Boundary Element Method on the Basis of Oseen's Fundamental Solution*

by Takeo KAJISHIMA**, Susumu MURATA***, and Yutaka MIYAKE****

The problem of a steady laminar flow past an arbitrary body is formulated using the boundary element method. First, surface integral equation is derived for the Oseen flow, and then, extended to the Navier-Stokes equations. In the latter case, the repetition of solving the surface integral equation is needed. The nonlinear terms in the Navier-Stokes equations are regarded as the Oseenlets distributed in the flow field, and surface stresses are obtained numerically. As an example, the flow past a two-dimensional flat plate is solved. The numerical results are coincident with the experimental data for the Reynolds numbers smaller than 100. But the proposed method is not efficient for a high Reynolds number flow, because it is based on the Oseen solutions. This technique can be applied to a three dimensional flow past an arbitrary body, and expected to make up the Reynolds number region between the Oseen flow and the boundary layer approximation.

Key words : Fluid Mechanics, Numerical Analysis, Boundary Element Method, Navier-Stokes Equation, Laminar Flow

1. Introduction

A lot of efforts have been devoted to solving the Navier-Stokes equations as accurately as possible. When Reynolds number is so small that the governing equations can be reduced to linear equations, a variety of analytical treatments have been established. But the Reynolds number range in which the linear approximation can be applied is extremely narrow. On the other hand, when the Reynolds number is sufficiently high, the flow field can be solved satisfactorily, being treated separately in main flow region and in boundary layer region. But, in the intermediate range between the very low and the moderately high Reynolds numbers, numerical methods must be applied because analytical solutions are not available.

Because of the nonlinearity of the Navier-Stokes equations, finite difference methods and finite element methods seem to be mainly applied for the numerical technique. To solve the potential flow, singularity methods are widely applied together with finite element methods, and especially for the outer flow problems boundary element methods (i.e. panel methods) are exclusively employed. Boundary element method needs only surface integral with respect to unknown variables, and, from this viewpoint, it has an advantage over finite element method which treats

volume integral. In the case of a moderately low Reynolds number flow with weak nonlinearity, it is expected that numerical solution is obtained easily by applying the boundary element method iteratively.

The aim of the present study is to apply the boundary element method to analyze a viscous flow. We establish the formulation of the boundary element method for Oseen's linearized equations, and then extend it to solving the Navier-Stokes equations of motion, in this report. While it is possible to form the integral equations on the basis of Stokes fundamental solution, it is expected that Oseen approximation is more advantageous because it is applicable for higher Reynolds number range and makes rapid convergence of the iterative procedure. For the Stokes flow the formulation by so-called direct method is available⁽¹⁾, and for the Oseen flow the solution based on the indirect method is presented for the two-dimensional flow⁽²⁾. Nevertheless the formulation by the direct method of the boundary element method on the basis of Oseen's fundamental solution has not been presented yet.

As an example of application, we calculate the flow around a two-dimensional flat plate of finite length set parallel to a uniform stream. For this problem, the boundary layer approximation can be applied, in the high Reynolds number case, and furthermore even at so low a Reynolds number as $Re=10$, a reasonable solution can be obtained by applying Kuo's modification⁽³⁾ which corrects the effect of the growth of the boundary layer thickness. Moreover the singularity near the leading and trailing edges has been carefully examined⁽⁴⁾⁽⁵⁾, yielding the characteristics of the flow field near the edges in detail. In contrast to the method based on the boundary layer approximation,

* Received, 1st October, 1984.

** Graduate Student, Graduate School of Osaka University, (2-1 Yamada-Oka Suita Osaka Japan).

*** Professor, Toyota Technical Institute, (2-12-1 Hisakata Tenpaku-ku Nagoya Japan).

**** Professor, Dept. of Mech. Engng., Osaka University.

our method is to extend Oseen's solution to higher Reynolds number range. With respect to this problem the method of Dennis et al. (6) is known to be efficient in a wide Reynolds number range. But our method has an advantage of applicability to an arbitrary shape and simplicity of the procedure over the Dennis' method.

2. Integral Representation of the Velocity Vector

A steady flow of a viscous, incompressible fluid past an arbitrary body is dealt with, as shown in Fig.1.

2.1 Formulation on the basis of Oseen's equation

Oseen's linearized equations of motion for the steady flow is described by

$$U \frac{\partial v'_i}{\partial y_1} = -\frac{1}{\rho} \frac{\partial p}{\partial y_i} + \nu \frac{\partial^2 v'_i}{\partial y_j \partial y_j} + X_i, \quad (1)$$

where U is uniform flow in the direction 1; v'_i , velocity variance from U , (so $v_i = \delta_{i1}U + v'_i$); p , the static pressure and X_i is the body force. Introducing the stress tensor

$$\tau_{ij} = -\delta_{ij}p + \mu \left(\frac{\partial v'_i}{\partial y_j} + \frac{\partial v'_j}{\partial y_i} \right) \quad (2)$$

in eq.(1), we obtain

$$\frac{\partial}{\partial y_j} \left(\delta_{j1} U v'_i - \frac{1}{\rho} \tau_{ij} \right) = X_i. \quad (3)$$

When a concentrated force in the direction k is applied at the point y , the Oseen flow is written as

$$U \frac{\partial \lambda_{i,k}}{\partial x_1} = -\frac{1}{\rho} \frac{\partial q_k}{\partial x_i} + \nu \frac{\partial \lambda_{i,k}}{\partial x_j \partial x_j} + \frac{1}{\rho} \delta_{ik} \delta(x-y) \quad (4)$$

where $\lambda_{i,k}$ and q_k are velocity and pressure which represent Oseen's fundamental solution, and $\delta(x)$ is Dirac's impulse function. As before, using the stress tensor

$$T_{ij,k} = -\delta_{ij} q_k + \mu \left(\frac{\partial \lambda_{i,k}}{\partial x_j} + \frac{\partial \lambda_{j,k}}{\partial x_i} \right) \quad (5)$$

in eq.(4), the equation similar to eq.(3) may be written as

$$\frac{\partial}{\partial x_j} \left(\delta_{j1} U \lambda_{i,k} - \frac{1}{\rho} T_{ij,k} \right) = \frac{1}{\rho} \delta_{ik} \delta(x-y). \quad (6)$$

Multiplying eq.(3) by $\lambda_{i,k}(x-y)$ and eq.(6) by $v'_i(y)$, and subtracting, we get

$$\delta(x-y) v'_k(y) = \frac{\partial}{\partial y_i} \{ \lambda_{k,i}(x-y) \tau_{ij}(y) + T_{ij,k}(x-y) v'_i(y) \}$$

Fig. 1 Body in a uniform flow

$$-\delta_{j1} \rho U \lambda_{k,i}(x-y) v'_i(y) + \rho \lambda_{k,i}(x-y) X_i(y). \quad (7)$$

Integrating eq.(7) in the whole flow field Ω and using divergence theorem, we obtain the integral representation for the velocity vector as follows

$$v'_k(x) = \iint_{S_p} n_j(y) \{ \lambda_{k,i}(x-y) \tau_{ij}(y) + T_{ij,k}(x-y) v'_i(y) - \delta_{j1} \rho U \lambda_{k,i}(x-y) v'_i(y) \} dS_y + \rho \iiint_{\Omega} \lambda_{k,i}(x-y) X_i(y) dV_y, \quad (8)$$

where S_p means the surface of the body, and n_j indicates unit length vector inward normal to the body. Oseen's fundamental solutions $\lambda_{i,k}$, q_k and $T_{ij,k}$ are already known (7). Above equation yields the surface integral equations that satisfy the no-slip velocity boundary condition.

2.2 Extension to solving the Navier-Stokes equations

Next, the form (8) is extended to the case of the Navier-Stokes equations

$$U \frac{\partial v'_i}{\partial y_1} + v'_j \frac{\partial v'_i}{\partial y_j} = -\frac{1}{\rho} \frac{\partial p}{\partial y_i} + \nu \frac{\partial^2 v'_i}{\partial y_j \partial y_j} + X_i. \quad (9)$$

When the nonlinear terms

$$X'_i = -v'_j \frac{\partial v'_i}{\partial y_j} \quad (10)$$

in the left hand side of above equation are regarded as Oseenlets distributed densely in the flow field, they can be treated as a part of the body force X_i . Then, the volume integral term $X_i(y)$ in eq.(8) includes these terms. The solution of eq.(8) thus modified gives a flow based on the Navier-Stokes equations. Since X'_i are unknown variables, iterative procedure is indispensable.

3. Flow around a Two-dimensional Flat Plate

3.1 Surface integral equations

A flow past a two dimensional flat plate, as shown in Fig.2, is to be calculated. Boundary conditions on the surface are represented as

$$U + v'_1 = 0, \quad v'_2 = 0 \quad (11)$$

With the aid of eq.(11) and $n_1=0$ on the plate, the integral representation of the velocity (8) becomes

$$v'_k(x) = \oint n_2 \{ \lambda_{k,i}(x-s) \tau_{i2}(s) - U T_{12,k}(x-s) \} ds + \rho \iint_{-\infty}^{\infty} \lambda_{k,i}(x-y) X'_i(y) dy_1 dy_2, \quad (12)$$

where the body force X_i is neglected. The integration of the first term of right hand side is performed both on the upper and lower surfaces of the plate. The drag force per unit length and width at the point $(s, 0)$ on the plate is represented by $\rho m(s)$. Since $n_2 = -1, \tau_{12} = \rho m/2$ on the upper surface, and $n_2 = 1, \tau_{12} = -\rho m/2$ on the under surface, $n_2(\lambda_{k,2}\tau_{22} - U\tau_{12,k})$ vanishes at any point along the plate when the contributions from both surfaces are considered collectively, and then eq. (12) becomes

$$v'_k(x) = -\rho \int_{-1}^1 \lambda_{k,1}(x-s)m(s)ds + \rho \int \int_{-\infty}^{\infty} \lambda_{k,i}(x-y)X'_i(y)dy_1dy_2. \quad (13)$$

The first term of the right hand side of above equation means the velocity caused by the Oseenlets which correspond to the drag distribution of the plate. And the second term is the velocity caused by the nonlinear terms (eq. (10)) which are treated as Oseenlets distributing over the flow field.

In eq. (13) $\lambda_{k,i}$ means the fundamental solutions of two dimensional Oseen flow, which are represented in the form $\lambda_{k,i} = 2\pi\rho U \lambda_{k,i}$ as

$$\begin{aligned} \lambda_{1,1}^* &= -\frac{x_1}{r^2} + ke^{kr_1} \left\{ K_0(kr) + \frac{x_1}{r} K_1(kr) \right\}, \\ \lambda_{1,2}^* &= \lambda_{2,1}^* = \frac{x_2}{r} \left\{ -\frac{1}{r} + ke^{kr_1} K_1(kr) \right\}, \\ \lambda_{2,2}^* &= \frac{x_1}{r^2} + ke^{kr_1} \left\{ K_0(kr) - \frac{x_1}{r} K_1(kr) \right\} \end{aligned} \quad (14)$$

where $r = |x|$, and K_0, K_1 are modified Bessel functions of the second kind. k is defined as

$$k = \frac{U}{2\nu} \quad (15)$$

and is 1/4 of the Reynolds number defined by plate length L and uniform velocity U as

$$R = \frac{UL}{\nu}. \quad (16)$$

On the plate v'_k must satisfy the boundary condition (11). As the flow is symmetric about the x axis, $v'_2 = 0$ must be satisfied on the plate. Applying these boundary conditions, the following integral equation is given

$$\begin{aligned} \rho \int_{-1}^1 \lambda_{1,1}(x-s)m(s)ds &= U \\ + \rho \int \int_{-\infty}^{\infty} \lambda_{1,i}(x-y)X'_i(y)dy_1dy_2. \end{aligned} \quad (17)$$

Fig. 2 Plate and the coordinate system

We can obtain drag distribution $m(s)$ by solving the above equation. As X'_i are unknown at the beginning, it is inevitable to neglect the second term of the right hand side of eq. (17) at the start of the calculation. The solution obtained in such a way is for Oseen's approximation. We calculate velocity distribution v'_k by eq. (13) and X'_i by eq. (10). Using the newly obtained X'_i , eq. (17) is solved until the convergence is achieved.

3.2 Numerical method

Hereafter the symbols (x, y) , (u', v') , (X', Y') are used in place of x_i , v_i' , X'_i .

3.2.1 The flow caused by Oseenlets on the plate

The skin friction distribution $m(s)$ is expressed by the modified Glauert series as follows.

$$m(s) = A_{-1} \tan \frac{\theta}{2} + A_0 \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin(n\theta), \quad (18)$$

where s is transformed to the angular measure θ as $s = -\cos\theta$ ($0 < \theta < \pi$). Truncating the series at the N -th term, we obtain the velocity induced at the point $z = (x+iy)$ by the Oseenlets distributed along the plate as follows.

$$u'(z) - iv'(z) = \frac{1}{2\pi U} \sum_{n=1}^N G_n(z) A_n. \quad (19)$$

$$G_{-1}(z) = \int_{-1}^1 g(z-s) \tan \frac{\theta}{2} ds,$$

$$G_0(z) = \int_{-1}^1 g(z-s) \cot \frac{\theta}{2} ds,$$

$$G_n(z) = \int_{-1}^1 g(z-s) \sin(n\theta) ds, \quad (n=1, 2, \dots, N) \quad (20)$$

$$g(z) = \frac{1}{z} - ke^{kr} \left\{ K_0(kr) + \frac{r}{z} K_1(kr) \right\}, \quad (\text{where } r = |z|) \quad (21)$$

When z is on the flat plate $(x, 0)$, $g(z)$ is reduced to

$$h(x) = \frac{1}{x} - ke^{kx} \left\{ K_0(k|x|) + \text{sgn}(x) K_1(k|x|) \right\} \quad (22)$$

and is used to obtain the velocity u' in eq. (20). In such a case, the symbol $H_n(x)$ is used for the integral of eq. (20) instead of $G_n(z)$.

3.2.2 The flow by Oseenlets in the flow field

Surface integral of the second term of the right hand side of eq. (13) is performed by splitting the flow field into grids. As the flow is symmetric about the x axis, grids for integration have to be generated over the upper half ($y > 0$). We set M grids in total, and express the representative point of m -th grid by $\zeta_m (= \xi_m + i\eta_m)$ and the area by a_m . Representing the Oseenlets at the point ζ_m by (X'_m, Y'_m) , we get the velocity at z as

$$u'(z) - iv'(z) = \frac{1}{2\pi U} \sum_{m=1}^M \{ C_{1,m}(z) (X'_m - iY'_m) \}$$

$$+C_{2,n}(z)(X'_n+iY'_n)\}. \quad (23)$$

$$C_{1,n}(z) = a_n \left[ke^{k(x-\xi_n)} \{K_0(kr_1) + \frac{r_2}{z_2} K_1(kr_2)\} - \frac{1}{z_2} \right],$$

$$C_{2,n}(z) = a_n \left[ke^{k(x-\xi_n)} \{K_0(kr_2) + \frac{r_1}{z_1} K_1(kr_1)\} - \frac{1}{z_1} \right],$$

(where $z_1 = z - \zeta_n$, $r_1 = |z_1|$,
 $z_2 = z - \bar{\zeta}_n$, $r_2 = |z_2|$) (24)

On the plate $(x,0)$, the above equation becomes

$$u'(x) = \frac{1}{\pi U} \sum_{n=1}^N \{D_{1,n}(x)X'_n + D_{2,n}(x)Y'_n\}. \quad (25)$$

$$D_{1,n}(x) = a_n \left[ke^{k(x-\xi_n)} \{K_0(kr_3) + \frac{x-\xi_n}{r_3} K_1(kr_3)\} - \frac{x-\xi_n}{r_3^2} \right],$$

$$D_{2,n}(x) = a_n \frac{\eta_n}{r_3} \left[\frac{1}{r_3} - ke^{k(x-\xi_n)} K_1(kr_3) \right],$$

(where $r_3^2 = (x-\xi_n)^2 + \eta_n^2$) (26)

3.2.3 Calculation of (X', Y')

We need not only (u', v') given by eqs. (19) and (23) but also the derivatives of these terms to obtain (X', Y') . They are exactly obtained by using the derivative of

$g(z)$ and $C_{i,n}(z)$ in eqs. (21) (24) in place of themselves.

3.2.4 Determination of A_n

Setting control points on the plate at

$$\theta_k = \frac{k}{N+3}\pi, \quad k=1, 2, \dots, N+2, \quad (27)$$

the following simultaneous equations about A_n are derived by applying the integral equation (17) at each control point.

$$\sum_{n=1}^N H_n(x_k) A_n = -2\pi U^2 - 2 \sum_{n=1}^N \{D_{1,n}(x_k)X'_n + D_{2,n}(x_k)Y'_n\},$$

(k=1, 2, \dots, N+2) (28)

where U is set equal to unity. The total drag of the flat plate per unit width is written as

$$D = \rho \int_{-1}^1 m(s) ds, \quad (29)$$

so the drag coefficient is represented by

$$C_D \left(\equiv \frac{D}{2\pi U^2} \right) = \frac{\pi}{2} (A_{-1} + A_0 + \frac{A_1}{2}), \quad (30)$$

3.2.5 Calculation procedure

First the Oseen solution is obtained as a first approximation by solving the simultaneous eqs. (28) putting $X'=Y'=0$. On the basis of velocity distribution of the Oseen flow, the grids for the surface integral are generated around the plate. The velocity (u', v') and nonlinear term

Fig. 3 Part of mesh for surface integral

Fig. 4 Drag coefficient vs. Reynolds number

(X', Y') are calculated at each grid point, simultaneous equations (28) are solved using the newly obtained (X', Y') , and these stages are repeated until convergence is achieved.

3.3 Numerical results

Fig.3 is the grid near the plate generated for the Reynolds number $R=10$. Considering the singularity of the flow close to the edges of the plate, the scale of minimum grids is set smaller than R^{-1} at the leading edge and smaller than $R^{-0.75}$ at the trailing edge. For example, when Reynolds number $R=10$, 436 grids are produced in the area of $-5 < x < 500, y < 8$. When $N=18$, iteration counts needed for the drag coefficient to converge within 0.1% were 7 when $R=10$, and 10 when $R=100$.

Fig.4 shows the relation between the Reynolds number R and the drag coefficient C_D . So far as R is smaller than 100, the present results show good agreement with experimental data by Janour⁽⁸⁾ and numerical result by Dennis et al.⁽⁶⁾. And the solution for the low Reynolds number approaches the value of Oseen's flow. On

the other hand, in the case $R > 100$, the normalized value of C_D by the present method becomes independent of R , as $R^{1/2}C_D = 1.9$, contrary to the experiment. The cause of disagreement is that our method is formed on the basis of Oseen's fundamental solution. For example the behaviour of $\lambda_{2,2}^*$ along x axis is represented in Fig.5. Maximum value is at the origin independently of Reynolds number, but the variation of $\lambda_{2,2}^*$ near the origin becomes steep at high Reynolds number. This fact makes numerical integration inaccurate in high Reynolds number region and necessitates very dense grids. Moreover, for high Reynolds number flow, the instability of calculation makes our method disadvantageous.

Fig.6 is a distribution of the skin frictions along the plate. Singularity at the leading edge ($x=-1$) and the trailing edge ($x=1$) is represented by the first and the second terms of the modified Glauert series shown in eq.(18). For high Reynolds number case, the difference between the Oseen solution and Navier-Stokes solution becomes accentuated.

Fig. 5 Example of Oseen's fundamental solution

Fig. 6 Skin friction distribution along the plate

Fig. 7 Velocity distribution around the plate

Fig. 8 Comparison of velocity distributions

Fig.7 is a flow field around the plate where the distribution of u over y axis at $x=0, \pm 1, \pm 2$ is shown. In a low Reynolds number flow, as the area where the influence of the plate manifests itself becomes larger because of the viscous diffusion, grids must be generated over a long distance. But meshes may be coarse for a low Reynolds number flow. Fig.8 represents the velocity distribution along y axis at the central section of the plate ($x=0$). It shows the comparison of present result for Navier-Stokes equation with Oseen solution or Blasius' one. Gradient of the velocity curve for the plate of N-S solution is larger than that of Blasius' curve and smaller than that of Oseen solution, and this result corresponds to Fig.4. The cause of this fact is that transportation of vorticity in the x direction is overestimated in Oseen approximation, but, for the case of low Reynolds number, as the error by the overestimation is not serious, the Oseen approximation gives a solution similar to the N-S flow. Blasius solution is found to be no more a good approximation even at $R=100$.

4. Conclusions

We applied the boundary element method to solve full Navier-Stokes equations of motion, and obtained the results as follows:

(1) The surface integral equation for the Oseen flow was derived by the direct

method, and then extended to solving full Navier-Stokes equations of motion.

(2) As an example, we solved a flow around the two-dimensional flat plate in a uniform flow, and showed the usefulness of the present method.

(3) Serious disagreement with experimental results appears at high Reynolds number $R > 100$, because the behaviour of Oseen's fundamental solution becomes complicated and steep.

(4) Oseen solution approximates the actual flow around the plate more reasonably than Blasius solution, even at the Reynolds number $R=100$.

The authors are grateful to Mr. Hisashi Minami (Sharp Co., Ltd.) for his helpful contribution.

References

- (1) Youngren, G.K., and Acrivos, A., *J. Fluid Mech.* 69-2 (1975), p.377.
- (2) Murata, S., Miyake, Y., and Tsujimoto, Y., *Bull. JSME* 20-145 (1977), p.819.
- (3) Kuo, Y.H., *J. Math. Phys.* 32 (1953), p.83.
- (4) Messiter, A.F., *SIAM J. Appl. Math.* 18-1 (1970), p.241.
- (5) Stewartson, K., *Mathematika* 16 (1969), p.106.
- (6) Dennis, S.C.R., and Dunwoody, J., *J. Fluid Mech.* 24-3 (1966), p.577.
- (7) Imai, I., *Fluid dynamics*, vol.1 (in Japanese) Shokabo (1973).
- (8) Janour, Z., *NACA Tech Memo.* No.1316 (1951).