which govern the m-value and the factors which influence the forming limit are discussed.

Key Words: Plasticity, Plane-strain Condition, Work Hardening Law, Logarithmic Work Hardening Rate, Non-polynomial Yield Function, Forming Limit.

1. Introduction

For predicting the forming limit of thin sheet alloys, several theories have been developed independently. Those are, for example, Swift's diffuse necking condition[1], Hill's localized necking condition[2], and Stören and Rice model[3], etc. In plane-strain condition, the theoretical maximum principal strain (ε₁) of forming limits predicted by different theories agree precisely with each other and coincide with n-value which is the work hardening exponent of Hollomon law equation[4] (σ = εεⁿ). However, poor agreement between the experimental forming limits and the theoretical forming limits has been reported[5]-[9].

The most commonly used yield criterion is well-known Hill's quadratic yield function[10]. According to this yield function, great difference is observed between experimental uniaxial stress-strain curve and effective stress-strain curve predicted from biaxial tension[11]. To accommodate the results for biaxial tension, Hill has proposed a non-polynomial yield function containing a new parameter m[12]. It is, however, difficult to obtain the m-value so that only a few results is reported. Mellor and Parma[13] determined the m-value by comparing the experimental work hardening characteristics for uniaxial tension with balanced biaxial tension, while Wagoner[14] determined by comparing that for uniaxial tension with plane-strain tension. The theoretical limit strain may be affected considerably by the m-value.

The dependence of work hardening exponent on strain is discussed from another reason of the poor agreement between theoretical and experimental limit strain in plane-strain condition[15],[16]. For 70/30 brass, an approach proposed by Ghosh[15], for example, uses n-value in post-uniform elongation where n-value is clearly lower than that in uniform elongation. However, n-value changes continuously with strain and the limit effective strain varies with strain condition, so that this approach seems to be inadequate.

In the present paper, the work hardening behaviors of thin sheet alloys for uniaxial tension and plane-strain tension are examined and the new parameter m is determined by applying the method proposed by Wagoner[14]. Then, in order to establish the limit strain in plane-strain condition, the role of the variations of material properties, i.e., yield function and work hardening behavior, upon the limit strain in plane-strain condition is carefully examined.
2. Experiment and Analysis

2.1 Experimental Procedure

Three test materials were commercial 70/30 brass, OFHC copper, and 1100-O aluminum having a nominal thickness of 1.0mm. Standard JIS 6 sheet uniaxial tensile specimen and plane-strain tensile specimen shown in Fig.1 were machined with tensile axes parallel to the rolling direction (RD), transverse direction (TD), and 45deg-to-rolling direction (45). The true strain rate was 10^-3 second per. For uniaxial tensile test, the deformation of grid occurs with 2mm diameter contacting on the specimen surface were recorded by the photographs. Then the axial and transverse strain (εi and εj) and plastic anisotropy parameter (r-value) were directly obtained from the photographs. For plane-strain tensile test, the deformation of grid with 1mm long by 0.5mm wide contacting on the specimen surface between the notch root were also recorded by the photographs. These photographs were used to calculate axial and transverse strain distributions. Here assumed the constancy of volume (which gives εi=εj=ε) and considered only normal anisotropy. The directional mechanical properties of the sheet were averaged and the average values from three directions in the sheet were given by

\[
X = \frac{1}{3}(X_{10} + 2X_{40} + X_{60}) \quad \text{(1)}
\]

where the subscripts identified test directions. The results of the uniaxial tests were summarized in Table 1.

2.2 Yield Criteria

For Hill's quadratic yield function [10], assuming only normal anisotropy, the effective stress and effective strain are given by

\[
\sigma = \frac{1}{2(1+r)}[(1+2r)\sigma_0 + \sigma_1^1 + \sigma_2^2] \quad \text{(4)}
\]

\[
\varepsilon = \frac{1}{2(1+r)}[(1+2r)(\varepsilon_0^0 + \varepsilon_1^1 + \varepsilon_2^2)] \quad \text{(5)}
\]

where, \(a = m/(m-1)\), \(x = (1+2r)^{1/(m-1)}\).

Eqs. (4) and (5) are presented as specialization of Eqs. (4) and (5) with the m parameter fixed at 2.0, respectively.

2.3 Plane-strain Tests Analysis

The effective stress-strain curves for plane-strain tension based on Hill's quadratic yield function, Eqs. (2) and (3), are obtained by the method proposed by Wagoner[14]. The outline is written as below.

For each set of grid readings, two symmetric smooth curves, which represent the distributions of axial and transverse strain (εi and εj, respectively) along the transverse center line at the notch, are generated. The width at the notch is separated into a central region (approximately plane-strain region), \(|\varepsilon_0/\sigma_0|<1/5\), and two edge regions, \(|\varepsilon_0/\sigma_0|>1/5\). The distributions of effective stress are calculated from these axial and transverse strain distributions. The distributions of effective stress are estimated applying the uniaxial stress-strain curve. Then, the average of the apparent effective stress in central re-

\[
\sigma = \frac{1}{2(1+r)}[(1+2r)\sigma_0 + \sigma_1^1 + \sigma_2^2] \quad \text{(4)}
\]

\[
\varepsilon = \frac{1}{2(1+r)}[(1+2r)(\varepsilon_0^0 + \varepsilon_1^1 + \varepsilon_2^2)] \quad \text{(5)}
\]

Table 1 Summary of data in uniaxial tension test.

<table>
<thead>
<tr>
<th></th>
<th>Brass</th>
<th>Copper</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>UTS, MPa</td>
<td>607.</td>
<td>368.</td>
<td>142.</td>
</tr>
<tr>
<td>δ, %†</td>
<td>86.2</td>
<td>63.0</td>
<td>53.3</td>
</tr>
<tr>
<td>r</td>
<td>1.02</td>
<td>1.90</td>
<td>0.85</td>
</tr>
<tr>
<td>c, MPa</td>
<td>752.</td>
<td>463.</td>
<td>164.*</td>
</tr>
<tr>
<td>n</td>
<td>0.535</td>
<td>0.339</td>
<td>0.268*</td>
</tr>
<tr>
<td>c_e, MPa</td>
<td>772.</td>
<td>470.*</td>
<td>164.</td>
</tr>
<tr>
<td>ε_e</td>
<td>0.036</td>
<td>0.029*</td>
<td>0.0</td>
</tr>
<tr>
<td>n_e</td>
<td>0.625</td>
<td>0.382*</td>
<td>0.268</td>
</tr>
<tr>
<td>S, MPa</td>
<td>782.*</td>
<td>479.</td>
<td>156.</td>
</tr>
<tr>
<td>A</td>
<td>0.870*</td>
<td>0.708</td>
<td>0.643</td>
</tr>
<tr>
<td>B</td>
<td>1.90*</td>
<td>2.26</td>
<td>3.35</td>
</tr>
</tbody>
</table>

†: total elongation in 40mm.
*: most suitable.
c,n : Hollomon Eq. parameters, Eq. (10).
c_e,ε_e,n_e : Swift Eq. parameters, Eq. (9).
S,A,B : Voce Eq. parameters, Eq. (8).
The effective stress strain in plane-strain test \( (\sigma^{m}) \) is obtained by the reverse calculation from the uniaxial stress-strain curve. The average axial stress in plane-strain test is calculated after subtracting the estimated load supported by the edge regions from the total recorded load. The remaining load is divided by the central region's cross-sectional area to obtain the average axial stress. This average axial stress is converted to the effective stress in plane-strain test \( (\sigma^{0}) \) by assuming that the entire center section is sufficiently close to plane-strain condition.

2.4 m-evaluation Method

To obtain the effective stress \( (\sigma^{m}) \) and strain \( (\varepsilon^{m}) \) based on the non-polynomial yield function, Eqs. (4) and (5), from the axial stress and strain of plane-strain test, it is necessary to introduce the multiplication factors, \( f_{1}^{m} \) and \( f_{2}^{m} \), respectively. These factors are given by

\[
f_{1}^{m} = \frac{\sigma^{m}}{\sigma_{1}} = \frac{2\kappa}{(1+x)(1+r)^{y}} \tag{6}
\]

\[
f_{2}^{m} = \frac{\varepsilon^{m}}{\varepsilon_{1}} = \frac{2\kappa}{(1+x)(1+r)^{y}} \tag{7}
\]

where, \( \beta = 1/(m-1) \), \( \gamma = (m-1)/m \), \( \kappa = (1+2r)^{y} \).

A point on the plane-strain \( \sigma^{m}, \varepsilon^{m} \) curve can be translated to \( \sigma^{u}, \varepsilon^{u} \) curve by varying \( m \)-value because the multiplication factor of \( f_{1}^{m} \) and \( f_{2}^{m} \) are varied with respect to change in \( m \)-value. Evaluation of \( m \)-value is achieved by the two types of comparisons of \( \sigma - \varepsilon \) curve for uniaxial tension with \( \sigma^{m}, \varepsilon^{m} \) curve for plane-strain tension as follows.

1. The relation between \( m \)-value and strain is obtained so that the current value of \( m \) is found by coincidence uniaxial \( \sigma - \varepsilon \) curve with plane-strain \( \sigma^{m}, \varepsilon^{m} \) curve whose coefficients depend on all the previous solution (Wagoner's rigorous analysis[14]).

2. Constant \( m \)-value is obtained where the best correlation between uniaxial \( \sigma - \varepsilon \) curve and plane-strain \( \sigma^{m}, \varepsilon^{m} \) curve can be achieved.

3. Results and Discussion

3.1 Plane-strain Tensile Test

Figure 2 shows the axial and transverse strain distributions for an RD specimen of 70/30 brass at two deformations during a plane-strain test. The figure represents the half-width at the notch to take advantage of symmetry. The arrows in the figure indicate the boundary of plane-strain region. The average strain ratio of the region assumed plane-strain condition in the strain range tested is -0.065, -0.060, and -0.035 for brass, copper, and aluminum, respectively, and the variation of the average strain ratio with respect to tensile directions in the sheet is little observed. The divergences of stress value caused by this shifts from plane-strain condition remains less than 3%.

Fig. 2 An example of the experimental axial and transverse strain distributions during a plane-strain tension.

Fig. 3 Comparison of effective stress-effective strain curves in uniaxial tension and in plane-strain tension using Hill's quadratic yield function.
Figure 3 exhibits comparisons of the effective stress—effective strain curves obtained by uniaxial test (solid line) and those in plane-strain test (broken line) based on Hill's quadratic yield function. The effective stress in plane-strain test is lower than the uniaxial stress at every strain level in all materials or tensile directions in the sheet. If the Hill's quadratic yield function was precise, these curves would coincide at all strains. The divergences indicate that different effective stress exists for different strain states. The parameters representing the work hardening behavior of materials are work hardening rate \(\frac{d\sigma}{d\varepsilon}\), normalized work hardening rate \(\frac{1}{\sigma} \frac{d\sigma}{d\varepsilon}\), and logarithmic work hardening rate \(\frac{d\ln\sigma}{d\ln\varepsilon}\). They are plotted against effective strain in Fig. 4. The variations of work hardening rates and normalized work hardening rates are both distributed with a profile decreasing toward zero with strain. While, within the statistical scatter, the distributions of logarithmic work hardening rates show remarkable difference among the materials tested, i.e., a profile taking maximum for brass, a profile increasing toward a certain value with strain for copper, and nearly constant for aluminum. Well-known empirical work hardening law equations which satisfy the above mentioned characteristics are Voce law equation \(\sigma = S(1 - \frac{A}{\exp(B\varepsilon)})\) (saturation law) for brass, Swift law equation \(\sigma = C_0 (\varepsilon + C_1)^n\) (modified power law) for copper, and Hollomon law equation \(\sigma = C_0\varepsilon^n\) (power law) for aluminum, respectively,

\[
\sigma = S(1 - \frac{A}{\exp(B\varepsilon)}) \\
\sigma = C_0 (\varepsilon + C_1)^n \\
\sigma = C_0\varepsilon^n
\]

where, \(S, A, B, C_0, C_1, n,\) and \(c\) are positive constants. Average of these coefficients from three directions in the sheet are all calculated statistically with considering the relation between logarithmic work hardening rate and strain, and are summarized in Table 1 and 2. The stress-strain relations of the best-fit work hardening law equations are plotted in Fig. 5 where the solid and single dotted lines indicate the uniaxial and plane-strain test, respectively. Plots of the corresponding work hardening behaviors calculated from the best-fit work hardening law equations are shown in Fig. 4 where the

| Table 2 Summary of data in plane-strain tension test. |
|-------------|------------|-------------|
|            | Brass      | Copper      | Aluminum   |
| \(c_{\text{max}}\) | 0.444      | 0.360       | 0.251       |
| \(c_1\) \(\,\text{MPa}\) | 657.       | 426.        | 155.        |
| \(c_2\) \(\,\text{MPa}\) | 0.833      | 0.025       | 0.283       |
| \(c_3\) \(\,\text{MPa}\) | 2.16       | 0.379       | —           |
| \(m_{\text{const}}\) | 2.56       | 2.91        | 3.20        |
| \(m_{1}\) \(\uparrow\) | 1.59       | 2.83        | 2.28        |
| \(m_{2}\) \(\uparrow\) | 2.31       | 0.005       | 0.334       |
| \(m_{3}\) \(\uparrow\) | 0.610      | -1.39       | -0.521      |

*\(c_1=S, c_2=A, c_3=B\) in Eq. (8) for Brass, \(c_1=c_4, c_2=c_5, c_3=n\) in Eq. (9) for Copper, \(c_1=c, c_3=n\) in Eq. (10) for Aluminum. 

\(\uparrow\)parameters in Eq. (11).

Fig. 4 Comparison of work hardening rate, normalized work hardening rate, and logarithmic work hardening rate with effective strain.
solid and broken lines indicate the uniaxial and plane-strain test, respectively. Both agree well with the experimental results within the experimental scattering. It should be noted that there is little difference among the stress-strain curves described by three empirical work hardening law equations, except at low strains (<0.1), as shown by others, e.g., in Ref.18.

3.2 Parameter m

Figure 6 illustrates the variation of parameter m with respect to effective strain (solid line). The analysis and calculation procedure for finding m-value as a function of strain is presented in Sec.2.4. The fitness of the relation between m-value and strain is examined by using several types of empirical curves. The most suitable curve, shown by the broken line in the figure, is

$$m = m_1 + m_2 e^{m_3}$$

where $m_1$, $m_2$, and $m_3$ are constants and these values are given in Table 2. Excellent agreement between m-value analyzed and calculated using Eq.(11) is observed for copper and aluminum and a little difference is observed for brass. The m-value is increasing function with strain for brass, and this tendency agrees with previous estimation of m-value by Wagoner [19]. In contrast, those are decreasing function with strain for copper and aluminum. These qualitative differences does not depend on the difference of the work hardening law equations used, because it affects only the difference of the absolute value of m. Comparing the work hardening behaviors shown in Fig.4, the logarithmic work hardening rates for plane-strain tension are clearly lower than that for uniaxial tension in the strain range tested for brass and are higher for copper and aluminum. Then, the qualitative tendency of m-value with strain seems to depend on the relative values of logarithmic work hardening rates for plane-strain tension to those for uniaxial tension.

The broken line in Fig.5 shows the stress-strain curve for plane-strain tension calculated by using the constant value of m (shown in Table 2 as $m_{const}$) which are the best fitness with that for uniaxial tension. Figure shows in good agreement between the broken and solid lines for brass and in excellent agreement for copper and aluminum. From this results, the nessessity to calculate m-value as a function of strain is obscure.

3.3 Forming Limit in Plane-Strain Condition

Criteria of diffuse[1] and localized [2] necking in plane-strain condition coincide with each other and is given by the strain when deviation of yield criteria induced by strain increment under constant load coincides with increment of work hardening rates. This condition is analytically solved in case when quadratic
yield function (Eqs. (2) and (3)) is applied, and is expressed in terms of logarithmic work hardening rate

\[ d \ln \varepsilon_c = \frac{d \varepsilon_c}{\varepsilon_c} = \frac{1}{1 + r} \]

Then, the critical strain \( \varepsilon_c \) and \( \varepsilon_{c0} \) are

\[ \varepsilon_c = \frac{1 + r}{\sqrt{1 + 2r}} \frac{d \ln \varepsilon_c}{d \ln \varepsilon_c} \]

\[ \varepsilon_{c0} = \frac{1}{1 + r} \frac{d \ln \varepsilon}{d \ln \varepsilon} \]

\( \varepsilon_c \) can be reduced to, when stress-strain curve is well represented by Voce law equation,

\[ \varepsilon_c = \frac{1}{B} \ln \left( \frac{1 + r}{1 + 2r} B + 1 \right) \]

by Swift law equation,

\[ \varepsilon_c = \frac{1 + r}{\sqrt{1 + 2r}} \ln \varepsilon_c - \varepsilon_{c0} \]

by Hollomon law equation,

\[ \varepsilon_c = \frac{1 + r}{\sqrt{1 + 2r}} n \]

The comparison of Eqs. (15)-(17) reveals that it is inadequate to calculate the forming limit by using the ambiguous m-value obtained from post-uniform elongation.

If non-polynomial yield function (Eqs. (4) and (5)) is used, the limit strain \( \varepsilon_c \) and \( \varepsilon_{c0} \) are only numerically solved. Especially m-value is the function of strain as shown in Fig. 6, then it is required to calculate the limit strain by successive summation from \( \varepsilon = \varepsilon_0 = 0 \). To avoid the difficulty involved in the evaluation of the qualitative effect of m-value on limit strain, constant m-value is examined and the results are shown in Fig. 7. Figure 7 illustrates relation between logarithmic work hardening rate and effective strain. Solid lines, being independent on work hardening law equation, are plots of the variation of the logarithmic work hardening rate with limit effective strain for three different m-values (i.e., 1.4, 2.2, and 3.2) at r=1.0 and three different r-values (i.e., 0.7, 1.0, and 1.4) at m=2.0.

The logarithmic work hardening rate, in other words instantaneous n-value, coincide with \( \varepsilon_{c0} \) calculated by using non-polynomial yield function as well as quadratic yield function. The relation expressed by the broken, single dotted, and double dotted curves in Fig. 7 are directly obtained from the Voce, Swift, and Hollomon law equations, respectively, whose coefficients are presented in Table 1. The intersecting points of the solid line with other curves give theoretical \( \varepsilon_c \), \( \varepsilon_{c0} \), and instantaneous m-value and those are depended on yield criteria or work hardening law equation, in general. According to Hollomon law equation, only the \( \varepsilon_c \) depends on yield criteria and the \( \varepsilon_c \) decreases with increasing m-value. It is corresponding to the decrease of r-value as shown in the figure. According to Voce or Swift law equation, the dependence of theoretical \( \varepsilon_c \) on m-value is complicated, however, the \( \varepsilon_c \) depends strongly on yield criteria qualitatively.

Theoretical limit strains in plane-strain conditions are calculated for m-value as a function shown in Eq. (11), \( m_{const} \) shown in Table 2, and m=2.0 by using work hardening law of Eqs. (8)-(10). Relative error of theoretical \( \varepsilon_{c0} \) to experimental \( \varepsilon_{c0} \) is examined in order to evaluate the dependence of \( \varepsilon_{c0} \) on m-value and work hardening law equation. The histograms are presented in Fig. 8 where the value calculated from Swift law equation for aluminum is omitted because Swift and Hollomon law equations agree precisely each other. The relative differences among the each set of results calculated by three functions of parameter m are small so that the effect on m-value on theoretical \( \varepsilon_{c0} \) seems to be neglected. For brass

![Graph](image)

**Fig. 7** Effect of empirical work hardening law equation and parameter m on forming limit in plane-strain condition.

**Fig. 8** Comparison of relative error of theoretical \( \varepsilon_{c0} \) value to that of measured.
the theoretical $\varepsilon_{1c}$ are greater than experimental $\varepsilon_{1c}$ and the least error is attained in Voce law equation and is less than 10%. The dependence on empirical equation is little observed for copper, because all errors are less than ±10%. For aluminum the theoretical $\varepsilon_{1c}$ predicted by using Voce equation is overestimated and the error attained in Hollomon equation is less than 10%. It is qualitative agreement with the results reported here-tofore that the theoretical $\varepsilon_{1c}$ using Hollomon law equation is less than m-value for brass, nearly equal to m-value for copper and aluminum[5],[6],[9]. The theoretical $\varepsilon_{1c}$ calculated by using the best-fit work hardening law equations are in excellent agreement with experimental $\varepsilon_{1c}$. These results imply an importance of selection of empirical work hardening law equation for the forming limit prediction of thin sheet alloys, because the theory of forming limit is evaluated with considering not only stress-strain relation but also the increments of stress and strain.

4. Conclusions

(1) At every strain level, the effective stress for plane-strain tensile test based on Hill's quadratic yield function is lower than that for uniaxial test, in all materials and this means that Hill's quadratic yield function has strain state dependence.

(2) The parameter $m$ of Hill's non-polynomial yield function is calculated by applying Wagner's rigorous method and is confirmed to be the function of strain, however, it may be assumed constant because the difference between uniaxial stress-strain curve and plane-strain effective stress-strain curve calculated using constant m-value can be almost neglected.

(3) Theoretical $\varepsilon_{1c}$ of forming limit in plane-strain condition are almost independent of m-value and strongly dependent on relation between logarithmic work hardening rate and effective strain (i.e., empirical work hardening law equation). So, the use of best-fit empirical work hardening law equation is necessary to evaluate the theoretical forming limit.

(4) The best-fit work hardening law equation seems to be Voce law equation (satu-

References