Finite Amplitude Waves on an Elastic Plate Horizontally Separating Two Different Fluid Streams*

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Nonlinear waves on a thin elastic plate horizontally separating an upper fluid stream and a lower one are analyzed theoretically. The lower stream is denser. The elastic plate is governed by the equation for large deflections including the in-plane forces due to the longitudinal deformations of the plate. Two fluid streams are assumed to be incompressible and inviscid, but on the elastic plate the balance of the normal component of force is expressed by taking the viscous stress into account. Some progressive waves of finite amplitude are found on the elastic thin plate. The nonlinear elevation of the elastic plate is obtained up to the third order approximation in the wave amplitude. The unsteady amplitude of waves tend to a finite limit after a long time under the effect of the viscosity of the fluid.

Key Words : Nonlinear Stability, Elastic Plate, Finite Amplitude Wave, Successive Approximation, Equilibrium Amplitude

1. Introduction

Covering with a thin plate is generally useful as the device to keep off dust and retain warmth. In such cases, a wave motion is well known to be frequently generated by a wind. In this paper, the form of a wave motion generated on a thin plate by a fluid stream is theoretically found.

The problem of a wave generated by wind on the surface of water, that is, of the instability of Kelvin-Helmholtz type has been studied from long ago as one of the fundamental problems on the water waves.⁽¹⁾ The problem of this kind has been well investigated taking the nonlinear effects into account.⁽²⁾⁻⁽⁴⁾ In these investigations, however, the fluid is usually assumed to be ideal. The nonlinear theory is scarcely treated taking account of the viscous effect.^{(3),(5)} In such nonlinear theories, Landau's coefficient is obtained by using an expansion procedure of the wave amplitude, and the amplitude of the wave motion is found from the nonlinear evolution equation. Since the viscosity is neglected in such nonlinear theories, a solution with a constant amplitude at any time has not been obtained from that equation. That is, the amplitude of the wave motion is theoretically known to vary in time at a finite equilibrium amplitude.⁽²⁾ On the other hand, it is

well known from experiments that if the velocity of the flow considered is steady and not large, a wave motion arising in the flow has a constant amplitude. The inconsistency between the theory and the fact is clearly caused by neglecting the viscosity of the fluid. In the study of this kind, therefore, it would be important to take the dissipative effect into account. In the problem considered here, that is, in the study of a wave motion in a system consisting of the fluids and an elastic plate, the viscosity has been taken into account. It is, however, very difficult to take account of the viscosity in the nonlinear theories of such problems. Only a wave motion on the free surface of a thin liquid layer has been studied with difficulty by the nonlinear theory taking account of the viscosity.^{(3),(5)}

Now, it is usual in the motion of an elastic plate that only the displacement in the direction normal to the surfaces of the plate is examined and that in the direction tangent to the plate is neglected. That is, only the fluid forces in the direction normal to the surfaces of the plate are necessary to derive the equation of motion of the elastic plate. Thus, a method which yields good results in the problems of solving the deformation of the interface between two fluids is applied in this paper. That is, it is assumed that the fluid flow is regarded as a flow of an ideal fluid and on the boundary of the fluid only the viscous stress corresponding to the velocity distribution based on the flow of the ideal fluid is acting.⁽⁸⁾ A wave motion on a thin elastic plate horizontally separating such two fluid streams that the upper layer is lighter than the lower one is analyzed using this method. The system considered here is stable if both the upper and the lower layers are at rest.

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The equation governing the amplitude of the wave motion is derived up to the third order of the magnitude of the amplitude, and the growing process from a linear wave to a finite amplitude wave is examined in the linearly unstable region. Especially, the transition from the infinitesimal wave to the nonlinear wave with an equilibrium amplitude is studied. The equation of motion of the plate with a large deflection taking account of the in-plane forces caused by the displacement in the own plane is applied in this analysis. It should be noticed that the Karman equation governing the thin plate, which is usually applied in the case of a large deflection of the plate, has not been used in this paper, because it does not permit a solution in the form of a progressive wave.⁽⁹⁾

2. Governing Equations

Consider two fluid layers flowing horizontally in any direction at any velocities as in the case of water in a reservoir subjected to wind. Suppose that these layers are separated from each other with a thin elastic plate. In this case, under what condition a wave motion arises and through what process an infinitesimal wave grows into a finite amplitude wave are theoretically examined in the following. The separating plate is assumed to be made of a light material as rubber and plastic. Then the wavelength of a wave motion arising on such a plate is generally not very long. Therefore, both the horizontal extent and the vertical one are assumed to be infinite. The flow of the fluid layers is regarded as an incompressible and irrotational flow. This assumption is suitable for the analysis of the problem on the wave motion. It should be, however, noticed that the effect of the viscosity is taken into account only in the expression of the force balance in the direction normal to the plate. That is, the normal stress corresponding to the velocity distribution caused by the irrotational flow is assumed to act on the plate. This assumption is known to yield good results about the damping effect on the water wave⁽¹¹⁾ and the deformation of the bubble.⁽⁸⁾ The plate separating the two fluid layers is a thin elastic plate. It is governed by the equation of the large deflection taking account of the in-plane force induced with a transverse displacement. This equation is usually applied to the problem of the panel flutter.⁽¹⁰⁾ The Cartesian coordinates are taken with x -axis along the neutral surface of the plate in the undisturbed state and y -axis upwards normal to it as shown in Fig.1. The motion in this system is assumed to be two-dimensional. The variables with respect to the lower layer and the upper one are identified with the subscripts 1 and 2 respectively. The reference scales are introduced as follows: the mean velocity $(U_1 + U_2)/2 (\equiv V)$ for the velocity, $V^2/g(\equiv l)$ for the length, l/V for the time t and V/l for the velocity

Fig.1 Schematic diagram and coordinates

potential, where, U_1 and U_2 are the mean velocities of the lower and upper layers, and g is the gravitational acceleration. The equations and the boundary conditions governing the velocity potentials ϕ_1 , ϕ_2 and the transverse displacement of the plate η caused by the disturbances are written in the nondimensional form as

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0, \quad (1)$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0, \quad (2)$$

$$y \rightarrow -\infty: \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_1}{\partial y} = 0, \quad (3)$$

$$y \rightarrow \infty: \frac{\partial \phi_2}{\partial x} = \frac{\partial \phi_2}{\partial y} = 0, \quad (4)$$

$$y=0: -\frac{\partial \phi_1}{\partial y} + \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} = Q \frac{\partial \phi_1}{\partial y} - \frac{\partial \eta}{\partial x} (1+Q) \frac{\partial \phi_1}{\partial x}, \quad (5)$$

$$\frac{\partial \phi_2}{\partial y} - \frac{\partial \eta}{\partial t} - U_2 \frac{\partial \eta}{\partial x} = -Q \frac{\partial \phi_2}{\partial y} + \frac{\partial \eta}{\partial x} (1+Q) \frac{\partial \phi_2}{\partial x}, \quad (6)$$

$$\begin{aligned} & \frac{1}{2}(1+\epsilon) \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) \\ & - \frac{1}{2}(1-\epsilon) \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right) \\ & + \epsilon \eta + D \frac{\partial^4 \eta}{\partial x^4} - N \frac{\partial^2 \eta}{\partial x^2} + m \frac{\partial^2 \eta}{\partial t^2} \\ & = \frac{n}{\lambda} \int_{-\infty}^{+\infty} \left(\frac{\partial \eta}{\partial x} \right)^2 dx \frac{\partial^2 \eta}{\partial x^2} \\ & - \frac{1+\epsilon}{2} \left[Q \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) \right. \\ & \left. + \frac{1}{2}(1+Q) \left(\left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial y} \right)^2 \right) \right] \\ & + \frac{1-\epsilon}{2} \left[Q \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right) \right. \\ & \left. + \frac{1}{2}(1+Q) \left(\left(\frac{\partial \phi_2}{\partial x} \right)^2 + \left(\frac{\partial \phi_2}{\partial y} \right)^2 \right) \right] \\ & - \frac{2(1+Q)}{1+(\partial \eta / \partial x)^2} \left[\frac{\mu_1}{R_*} \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \phi_1}{\partial x^2} \right. \\ & \left. - 2 \frac{\partial \eta}{\partial x} \frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_1}{\partial y^2} \right] - \frac{\mu_2}{R_*} \left(\frac{\partial \eta}{\partial x} \right)^2 \\ & \times \left[\frac{\partial^2 \phi_2}{\partial x^2} - 2 \frac{\partial \eta}{\partial x} \frac{\partial^2 \phi_2}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial y^2} \right], \quad (7) \end{aligned}$$

where,

$$Q = \eta \frac{\partial}{\partial x} + \frac{\eta^2}{2!} \frac{\partial^2}{\partial x^2} + \frac{\eta^3}{3!} \frac{\partial^3}{\partial x^3} + \dots \quad (8)$$

Here, the following nondimensional

parameters are introduced,

$$\begin{aligned}\epsilon &= \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1 + \bar{\rho}_2}, \quad R_e = \frac{2(\bar{\rho}_1 + \bar{\rho}_2) V l}{\bar{\mu}_1 + \bar{\mu}_2}, \\ \mu_1 &= \frac{2\bar{\mu}_1}{\bar{\mu}_1 + \bar{\mu}_2}, \quad \mu_2 = \frac{2\bar{\mu}_2}{\bar{\mu}_1 + \bar{\mu}_2}, \\ D &= \frac{\bar{D}}{(\bar{\rho}_1 + \bar{\rho}_2) V^2 l^3}, \quad N = \frac{\bar{N}}{(\bar{\rho}_1 + \bar{\rho}_2) V^2 l}, \\ m &= \frac{\bar{m}}{(\bar{\rho}_1 + \bar{\rho}_2) l}, \quad n = \frac{\bar{E} \bar{h}}{2(\bar{\rho}_1 + \bar{\rho}_2) V^2 l},\end{aligned}$$

where, $\bar{\rho}_1$ and $\bar{\rho}_2$ are the densities of the fluids, $\bar{\mu}_1$ and $\bar{\mu}_2$ the viscosities of the fluids, $\bar{D}$ the stiffness, $\bar{N}$ the in-plane force given externally, $\bar{m}$ the surface density, $\bar{E}$ Young's modulus, $\bar{h}$ the thickness of the plate and λ the wavelength of the disturbance. In the above equations, Eqs.(1) and (2) are equations of the irrotational motion, Eqs.(3) and (4) are boundary conditions that the disturbances diminish at infinity, Eqs. (5) and (6) kinematic conditions transforming into those applied at $y=0$ that the plate moves together with the fluids and Eq.(7) the equation of motion of the plate taking account of the fluid forces due to the pressure and the viscous stress and applied at $y=0$. If the equation of motion of the plate Eq.(7) is replaced with the equation of the force balance in the direction normal to the interface, it is reduced to an equation governing the nonlinear instability of the Kelvin-Helmholtz type.⁽²⁾ The governing equations in this analysis are, however, quite different from those governing the Kelvin-Helmholtz instability, because the equation governing the plate is introduced and the viscous effect is taken into account. Eqs.(1) to (7) are solved taking account of the nonlinear terms in the following sections.

3. Critical Condition that a Wave Motion First Arises

If, in the differential system of Eqs.(1) to (7), all terms in the right-hand side are set equal to zero, a homogeneous differential system is derived. The neutral stability curve separating the stable region from the unstable one is found from its system as an eigen-value problem. If the wave number and the complex growth rate are denoted with α and s respectively, the solutions of Eqs.(1) to (4) can be written as

$$\phi_1 = a_1 e^{i\alpha x} e^{s y} e^{i t}, \quad \dots \dots \dots (9)$$

$$\phi_2 = a_2 e^{i\alpha x} e^{-s y} e^{i t}, \quad \dots \dots \dots (10)$$

Expressing the displacement of the plate η as

$$\eta = \bar{\eta} e^{i\alpha x} e^{i t}, \quad \dots \dots \dots (11)$$

substituting it into Eqs.(5) to (7), and neglecting the right-hand side of these equations, the following eigen-relation is found:

$$\begin{aligned}s &= [-i\alpha\{(1+\epsilon)U_1 + (1-\epsilon)U_2\} \\ &\quad \pm (\alpha^2(1-\epsilon^2)(U_1 - U_2)^2 \\ &\quad + 2\alpha^3 m\{(1+\epsilon)U_1^2 + (1-\epsilon)U_2^2\} \\ &\quad - 4Ma(\epsilon + \alpha^2 N + \alpha^4 D)^{1/2})^{1/2}]/2M, \quad \dots \dots \dots (12)\end{aligned}$$

where,

$$M = 1 + m\alpha.$$

Here, a_1 , a_2 and $\bar{\eta}$ in Eqs. (9) to (11) are constants. Whether or not a disturbance grows is determined according to the sign of the real part of s . Therefore, the critical condition written in terms of the density difference ϵ is found as

$$\begin{aligned}\epsilon_c &= [(a^2 m(U_1^2 - U_2^2) - 2M) \\ &\quad + ((a^2 m(U_1^2 - U_2^2) - 2M)^2 \\ &\quad + (U_1 - U_2)^2(a^2(U_1 - U_2)^2 \\ &\quad + 2a^3 m(U_1^2 + U_2^2) - 4Ma(a^2 N \\ &\quad + a^4 D)^{1/2})/a(U_1 - U_2)^2]^{1/2} \quad \dots \dots \dots (13)\end{aligned}$$

The corresponding phase velocity v_c is found from the imaginary part of s to be

$$v_c = ((1+\epsilon_c)U_1 + (1-\epsilon_c)U_2)/2M, \quad \dots \dots \dots (14)$$

The system is stable for the case of $\epsilon > \epsilon_c$ and unstable for the case of $\epsilon < \epsilon_c$. In this paper, the system is assumed to be stable if both fluids are at rest. Then ϵ is supposed to be always positive.

4. Governing Equations in Moving Coordinates

If the density difference ϵ is smaller than the critical one ϵ_c obtained in the section 3, a disturbance grows and it travels in the x direction with the phase velocity v . Then, the coordinates moving in the x direction with the velocity v are used for the discussion of the growing process of the disturbance. That is, replacing x with $x - vt$ and $\partial/\partial t$ with $\partial/\partial t - v\partial/\partial x$ respectively, the conditions (5) to (7) at $y=0$ are written as

$$y=0: Lx = N, \quad \dots \dots \dots (15)$$

where, L is a 3×3 matrix constituted of linear differential operators of which the coefficients are

$$a_{11} = -\frac{\partial}{\partial y}, \quad a_{12} = 0, \quad a_{13} = (U_1 - v_c)\frac{\partial}{\partial x},$$

$$a_{21} = 0, \quad a_{22} = \frac{\partial}{\partial y}, \quad a_{23} = -(U_2 - v_c)\frac{\partial}{\partial x},$$

$$a_{31} = (1+\epsilon_c)(U_1 - v_c)\frac{\partial}{\partial x},$$

$$a_{32} = -(1-\epsilon_c)(U_2 - v_c)\frac{\partial}{\partial x},$$

$$a_{33} = 2\left(\epsilon_c + D\frac{\partial^4}{\partial x^4} - N\frac{\partial^2}{\partial x^2} + mv_c^2\frac{\partial^2}{\partial x^2}\right).$$

x is a column vector with the elements ϕ_1 , ϕ_2 and η , and N is a column vector with the nonlinear elements as

$$\begin{aligned}(N)_1 &= -\frac{\partial \eta}{\partial t} + (v - v_c) \frac{\partial \eta}{\partial x} + Q \frac{\partial \phi_1}{\partial y} \\ &\quad - \frac{\partial \eta}{\partial x} (1 + Q) \frac{\partial \phi_1}{\partial x}, \\ (N)_2 &= \frac{\partial \eta}{\partial t} - (v - v_c) \frac{\partial \eta}{\partial x} - Q \frac{\partial \phi_2}{\partial y} \\ &\quad + \frac{\partial \eta}{\partial x} (1 + Q) \frac{\partial \phi_2}{\partial x}, \\ (N)_3 &= \frac{2\pi}{\lambda} \left(\int_x^{x+\lambda} \left(\frac{\partial \eta}{\partial x} \right)^2 dx \right) \frac{\partial^2 \eta}{\partial x^2} \\ &\quad - 2(\varepsilon - \varepsilon_c) \eta - 2m \left\{ \frac{\partial^2 \eta}{\partial t^2} - 2v \frac{\partial^2 \eta}{\partial t \partial x} \right. \\ &\quad \left. + (v^2 - v_c^2) \frac{\partial^2 \eta}{\partial x^2} \right\} - (1 + Q) \left\{ (1 \right. \\ &\quad \left. + \varepsilon) \frac{\partial \phi_1}{\partial t} - (1 - \varepsilon) \frac{\partial \phi_2}{\partial t} \right\} \\ &\quad - Q \left\{ (1 + \varepsilon_c)(U_1 - v_c) \frac{\partial \phi_1}{\partial x} \right. \\ &\quad \left. - (1 - \varepsilon_c)(U_2 - v_c) \frac{\partial \phi_2}{\partial x} \right\} - \frac{1}{2} (1 + Q) \\ &\quad \times \left[(1 + \varepsilon) \left\{ \left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial y} \right)^2 \right\} \right. \\ &\quad \left. - (1 - \varepsilon) \left\{ \left(\frac{\partial \phi_2}{\partial x} \right)^2 + \left(\frac{\partial \phi_2}{\partial y} \right)^2 \right\} \right] \\ &\quad + ((1 + \varepsilon_c)(v - v_c) - (\varepsilon - \varepsilon_c)(U_1 - v_c) \\ &\quad + (\varepsilon - \varepsilon_c)(v - v_c))(1 + Q) \frac{\partial \phi_1}{\partial x} \\ &\quad + \{ -(1 - \varepsilon_c)(v - v_c) - (\varepsilon - \varepsilon_c)(U_2 \\ &\quad - v_c) + (\varepsilon - \varepsilon_c)(v - v_c) \} (1 \\ &\quad + Q) \frac{\partial \phi_2}{\partial x} - \frac{4}{1 + (\partial \eta / \partial x)^2} \frac{1 + Q}{R_s} \\ &\quad \times \left[\mu_1 \left\{ \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \phi_1}{\partial x^2} - 2 \frac{\partial \eta}{\partial x} \frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_1}{\partial y^2} \right\} \right. \\ &\quad \left. - \mu_2 \left\{ \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \phi_2}{\partial x^2} - 2 \frac{\partial \eta}{\partial x} \frac{\partial^2 \phi_2}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial y^2} \right\} \right].\end{aligned}$$

Substituting the formal solutions satisfying Eqs.(1) to (4) into Eq.(15) and neglecting the right-hand side of itself, the steady solution corresponding to the neutral stable state, which is determined in the section 3 setting the real part of s equal to zero, is easily found. The solution of Eq.(15) taking account of nonlinear terms is found in the form of an expansion series with respect to the amplitude $A(t)$ of a disturbance of which the leading term is equal to the steady linear solution obtained above. The behavior of the disturbance with a small but finite amplitude is discussed in the following.

5. Adjoint Equation and its Solution

As mentioned in the section 4, how the steady linear solution is corrected by taking account of the nonlinear terms in the right-hand side of Eq.(15) is a problem to be solved. In the course of such a nonlinear analysis, an additional condition called the solvability condition is necessary to solve the problem considered. Then, the inner product is defined and an adjoint operator is derived in the following.

Now, the inner product of a column vector x with element ϕ_1 , ϕ_2 , η and a column vector x' with element ϕ'_1 , ϕ'_2 , η' is assumed to be defined as

$$(x, x') = \frac{1}{\lambda} \int_x^{x+\lambda} (\phi_1 \phi'_1 + \phi_2 \phi'_2 + \eta \eta') dx \quad \dots\dots\dots (16)$$

Using this definition of the inner product, the adjoint operator L^* corresponding to L is defined as

$$(x', Lx) = (L^* x', x) \quad \dots\dots\dots (17)$$

and the elements a_{ij}^* of L^* are found as follows:

$$\begin{aligned}a_{11}^* &= -\frac{\partial}{\partial y}, \quad a_{12}^* = 0, \quad a_{13}^* = -(U_1 - v_c) \frac{\partial}{\partial x}, \\ a_{21}^* &= 0, \quad a_{22}^* = \frac{\partial}{\partial y}, \quad a_{23}^* = (U_2 - v_c) \frac{\partial}{\partial x}, \\ a_{31}^* &= -(1 + \varepsilon_c)(U_1 - v_c) \frac{\partial}{\partial x}, \\ a_{32}^* &= (1 - \varepsilon_c)(U_2 - v_c) \frac{\partial}{\partial x}, \\ a_{33}^* &= 2 \left(\varepsilon_c + D \frac{\partial^4}{\partial x^4} - N \frac{\partial^2}{\partial x^2} + m v_c^2 \frac{\partial^2}{\partial x^2} \right) \quad \dots\dots\dots (18)\end{aligned}$$

Thus, the adjoint equation $L^* x' = 0$ satisfying Eqs.(1) to (4) is found to have the following two solutions:

$$x^* = \begin{pmatrix} (1 + \varepsilon_c)(U_1 - v_c) e^{i\omega} \cos \alpha x \\ -(1 - \varepsilon_c)(U_2 - v_c) e^{-i\omega} \cos \alpha x \\ -\sin \alpha x \end{pmatrix} \quad \dots\dots (19)$$

$$x^{*'} = \begin{pmatrix} (1 + \varepsilon_c)(U_1 - v_c) e^{i\omega} \sin \alpha x \\ -(1 - \varepsilon_c)(U_2 - v_c) e^{-i\omega} \sin \alpha x \\ \cos \alpha x \end{pmatrix} \quad \dots\dots (20)$$

6. Equations Governing Amplitude

Consider the nonlinear effect of the right-hand side of Eq.(15) on the amplitude $A(t)$ of the fundamental wave as

$$x^{(1)} = \begin{pmatrix} (U_1 - v_c) e^{i\omega} \cos \alpha x \\ -(U_2 - v_c) e^{-i\omega} \cos \alpha x \\ \sin \alpha x \end{pmatrix} A(t) \quad \dots\dots (21)$$

which is the same as the eigen-function of the linearized equation of Eq.(15). In the following, under the assumption that the amplitude $A(t)$ is small, it is approximately found up to the third order of A .

Now, substituting the first approximate solution $x^{(1)}$ into the right-hand side of Eq.(15), and using the notations $\sin n\alpha x \equiv S_n$ and $\cos n\alpha x \equiv C_n$ ($n = 1, 2, 3$), the components of vector N , $(N^{(1)})_1$, $(N^{(1)})_2$, $(N^{(1)})_3$ are written as follows:

$$\begin{aligned}(N^{(1)})_1 &= -AS_1 + a(v - v_c)AC_1 \\ &\quad + a^2(U_1 - v_c)A^2S_2 \quad \dots\dots\dots (22)\end{aligned}$$

$$\begin{aligned}
(N^{(1)})_2 &= \dot{A}S_1 - \alpha(v - v_c)AC_1 \\
&+ \alpha^2(U_2 - v_c)A^2S_2, \dots\dots\dots(23) \\
(N^{(1)})_3 &= -2(\varepsilon - \varepsilon_c)\dot{A}S_1 + 2m\{2\alpha v \\
&\times \dot{A}C_1 + \alpha^2(v^2 - v_c^2)AS_1\} - (1 + \varepsilon)(U_1 \\
&- v_c)\dot{A}C_1 - (1 - \varepsilon)(U_2 - v_c)\dot{A}C_1 \\
&+ (\alpha^2/2)(1 + \varepsilon_c)(U_1 - v_c)^2A^2(1 \\
&- C_2) - (\alpha^2/2)(1 - \varepsilon_c)(U_2 - v_c)^2 \\
&\times A^2(1 - C_2) - (\alpha/2)\{(1 + \varepsilon_c)(v \\
&- v_c) - (\varepsilon - \varepsilon_c)(U_1 - v_c) + (\varepsilon - \varepsilon_c)(v \\
&- v_c)\}\{\alpha(U_1 - v_c)A^2(1 - C_2) \\
&+ 2(U_1 - v_c)AS_1\} - (\alpha/2)\{-(1 \\
&- \varepsilon_c)(v - v_c) - (\varepsilon - \varepsilon_c)(U_2 - v_c) \\
&+ (\varepsilon - \varepsilon_c)(v - v_c)\}\{\alpha(U_2 - v_c) \\
&\times A^2(1 - C_2) - 2(U_2 - v_c)AS_1\} - (\alpha^2 \\
&/2)(1 + \varepsilon)(U_1 - v_c)^2A^2 + (\alpha^2/2) \\
&\times (1 - \varepsilon)(U_2 - v_c)^2A^2 - (6\mu_1/R_e) \\
&\times \alpha^3(U_1 - v_c)A^2S_2 + (6\mu_2/R_e) \\
&\times \alpha^3(U_2 - v_c)A^2S_2, \dots\dots\dots(24)
\end{aligned}$$

The condition that the second approximate equation $Lx^{(2)} = N^{(2)}$ is solvable, of which the right-hand side is given with Eqs.(22) to (24), is expressed using Eqs.(19) and (20) as $(x^*, N^{(2)}) = 0$ and $(x^*, N^{(2)}) = 0$. The following expressions are derived from this condition:

$$(\varepsilon - \varepsilon_c)A = O(A^3), \dots\dots\dots(25)$$

$$(v - v_c)A = O(A^3), \dots\dots\dots(26)$$

$$\begin{aligned}
&\{(1 + \varepsilon_c)(U_1 - v_c) + (1 - \varepsilon_c)(U_2 - v_c) \\
&- 2amv\}A = O(A^3), \dots\dots\dots(27)
\end{aligned}$$

Taking account of Eqs.(25) to (27) and retaining only the terms of the second order of A in the right-hand side of the second approximate equations, the solution $x^{(2)}$ is found as

$$\begin{aligned}
(x^{(2)})_1 &= \frac{A}{\alpha}e^{i\alpha v}S_1 - \frac{\alpha}{2}(U_1 - v_c)(1 + B_1) \\
&\times A^2e^{2i\alpha v}S_2, \dots\dots\dots(28)
\end{aligned}$$

$$\begin{aligned}
(x^{(2)})_2 &= -\frac{\dot{A}}{\alpha}e^{-i\alpha v}S_1 - \frac{\alpha}{2}(U_2 - v_c)(1 - B_1) \\
&\times A^2e^{-2i\alpha v}S_2, \dots\dots\dots(29)
\end{aligned}$$

$$(x^{(2)})_3 = \frac{\alpha}{2}B_1A^2C_2, \dots\dots\dots(30)$$

where,

$$\begin{aligned}
B_1 &= \alpha\{(1 + \varepsilon_c)(U_1 - v_c)^2 - (1 - \varepsilon_c)(U_2 \\
&- v_c)^2\} / [2(\varepsilon_c + 16\alpha^4D + 4\alpha^2N \\
&- 4\alpha^2mv_c^2) - 2\alpha\{(1 + \varepsilon_c)(U_1 \\
&- v_c)^2 + (1 - \varepsilon_c)(U_2 - v_c)^2\}].
\end{aligned}$$

Next, substituting $x^{(1)} + x^{(2)}$ into the right-hand side of Eq.(15), the third approximate equations are written as

$$\begin{aligned}
(N^{(2)})_1 &= -\alpha(1 + B_1)A\dot{A}C_2 + \alpha(v \\
&- v_c)AC_1 - (\alpha^2/8)(1 + 6B_1)(U_1 \\
&- v_c)A^3C_1 + (9\alpha^3/8)(1 + 2B_1)(U_1 \\
&- v_c)A^3C_3, \dots\dots\dots(31)
\end{aligned}$$

$$\begin{aligned}
(N^{(2)})_2 &= -\alpha(1 - B_1)A\dot{A}C_2 - \alpha(v \\
&- v_c)AC_1 + (\alpha^2/8)(1 - 6B_1)(U_2 \\
&- v_c)A^3C_1 - (9\alpha^3/8)(1 - 2B_1)(U_2 \\
&- v_c)A^3C_3, \dots\dots\dots(32)
\end{aligned}$$

$$\begin{aligned}
(N^{(2)})_3 &= -\alpha^2nA^2S_1 - 2(\varepsilon - \varepsilon_c)AS_1 \\
&- (2/\alpha)\dot{A}S_1 - 2m\{\dot{A}S_1 + 4\alpha^2v \\
&\times B_1A\dot{A}S_2 - \alpha^2(v^2 - v_c^2)AS_1\} \\
&+ \alpha(1 + \varepsilon_c)(U_1 - v_c)B_1A\dot{A}S_2 + \alpha(1 \\
&- \varepsilon_c)(U_2 - v_c)B_1A\dot{A}S_2 - (\alpha^3/8)(1 \\
&+ \varepsilon_c)(U_1 - v_c)^2(5 + 2B_1)A^3S_1 - (\alpha^3 \\
&/8)(1 - \varepsilon_c)(U_2 - v_c)^2(5 - 2B_1)A^3S_1 \\
&+ (\alpha^3/8)(1 + \varepsilon_c)(U_1 - v_c)^2 \\
&\times (7 + 10B_1)A^3S_3 + (\alpha^3/8)(1 - \varepsilon_c) \\
&\times (U_2 - v_c)^2(7 - 10B_1)A^3S_3 \\
&- \alpha\{(1 + \varepsilon_c)(v - v_c) - (\varepsilon - \varepsilon_c)(U_1 \\
&- v_c)\}(U_1 - v_c)AS_1 - \alpha\{(1 - \varepsilon_c)(v \\
&- v_c) + (\varepsilon - \varepsilon_c)(U_2 - v_c)\}(U_2 - v_c) \\
&\times AS_1 - (2\mu_1/R_e)\{2\alpha\dot{A}S_1 - \alpha^2(U_1 \\
&- v_c)(1 + 4B_1)A^2S_2\} - (2\mu_2/R_e) \\
&\times \{2\alpha\dot{A}S_1 + \alpha^2(U_2 - v_c)(1 - 4B_1)A^2S_2\}, \\
&\dots\dots\dots(33)
\end{aligned}$$

Applying the solvability condition to the nonhomogeneous terms in the right-hand side of Eqs.(31) to (33), the equation governing the amplitude to be found is derived with the degree of accuracy up to the third order of A as

$$\dot{A} = \frac{\alpha}{2M}(aA + bA^3 + c\dot{A}), \dots\dots\dots(34)$$

where,

$$\begin{aligned}
a &= -2(\varepsilon - \varepsilon_c) + 2\alpha^2m(v^2 - v_c^2) \\
&- \alpha(v - v_c)\{(1 + \varepsilon_c)(U_1 - v_c) \\
&+ (1 - \varepsilon_c)(U_2 - v_c)\} - \alpha\{(1 + \varepsilon_c)(v \\
&- v_c) - (\varepsilon - \varepsilon_c)(U_1 - v_c)\}(U_1 - v_c) \\
&- \alpha\{(1 - \varepsilon_c)(v - v_c) + (\varepsilon - \varepsilon_c)(U_2 \\
&- v_c)\}(U_2 - v_c), \\
b &= -\alpha^2n - (\alpha^2/2)(1 - B_1)(1 + \varepsilon_c) \\
&\times (U_1 - v_c)^2 - (\alpha^2/2)(1 + B_1)(1 - \varepsilon_c) \\
&\times (U_2 - v_c)^2, \\
c &= -(4\alpha/R_e)(\mu_1 + \mu_2).
\end{aligned}$$

It is noticed that the phase velocity in the linear case remains unchanged in the degree of accuracy up to the third order of A as

$$v = \{(1 + \varepsilon)U_1 + (1 - \varepsilon)U_2\} / 2M, \dots\dots\dots(35)$$

When the inhomogeneous nonlinear equations, Eq.(15), with Eqs.(31) to (33) are integrated taking account of the solvability condition (34), the following third approximate solution is obtained:

$$\begin{aligned}
(x^{(3)})_1 &= \{-(v - v_c)A + (\alpha^2/8)(1 + 6B_1) \\
&\times (U_1 - v_c)A^3\}e^{i\alpha v}C_1 + \{(U_1 - v_c)(B_{21} \\
&\times A^2 + B_{22}A\dot{A}) + (1 + B_1)A\dot{A}/2\}e^{2i\alpha v}C_2 \\
&+ (U_1 - v_c)\{B_{23} - (3\alpha^2/8)(1 + 2B_1)\} \\
&\times A^3e^{3i\alpha v}C_3, \dots\dots\dots(36) \\
(x^{(3)})_2 &= \{(v - v_c)A - (\alpha^2/8)(1 - 6B_1) \\
&\times (U_2 - v_c)A^3\}e^{-i\alpha v}C_1 - \{(U_2 - v_c)
\end{aligned}$$

$$\begin{aligned}
& \times (B_{11}A^2 + B_{12}AA) - (1 - B_1)AA/2 \\
& \times e^{-2\alpha y} C_2 - (U_1 - v_c) \{B_{13} - (3\alpha^2/8) \\
& \times (1 - 2B_1)\} A^3 e^{-3\alpha y} C_3, \dots \dots \dots (37) \\
(x^{(3)})_3 = & (B_{11}A^2 + B_{12}AA)S_2 \\
& + B_{13}A^3 S_3, \dots \dots \dots (38)
\end{aligned}$$

where,

$$\begin{aligned}
B_{11} = & \alpha^2 \{ (\mu_1/R_e)(1 + 4B_1)(U_1 - v_c) \\
& - (\mu_2/R_e)(1 - 4B_1)(U_2 - v_c) \} \\
& / [2(\epsilon_c + 16\alpha^4 D + 4\alpha^2 N - 4\alpha^2 m v_c^2) \\
& - 2\alpha \{ (1 + \epsilon_c)(U_1 - v_c)^2 + (1 - \epsilon_c)(U_2 - v_c)^2 \}], \\
B_{12} = & \alpha \{ (1 + \epsilon_c)(1 + 2B_1)(U_1 - v_c) \\
& - (1 - \epsilon_c)(1 - 2B_1)(U_2 - v_c) - 8\alpha m \\
& \times v B_1 \} / [2(\epsilon_c + 16\alpha^4 D + 4\alpha^2 N \\
& - 4\alpha^2 m v_c^2) - 2\alpha \{ (1 + \epsilon_c)(U_1 - v_c)^2 \\
& + (1 - \epsilon_c)(U_2 - v_c)^2 \}], \\
B_{13} = & -(\alpha^3/4) \{ (1 + \epsilon_c)(1 + 4B_1)(U_1 \\
& - v_c)^2 + (1 - \epsilon_c)(1 - 4B_1)(U_2 - v_c)^2 \} \\
& / [2(\epsilon_c + 81\alpha^4 D + 9\alpha^2 N - 9\alpha^2 m \\
& \times v_c^2) - 3\alpha \{ (1 + \epsilon_c)(U_1 - v_c)^2 + (1 \\
& - \epsilon_c)(U_2 - v_c)^2 \}].
\end{aligned}$$

If the amplitude $A(t)$ is determined from Eq.(34), the motions of the fluids and the plate are given up to the third order of A with $x = x^{(1)} + x^{(2)} + x^{(3)}$. In particular, the deformation of the plate is obtained from Eqs.(21), (30) and (38) in the degree of accuracy mentioned above as

$$\eta = AS_1 + (\alpha/2)B_1A^2C_2 + (B_{11}A^2 + B_{12}AA)S_2 + B_{13}A^3S_3 + O(A^4), \dots \dots \dots (39)$$

7. Amplitude of a Finite Nonlinear Wave

The amplitude of a nonlinear wave arising in the system considered is governed by Eq.(34). Then, integrating this equation with respect to time, the amplitude A can be found. But, the governing equations contain numerous parameters involving the fluids and the plate. Then, it is assumed that the elastic plate is made of rubber, the lower fluid layer is at rest and both fluids of two layers are liquid. Under such conditions, the neutral stability curves in the ϵ - α plane are shown in Fig.2 (a) taking the thickness of the plate a parameter. In order to isolate the parameter involving the thickness of the plate from the other parameters, Young's modulus $\bar{E}$, the density $\bar{\rho}$, and thickness $\bar{h}$ are again made dimensionless as follows: $\bar{E}/(\rho_1 + \rho_2)V^2 \equiv E$, $\bar{\rho}/(\rho_1 + \rho_2) \equiv \rho$, and $\bar{h}/l \equiv h$. Poisson's ratio σ is introduced. In this figure, the system becomes unstable in the region bounded by the neutral stability curves and the α -axis, and stable in another region. If the lower liquid is water and the upper one is petroleum, the density difference becomes $\epsilon = 0.111$. In such interval of the

wave number that a straight line parallel to the α -axis is cut with a neutral stability curve, a disturbance is likely to develop. As the thickness of the plate increases, the unstable region in the wave number decreases and the wavelength becomes long. In Fig.2 (a), the case of $h = 0.2$ corresponds to the case of the velocity of petroleum 20.6 cm/s and the thickness of the plate 1.13 mm. In the unstable region of this case, the amplitude of the disturbance of the wave number $\alpha = 0.15$ is found by integrating Eq.(34) numerically under the initial condition, $t = 0$: $A = 0.001$, $\dot{A} = 0$. As a result, the amplitude is shown in Fig.2 (b) as a function of t and the relation of A and $\dot{A}$ is shown as Fig.2 (c) in the phase space. The growing process is described from Figs.2 (a) to (c) as follows: In the unstable region, the coefficient a in the right-hand side of Eq.(34) becomes positive, then the disturbance initially grows. However, the growth rate of the disturbance is depressed by the nonlinear term with a negative coefficient b and the amplitude becomes periodic in time at the equilibrium amplitude A_e . In the course of time, it tends to an equilibrium value decreasing the amplitude of the part of the time variation in the wave amplitude. The damping coefficient of the amplitude of the time variation is found from Eq.(34) to be proportional to the viscosity and to the square of the wavelength. The equilibrium amplitude is found from Eq.(34) setting $\dot{A}$ and $\ddot{A}$ equal to zero as

$$A_e = \sqrt{-a/b}, \dots \dots \dots (40)$$

It is shown in Fig.2 (d) over the unstable interval in the ϵ - α plane for the case of $\epsilon = 0.111$. It is found from this figure that a finite equilibrium amplitude A_e can be always obtained in the unstable interval of the wave number. It should be noticed that the wave number making the equilibrium amplitude maximum is not at the center of the unstable interval, but it is near the neutral stability state in the side of long wavelength. The wave number corresponding to the maximum equilibrium amplitude is almost independent of the plate thickness. Fig.2 (e) shows the effect of the in-plane force in the case of a rubber plate separating the upper oil layer from the lower water one. The interval of the wave number for the unstable disturbance and its amplitude decrease as the in-plane force increases. The wave number making the amplitude maximum increases slightly with an increasing in-plane force. However, the practical effect of in-plane force on the wave number having the maximum amplitude can be ignored. Fig.2 (f) shows the effect of the velocity difference $\Delta U = U_1 - U_2$. With a decreasing velocity difference the interval of the wave number for the unstable disturbance and the equilibrium amplitude decrease. The wave number having the maximum equilibrium amplitude

increases as the velocity difference decreases. The smaller the wavelength becomes, the larger the wave amplitude becomes. The equilibrium is found from Eq.(40) to be independent of the viscosity.

The amplitude is found from Fig.2 (b)

(a) Neutral stability curve

(b) Time variation in amplitude

(c) Amplitude behavior in phase plane

(d) Effect of plate thickness on equilibrium amplitude

to vary periodically with the equilibrium value after a long time. The time variation of the amplitude is examined in the following. The deviation of amplitude from the equilibrium value, $\Delta = A - A_e$, is found from Eq.(34) to oscillate in time with the frequency $\omega = \sqrt{\alpha a/M}$ and the damping coefficient $\beta = -\alpha C/4M$. These values are shown in Fig.2 (g) in the form of two kinds of the curves as a function of α . It is found from this figure that the frequency of the amplitude variation is maximum for the wavelength slightly longer than that at the center of the unstable region. Thus, a small disturbance arising in the unstable region grows into a nonlinear progressive wave with the finite amplitude periodic in time and the same phase velocity as that of the

(e) Effect of in-plane force on equilibrium amplitude

(f) Effect of velocity difference on equilibrium amplitude

(g) Frequency and damping coefficient of amplitude of time-dependent wave

Fig.2 Two liquid streams separated by a thin rubber plate
($E = 50000$, $\rho_r = 0.528$, $\sigma = 0.5$)

linear case. Its amplitude diminishes in the course of time and finally settles to an equilibrium value.

As an example of two layers consisting of a gas and a liquid, the equilibrium amplitude for $\varepsilon = 0.997$ corresponding to the air and water layers is shown in Fig.3. Since this case is more stable than the case of two liquid layers, the system becomes unstable only in the case of large gas stream and the unstable region shifts to the region of high wave numbers. However, the fundamental characteristics of the transition process can be supposed to hardly depend on whether the upper fluid layer is liquid or gas.

Fig.3 Equilibrium amplitude for case of gas-liquid layer

8. Conclusions

The behavior in the process of an infinitesimal wave growing into a finite amplitude wave arising on a thin elastic plate horizontally separating the upper light fluid stream from the lower heavy one is studied theoretically. The flows are treated as ideal fluid. But the viscous effect is taken into account on the surfaces of the plate. The elastic plate is assumed to be governed by the equation of motion for large deflection taking account of the in-plane force due to a transverse displacement. According to the result, (1) the equation governing the amplitude is derived by using coordinates moving with the same velocity as the phase velocity and the amplitude expansion procedure with the degree of accuracy up

to the third order of amplitude, (2) the phase velocity is the same as the linear case up to the third order of amplitude, (3) under the assumption that the material properties of the fluid and the plate are prescribed, there always exists a nonlinear wave with the finite equilibrium amplitude in the linearly unstable region, (4) the amplitude of the nonlinear wave is periodic in time, it oscillates near the equilibrium value and damps through the fluid viscosity, and finally the nonlinear wave settles to a steady progressive wave with a constant finite amplitude, (5) the maximum amplitude wave arises in the unstable wave number region near the neutral stability state in the long wave side, (6) the time variation in the amplitude of the unsteady nonlinear wave arises with the largest frequency in the unstable wave number region near the neutral stability state in the short wave side, and (7) the damping coefficient of the time variation in amplitude is proportional to the square of the wave number.

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